Course on Differential Equations for Engineers
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Lecture 54
Temperature in an infinite rod

Welcome back, last video we have seen we have defined 1 dimensional heat equation and initial data so that gives the initial value problem and we try to get this try to get the solution. We also have seen that the solution of this problem is unique so that if we construct a solution now that should be the unique solution, so how do we do this we do this construction based on the properties of the solutions of the heat equation, okay.

So we will see one of this you can see that one of them is translation in the spatial variable solution u of x minus yt is also a solution any derivative of this solution is also a solution and linear combination finite sum is also a solution and then also you take the convolution integral of spatial scaling with function. So basically convolution of solution with some suitable function g any suitable function gy so that the convolution should be finite, so that means the integral make sense that is also a solution, okay.

So we will just prove this properties and also if you use this scaling in the variables x and t with (())(1:38) any a positive u of root ax at is also a solution, so if u is a solution u with x and t variables are scaled with this factors is also a solution. So we can see all these things we will just show we will prove that these are solutions of the heat equation and then we would move on to construct the solution of the initial value problem, okay.

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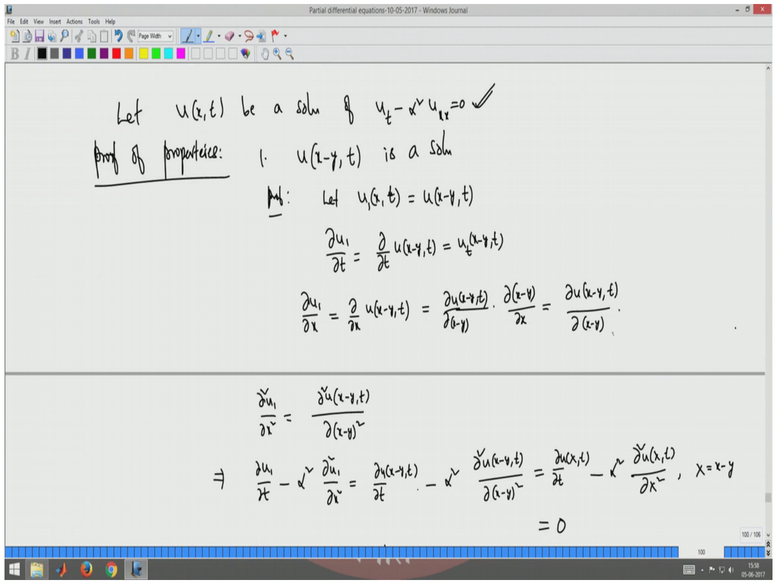

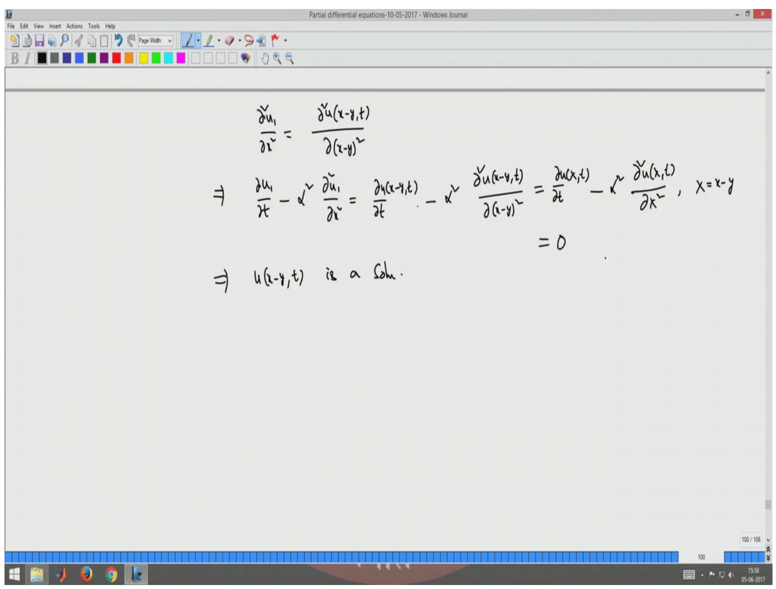

So let us start with showing that properties proves of properties thing is 1, 1 is u of x minus y t is also a is a solution so before I do this I assume that let u of x, t be a solution of heat equation that is ut minus alpha square uxx equal to 0, okay. So once you assume this one u this is also a solution so this proof is easy we can see that ut you need so you want to calculate ut so that is equal to so ut of so what is the meaning of ut so let you call this a u tilde or let us say u1 u1 of xy x, t is equal to u of x minus y, t, okay. So you want to check whether u1 is a solution so dough u1 by dough t is same as dough dough t of u of x minus y, t, okay.

So this is same as you simply ut r at x minus y, t that is okay, okay this is same so this is fine. So what happens dough u1 by dough x this is equal to dough dough x of u of x minus y, t so this is same as dough u by dough x minus y, okay half x minus y, t into dough x minus y by dough x so this is 1 so that implies is simply dough u x minus y, t by dough x minus y. So x minus y is the variable spatial variable.

Similarly you can calculate one more so you have a dough square u1 by dough x square is actually is equal to dough square u of x minus y, t as a function the derivative is also x minus y square, okay. So clearly this implies ut u1t basically dough u1 by dough t minus alpha square dough square u1 by dough x square is nothing but dough u by dough t of at of x minus y, t minus alpha square dough square u of x minus y, t by dough x minus y square, okay think of this is like this is think is dough dough t of u of capital X, T minus alpha square dough square u of x, t by dough x square so this exactly your wave equation for u where x is x minus y a new variable.

So this is actually equal to because u of x, t is a solution so because we know that this is a solution of u of x, t is a solution of heat equation so you have simply instead of x you have capital X everywhere, okay even the derivatives so that implies 0 so you have that implies u1 that is u of x minus y, t is a solution, so this is (())(5:49), okay.

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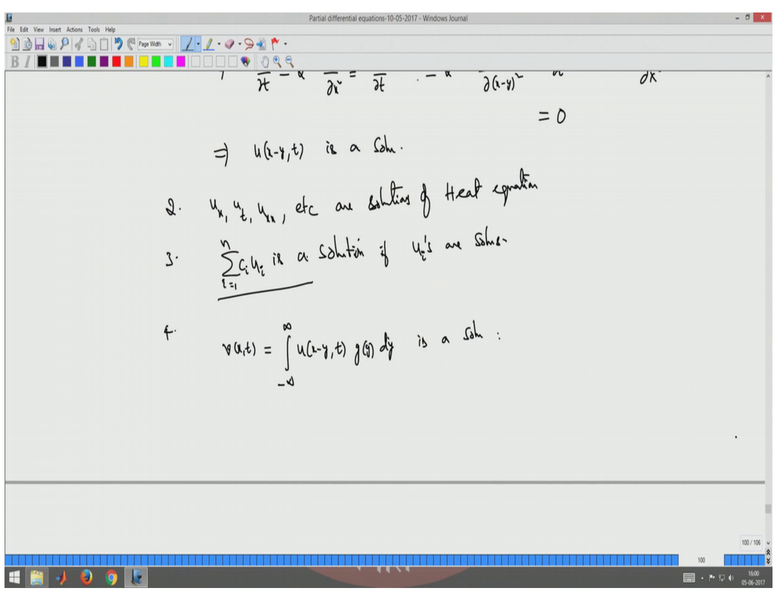

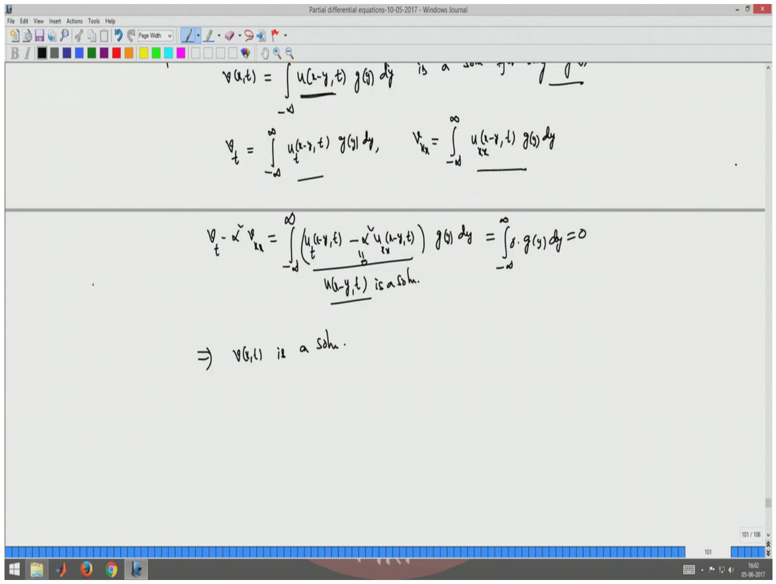

Second property is derivatives any derivatives this is trivial I will not do so this ux, ut, uxx and so on etcetera are solutions of heat equation. Third one is linear combination of this is a solution, right? So what you have is third one is if (())(6:15) are solutions u1, u2, un are solutions because it is a linear equation linear heat equation is a linear equation so linear combination of finite sum so finite sum of ui with ci i is from 1 to n is a solution if ui's are solutions.

If ui's are solution linear combination is also a solution that is also trivial because it is a linear equation you simply substitute and see you can verify that, okay. So the fourth one is this convolution integral let us call this v of x, t v of x, t is equal to minus infinity infinity u of x minus y, t g of y dy is a solution if so we have seen that solution for any g of y, okay any function g of y so that this makes sense.

So we know that this is solution already so you try to calculate vt, vt is equal to minus infinity infinity this is simply dough u dough t so simply ut at x minus y, t g of y dy you need vxx so vx first we calculate so this will give me what you get you get you simply get ux so uxx vxx is uxx uxx of x minus y, t g of y dy vx vxx both you simply take this derivative inside so this is integral with dy so you need not worry so simply dough square u by dough x square at x minus y, t so that is the meaning vxx.

So vt minus alpha square vxx is actually equal to integral minus infinity infinity you can now sum these two integrals you have vt x minus y, t minus alpha square uxx at x minus y, t into g of y dy and you see a this is we know that u of x minus y, t is a solution if u is the solution so this is equal to 0 this is integral minus infinity infinity 0 into g of y dy which is 0. So because u of x minus y, t is a solution so that is why is a solution if I use that means this is equal to 0.

So implies you have this convolution v that implies v of x, t is a solution, so this is the property four and you have the fifth one that is scaling so that is also easy to show that.

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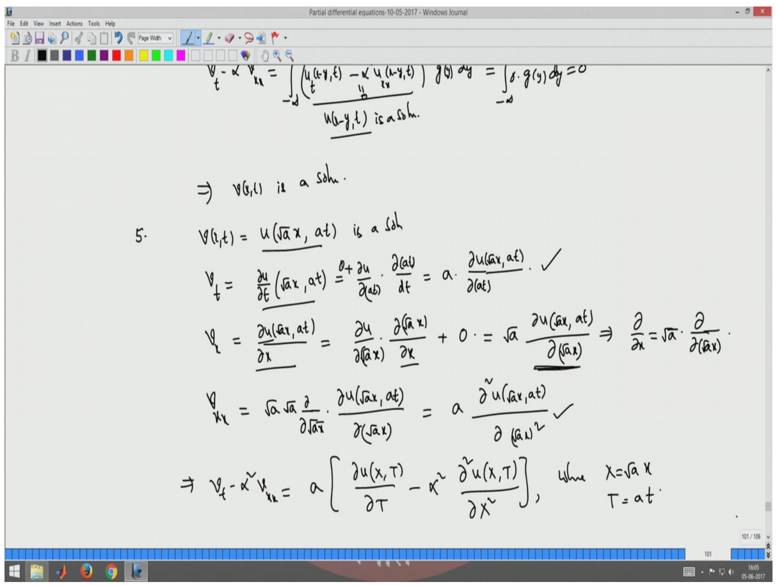

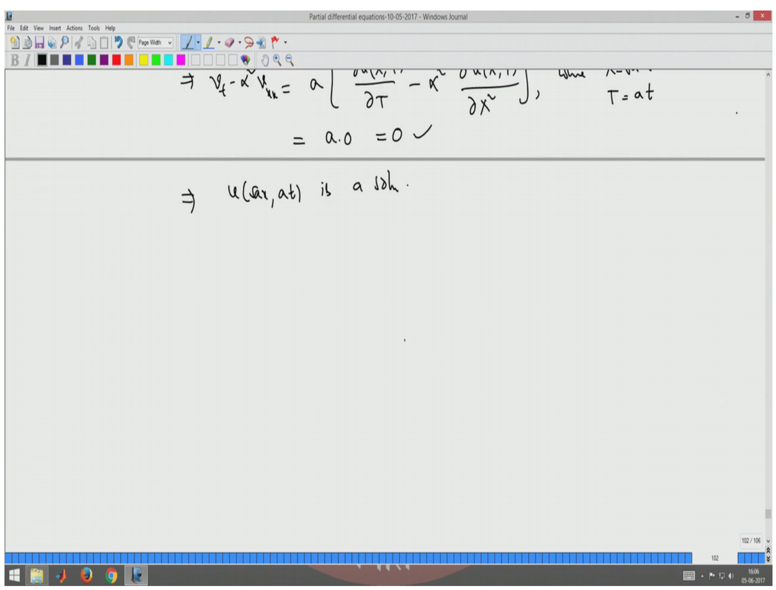

If u is a solution u of x, t is a solution what you need is v of x, t equal to u of or root ax, at is a solution this also we can show. So v, t is equal to so you have what you have is dough u by dough t of root ax at equal to dough u by dough at into dough at by dt, so this will give me a times dough u by dough at u at dough t ax at this is 1 vxx is dough u by vx start with vx so you have dough u by dough x of u at root ax at this is actually equal to dough u by dough root ax and this is the variable into dough root a into x with dough x so that is what is the meaning, okay.

And actually you have this one plus dough dough this is actually the first one even this one dough u by dough root ax into dough root ax divided by dough t so that is 0, so the first you have 0 part plus this one now this one plus now this is a 0 part dough u by dough at into dough at of dough x that is 0 so you have 0 so together is this one. So what you get is a root a this one is root a into dough u, u is now actually root ax at by dough root ax.

So if you do repeat one more time vxx it will be root a now if you differentiate this again one more time with respect to x dough dough x means dough dough root ax into root a that is what is dough dough x, okay. So this actually implies dough dough x means v is actually this, okay so from this one and this one you can see that dough dough x is actually root a times dough dough of root a into x that is what I have written dough dough x into whatever you have here.

So that is dough u root ax at by dough of root ax, so this is nothing but a times dough square u root ax root at that is a function of u into dough square root of ax square rather, okay. So this whole thing is a variable so now you can see that vt minus alpha square vxx is nothing but a is a common here here and here so a we take it out so you have so u dough u of capital X capital T by dough t dough t this capital minus alpha square dough square u of x, t divided by dough x square capital x square where x is root ax t is at that is what it is.

So now you now you know that u of x, t is a solution of the heat equation so this is equal to 0 a times 0 that is 0 so implies v so that implies v is root ax, at is a solution. So this is how we can solve we can control is how we have shown the properties of the solutions of heat equation.

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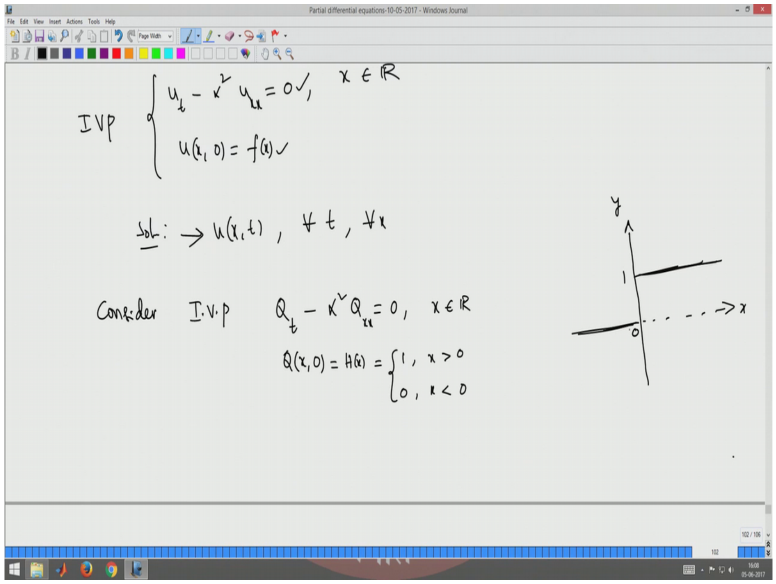
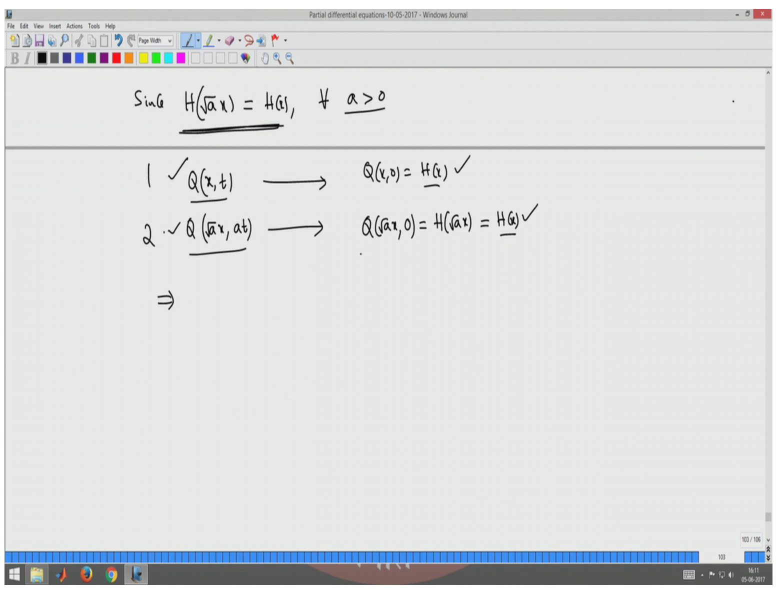




$$\Rightarrow \quad Q(x,t) = Q(\sqrt{a}\,x,\, at), \quad \forall\, a > 0$$

Let $a = \frac{1}{t}$, since $t > 0$, $a > 0$

$$\Rightarrow \quad Q(x,t) = Q\left(\frac{x}{\sqrt{t}}, 1\right) = q\left(\frac{x}{\sqrt{t}}\right) \checkmark$$

$$= g\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

Let $g(z) = q(2\alpha z)$

$$\Rightarrow q(z) = g\left(\frac{z}{2\alpha}\right)$$

$$Q(x,t) = g\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

Let $\frac{x}{2\alpha\sqrt{t}} = p$ ✓

$$Q(x,t) = g(p)$$

$$Q_t = \frac{\partial Q}{\partial t} = \frac{dg}{dp} \cdot \frac{dp}{dt} = g'(p) \cdot \frac{x}{2\alpha} \cdot \left(-\frac{1}{2}\right) \cdot t^{-3/2} = -\frac{x}{4\alpha\sqrt{t}} \frac{1}{t} g'(p)$$

$$Q_x = \frac{\partial Q}{\partial x} = \frac{dg}{dp} \cdot \frac{dp}{dx} = \frac{1}{2\alpha\sqrt{t}} \cdot g'(p)$$

$$Q_{xx} = \frac{1}{4\alpha^2 t} g''(p)$$

$$\Rightarrow Q_t - \kappa^2 Q_{xx} = 0 \quad \Rightarrow \quad -\frac{x}{4\kappa\sqrt{t}}\frac{1}{t}\,g'(p) - \kappa^2 \frac{1}{4\kappa^2 t}\,g''(p) = 0$$

$$\Rightarrow \quad g''(p) + 2p\,g'(p) = 0 \checkmark \ (\text{ODE}) \qquad e^{\int 2p\,dp} = e^{p^2}$$

$$\int_0^p \left[g'(p)\,e^{p^2}\right]' dp = 0 \quad \Rightarrow \quad g'(p)\,e^{p^2} = g'(0) = c_1$$

$$\Rightarrow \int_0^p g'(p)\,dp = \int_0^p c_1 e^{-p^2}\,dp$$

$$\Rightarrow \quad g(p) = c_1 \int_0^p e^{-s^2} + c_2$$

$$Q(x,t) = g\left(\frac{x}{2\kappa\sqrt{t}}\right) = c_1 \int_0^{\frac{x}{2\kappa\sqrt{t}}} e^{-p^2}\,dp$$

$$\Rightarrow \quad g(p) = C_1 \int_0^{p} e^{-s^2}\, ds + C_2$$

$$Q(x,t) = g\left(\frac{x}{2\kappa\sqrt{t}}\right) = C_1 \int_0^{\frac{x}{2\kappa\sqrt{t}}} e^{-s^2}\, ds + C_2 \quad ✓$$

$$Q(x,0) = H(x) \Rightarrow \begin{cases} C_1 \int_0^{\infty} e^{-s^2}\, ds + C_2 = 1 & , \text{ if } x > 0 \\ C_1 \int_0 \qquad\qquad = 0 & , \text{ if } x < 0 \end{cases}$$

$$Q(x,0) = H(x) \Rightarrow \begin{cases} C_1 \int_0^{\infty} e^{-s^2}\, ds + C_2 = 1 & , \text{ if } x > 0 \\ C_1 \int_0^{-\infty} e^{-s^2}\, ds + C_2 = 0 & , \text{ if } x < 0 \end{cases}$$

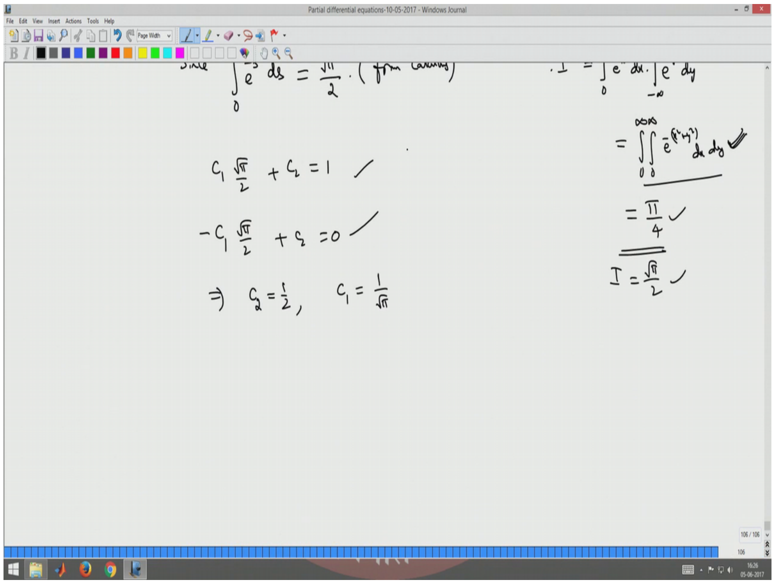
Since $\int_0^{\infty} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}$. (from calculus)

$$I^2 = \int_0^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy \quad ✓$$

$$= \frac{\pi}{4} \quad ✓$$

$$I = \frac{\sqrt{\pi}}{2} \quad ✓$$



So based on this we will try to find the solution of the initial value problem that is ut minus alpha square uxx equal to 0 x belongs to r this is the initial value problem so that is initially you have a profile fx.

So is an infinite rod both ends are infinite so we have infinite ends this is the initial value problem so initially that rod is having a temperature fx at every point x, x belongs to r and so we want to know what is the solution is actually is to find u of x, t for all times for every t and every x, okay and every x that is what you need to find so that is the solution so the solution is this one. So how do we find this solution? We construct the solution we know that this problem has a unique solution and based on this fives properties whatever we have shown for this wave equation we will just construct it.

So to start with what I do is I simply consider this solution this initial value problem I consider solving it this way. So what is that we do we consider so consider to solve to construct the solution we consider consider the same another initial value problem so we call this some u we call this for some Q, okay for some Q you consider this solution so consider this same problem similar problem same basically with initial data is different so you have alpha square Q of x, t Qxx equal to 0.

So the heat equation satisfied by the capital Q x belongs to r now I take the initial data in a different way so x0 as H of x, so what is H of x? H of x is actually Heaviside function 1 if positive, 0 if it is negative so such a function you consider and what you see is this is what you have so this is x and this is y this is x and this is y so you have Heaviside function is this is 1 and this is 0, okay.

So this is what is your function H(x), H(x) is defined for every x so if you consider this thing how do we solve this one. So the main why we choose this Heaviside function is translation so the scaling of this function H of ax is actually same as H of x for every a positive so you can easily see that, okay H of ax still x positive if as long as a is positive ax is always positive for x positive it is still 1 and x is positive, okay ax is positive since a is positive for every x positive this is true, okay.

So that is why this is the main idea so since this is true, okay what you have is Q of so since this one so let me take this scaling in this way so root ax so root a into x, okay that is what we have seen. So a scaling dilation or scaling variables root a with x and a with t for the solution of heat equation is also a solution that is the property number 5. So because of that since we have this Heaviside function is the same whatever maybe the scaling because of that what happens if basically x, t is satisfying the initial value problem with the initial data this is the initial so what is the initial data Q of x0 is Hx if I choose Q of root ax, at this is also a solution we know that this is also a solution of the heat equation and what is this initial data Q at root ax and t equal to 0 that is 0 which is equal to H of root ax by definition of this from here and we know that this is root H of root ax is Hx itself the Heaviside function itself.

So you have two solutions this is 1 and this is 2, two solutions of the heat equations with the same initial data you can see that initial data is same at t equal to 0 Q of x0 is Hx and Q of root ax 0 is also this one. So for this function t equal to 0 is Hx, this function t equal to 0 is also Hx, so initial data is same you have two solutions. Now because the solution is unique for the initial value problem for the heat equation which we have shown earlier with energy argument so this should be same so Q of x, t should be same as Q of root ax at for every t a positive since t is positive I can choose a as let I can choose a equal to 1 by t.

So since t is positive a is also positive that means a equal to 1 by t that is also positive. So this gives me Q of x, t equal to Q of so root ax times so 1, okay x by so sorry this is since I choose a equal to this so x by root t, x by root t is root a into x, okay I am just putting it here a so equal a equal to 1 by t equal to t that is 1. So this is equal to so now you call this some new function so I am looking for so is a simply a function of Q by x by root t 1 so this is some other function of x by root t, okay.

So we do not know exactly what this Q is and what this small q is, so that is how it is so you can look for solution in this form because because of this because this is true now if I choose a equal to this then my solution is actually becoming function of x by root t so which I am calling Q of x by root t. So once you have this I will try to change this into a new form so that is some so what you have is let g of z equal to q of 2 alpha z, so if I choose like this two alpha z then what happens to your q of z, q of z is g of z by two alpha, right? q of z is this, so if I write this one in terms of this g my g of z is x by root t so you have two alpha root t, okay.

So this is my Q of x, t so call this x by 2 alpha root t as a new variable p if you choose this one Q of x, t is g of p, okay. Now Q of t is satisfying wave equation so you calculate Qt so Qt is g dash of so this is like dough Q by dough t so this is equal to in terms of this dg by dp into dp by dt so what is this one if you actually calculate this is simply g dash of p into dp by this is your p, so p is a function of x and t x and root t.

So dough p by dough t so what is dough p by dough t, so you have a x by two alpha is constant 1 by 8 so t power so minus half into minus half into t power minus 3 by 2 so that is what you have. So you have finally minus half minus x by 4 alpha, right? x by 4 alpha you have here 1 by t so 4 alpha x by root t that you can put it t power 3 by 2 is I am writing t power 1 by root t into t 1 by t and what you have is g dash of p so that is what is 1.

And what you need is Qx, Qx is dough Q by dough x again dg by dp into dp by dx so this is simply 1 by 2 alpha so dp by dx is 1 by 2 alpha root t times g dash of p now Qxx becomes 1 by 4 alpha square t g double dash of p, okay that is what is so from this you can see that dough dough x is 1 by 2 alpha root t times dough dough p, right? So if you operate again dough dough p here so we take this out that comes out and we simply have this one so if you operate this on this gp Qx that is on this so if you operate that is simply dough dough p just x and only on this so that is what you get this one.

So Qx is this because Q of x, t satisfying the heat equation Qt minus alpha square Qxx is 0, okay so that implies what is my Qt? Qt is this Qxx is this you just put it together you get minus x by 4 alpha root t 1 by t g dash of p minus alpha square so minus alpha square and what you get is 1 by

4 alpha square t into g double dash of p equal to 0. So from this you can see that alpha square alpha square goes and what you get is simply so you get g so this can be minus minus goes you simply have g double dash of p and t, t also goes both sides so what you are left with is plus x by so with this x by 4 alpha root t is actually equal to x by 4 alpha root t is nothing but x by 4 alpha root t is what is my p no what is a p variable p is this one so this is your p x by 2 alpha root t is p so that you can use it.

So x by 2 alpha is as p, p divided by 2 t into g dash of p equal to 0, okay. So here I use this I write it as p divided by 2 1 by t, 1 by t goes so you have no t so you have only here p by 2, okay p by 2 or 2 by p 2 p, 2 p is actually 2 p, right? 2 p is, yeah 4, 4 also goes both sides t, t goes both sides what you have is x by alpha root t x by alpha root t is 2 p, so you have 2 p times g dash of e minus minus goes, okay as (())(26:36).

So this is what is the equation now, so you converted a partial differential equation into an ordinary differential equation ODE, so you can get your gp from this g dash of p this is a linear equation so how do you get this one g so you multiply the integrating factor, what is the integrating factor? e power integral 2 p d p so this is simply e power p square so e power p square is my integration factor so if this for this if you differentiate what you get is 0, okay.

Now I integrate both sides from 0 to p dp equal to 0 so this gives me g dash at p into e power p square equal to g dash at 0 and e power 0 is 1, so we have simply this is my constant let us say c1, okay. And now again you integrate again both sides integrate from 0 to p dp so this is a constant 0 to p dp now wait you cannot do this now so now you can write from this g dash of p as c1 times e power minus p square now you integrate both sides 0 to p so you have dp here integral 0 to p dp here.

If you do this one what you get is g at p minus g at 0 that you bring it to this side g at p g at 0 and here you have 0 to c1 comes out as a constant 0 to p e power minus p square dp, okay. So this is a g0 is also arbitrary constant this is because g is unknown so you call it c2 so this is what is your solution gp. So gp what is p? g of x by 2 what is your p, p is actually x by 2 alpha root t x by 2 alpha root t that is my g of p, what is g of p? g of p is g of this is exactly my Q of x, t.

So you have Q of x, t is this now you know what is this one this is actually c1 integral 0 to so this is simply dummy variable inside so you can write keep this p as a dummy variable and now this p is simply this has a function of x and t that is p is x by 2 alpha root t, okay this is same as writing instead of dummy variable I can put also s ds, right? So you do not have to use the same dummy variable even this one s square ds so that is how it is so you have this one.

So if you put p is this this is what you get plus c2, okay So this is your solution so so far I got my what is my Q of x, t so I got the solution of this heat equation in this form but I want I want q at 0 equal to H of x, H of x is equal to 1 if x is positive, so if x is positive I know that H of x is positive so what is H of x, so this implies so what is this one this is equal to 1 so Q at x0 is c1 times 0 to when you put x t is positive t is 0 x is positive so positive divided by t is 0 so this is going to be infinity e power minus a square ds so this is dummy variable so let us put this as ds plus c2 equal to now Hx H at x x positive is 1, so this is equation 1, okay this is what (())(30:53) do.

So now you take same Hx 0H of x from this again, so you have two cases if x is negative what happens? If x is negative H of x is 0 and here now c1 integral 0 to if x is negative and t is positive and t equal to 0 this is going to be minus infinity so now minus infinity e power minus a square ds plus c2 equal to 0. Now I know this integral, okay its value from the calculus since integral 0 to infinite e power minus a square ds you put it as double integral by just multiplying itself into suppose e power minus x square into e power minus y square so e power minus x square plus y square and use that circle x square plus y square, okay.

So use that polar coordinates and you can evaluate this integral as root pi by 2, okay root pi by 2 its value is this from calculus, okay. So the idea is to find this one so if you want to find e power minus x square dx you multiply with this e power minus y square dy this integral itself this is now double integral, okay e power minus x square plus y square dx dy. So this in polar coordinates you can evaluate and then so this is exactly what you want i is the required integral i square i square you will get it as pi by 4 so that is how you get it.

So if you evaluate this double integral in polar coordinates by converting in polar coordinates you get this one and finally when you take the square at both sides i will be root pi by 2 so that is how you evaluate this. Since this is the one this equations becomes c1 root pi by 2 plus c2 equal to 1 and what happens to other integral put s equal to minus s so what you get is minus c1 0 to infinity so minus c1 root pi by 2 plus c2 equal to 0.

So if you add it you get c2 will be 2 c2 equal to 1 so c2 is half once c2 is half and c1 is actually 1 by root pi, okay 1 by root pi so you solve these two equations you get this one so now I know once I applied basically initial condition so initial condition of this my new construct problem initial value problem, okay. So I know my solution is like this if I apply the initial value I could find what is my c1 and c2.

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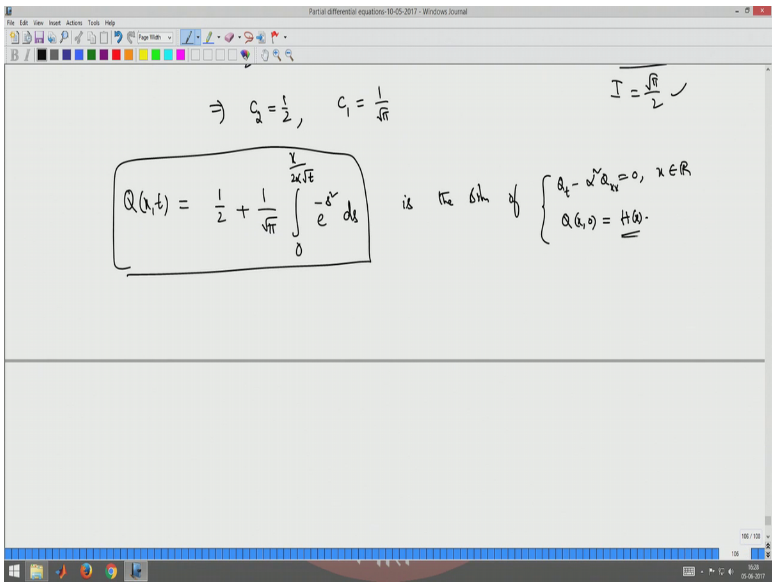

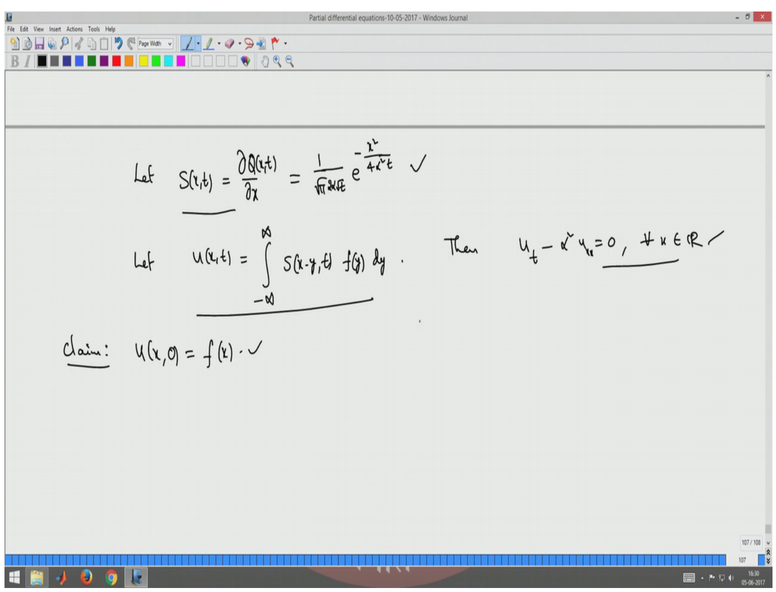

Partial differential equations-10-05-2017 - Windows Journal

Let $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, f(y)\, dy$. Then $u_t - \alpha^2 u_{xx} = 0, \ \forall x \in \mathbb{R}$ ✓

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha^2 t}} f(y)\, dy.}$$ ✓

Claim: $u(x,0) = f(x)$. ✓

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q(x-y,t)}{\partial x} f(y)\, dy$$

$$= -\int_{-\infty}^{\infty} \frac{\partial Q(x-y,t)}{\partial y} f(y)\, dy.$$

$Q(x-y,t)$

$$\frac{\partial Q(x-y,t)}{\partial x} = \frac{\partial Q(x-y,t)}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial x} = \frac{\partial Q}{\partial (x-y)}$$

$$+ \quad \frac{\partial Q(x-y,t)}{\partial y} = \frac{\partial Q(x-y,t)}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial y} = -\frac{\partial Q}{\partial (x-y)}$$

$$= 0$$

$$\frac{\partial Q(x-y,t)}{\partial x} = -\frac{\partial Q(x-y,t)}{\partial y}.$$

Partial differential equations-10-05-2017 - Windows Journal

Let $S(x,t) = \frac{\partial Q(x,t)}{\partial x} = \frac{1}{\sqrt{\pi}\, 2\alpha\sqrt{t}} e^{-\frac{x^2}{4\alpha^2 t}}$ ✓

Let $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, f(y)\, dy$. Then $u_t - \alpha^2 u_{xx} = 0, \ \forall x \in \mathbb{R}$ ✓

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha^2 t}} f(y)\, dy.}$$ ✓

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$$= -\int_{-\infty}^{\infty} \frac{\partial Q(x-y,t)}{\partial y} f(y)\, dy.$$ ✓

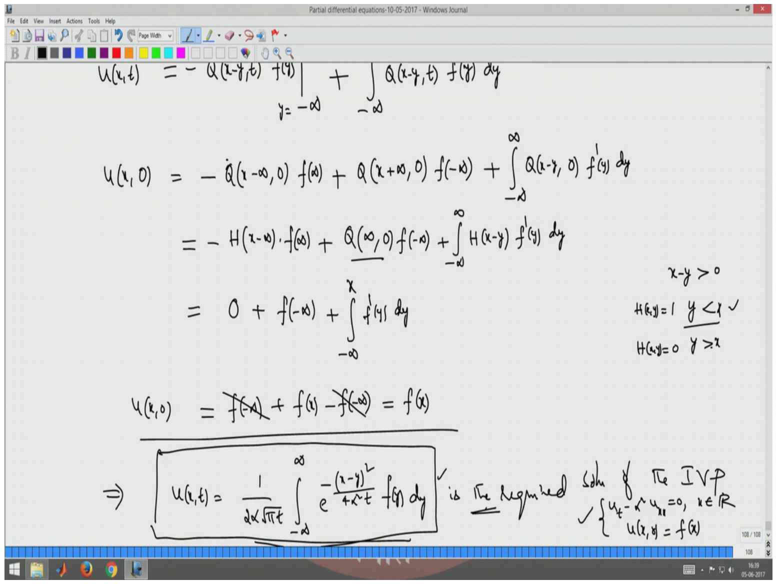
$Q(x-y,t)$

$$\frac{\partial Q(x-y,t)}{\partial x} = \frac{\partial Q(x-y,t)}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial x} = \frac{\partial Q}{\partial (x-y)}$$

$$+ \quad \frac{\partial Q(x-y,t)}{\partial y} = \frac{\partial Q(x-y,t)}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial y} = -\frac{\partial Q}{\partial (x-y)}$$

$$= 0$$

$$\frac{\partial Q(x-y,t)}{\partial x} = -\frac{\partial Q(x-y,t)}{\partial y}.$$



So just go on and substitute c1, c2 to see that Q of x, t is the solution that satisfies the initial value. So c2 is half plus c1 is root 1 by root pi integral 0 to x by 2 alpha root t e power minus a square ds. So this is my solution is the solution because why I am saying the solution because this problem initial value problem has a unique solution this is the solution that satisfies solution of question, t minus q alpha square Qxx equal to 0 x belongs to R with the initial data Q x, 0 equal to Hx, okay.

So this is the solution this is the solution of this initial value problem, now what you need is different so you need something else you need initial value problem with arbitrary initial data, okay. So what I do is to get to solve that problem actual initial value problem you consider simply dough Q by dough x of Q of x, t so this one you call this S of x, t I am calling S as x derivative of this function, okay this solution.

I constructed this solution with initial data is Heaviside function and this one if I differentiate with respect to x I call this S. Now you use the property 4 that is let u be u of x, t this is how it will be solution so I am just constructing you see that S of you take the convolution of this S, S x minus y, t is the property 4 with some function that function is the initial data that is f of y dy, f of x x is y is also between minus infinity this is the initial data you consider, okay.

So this then u satisfies you know that u satisfies the heat equation, okay then u this is what is the property number 4 ut minus alpha square uxx equal to 0 for every x belongs to infinity, okay this what we know already only thing you have to show is that u of we have to show that x this is with this convolution with this function we can show that x0 is actually fx, okay so claim is this if I show that this is true that means this is the solution that satisfies, okay so this is the solution with S being this is my solution of that initial data, okay.

So what is this one, this is actually you can calculate so S is dough Q by dough x this is 0 and this is 1 by root pi into e power minus so you differentiate with respect to x so you get 1 by so you differentiate what you get e power minus x square first of all you get e power minus x square by 4 alpha square t, okay and you have differentiate this with respect to x that means that will give you 1 by 2 alpha root t so 2 alpha root so this is my S of x, t, okay.

So that you translate it and put it here and you see that that will be the solution. So what is that solution? So this implies u of x, t is actually equal to 1 by 2 alpha root pi t this is minus infinity infinity, okay S of x minus y so you have e power wherever x is the we put x minus y whose square by 4 alpha square t that is what is my into f of y dy. So this is my this function this is the solution of heat equation.

I just showed that u at x, 0 is actually equal to fx, okay how do I show this how do I show this you consider you start with u of x, t this one as this function that is minus infinity infinity what is S, S is dough Q by dough x, okay S of x minus y so you have wherever x is that dough Q by dough x Q of x minus y, t, okay into f of y dy that is my u of x, t. Now we just observe that Q of x minus y if you have this as function of x, t, okay you have this one, so for example this is your function what is the what is the dough Q by dough x of x minus y x minus y, t, okay is equal to or simply okay this is the equal to dough Q by dough x minus y into this function at x minus y, t, okay into dough x minus y by dough x.

Similarly, dough Q by dough y is dough Q by dough x minus y into dough x minus y by dough y, okay this you can take f or x, y. Now if you add them together so what you get is the one is plus other one is minus so when you add so let me write it so I do not want to confuse you so dough Q by dough y for a function of x minus y, t is actually equal to dough Q by dough Q function is x minus y by 2 into dough of x minus y into dough of x minus y by dough y and this value is 1 this value is dough Q by dough x minus y this is minus dough Q by dough x minus y.

So when you add them plus plus is equal to 0 so this means dough Q by dough x and it is a function of x minus y, t is actually equal to from this left hand side you can see that minus dough

Q by dough y when it is function of x minus y, t so that is what I am going to replace here. So I will have minus infinity so minus dough Q by dough y and it is a function of x minus y, t f of y dy. Now why I did this? Now I can now integrate by parts this is the derivative with respect to y, y is the integration variable so I can now do integration by part so you see that Q x minus y, t into f of y now between minus infinity infinity minus minus plus, okay into integral minus infinity infinity Q of x minus y, t, okay into f dash of y assume that this function f is differentiable so you have a dy, okay the initial data is differentiable, right?

That is expected because it is a initial temperature temperature is always is a continuous function so you can actually it is differentiable so smooth you cannot have suddenly you cannot have a temperature changing and piece wise continuous cannot be even just continuous it should be differentiable, okay it is reasonable to assume that initial data of the initial temperature of the rod is a continuous function so a differentiable function.

So what you get is Q this is what I have, okay so you can now write what is this one so what you get is this is my u of x, t from this I try to calculate what is my u of x, 0. So what is u of my x, 0 is? Minus a Q at x is fixed once you fix x minus y see it is y, y equal to minus infinity to y equal to plus infinity. When you put y equal to infinity x minus infinity Q is Q at infinity at t equal to 0, okay. So this is Q at infinity x is infinity so x minus infinity to 0 into f at infinity minus minus plus Q at x minus y is minus y minus infinity so it is going to be plus infinity 0 at t equal to 0 into f at infinity f at minus infinity, okay plus this one as it is so this one is Q of x minus y 0 f dash of y dy this is what you have. So what is this one? Now Q at t equal to 0 is minus H of x minus infinity so this is negative for any once any finite x value x minus infinity is negative value.

So at minus infinity this is simply 0 f at infinity this is actually remains so this is x plus infinity infinity 0 f at minus infinity plus this one Q at Q at x minus y is H of x minus y, 0 so H of x minus y f dash of y dy, okay. So this is 0 plus this will be 1 so you have f at minus infinity plus this is now from x minus y is positive y H of x minus y is 1 if this is positive, that means if y is less than x this is 1 otherwise it is 0 y is greater than or x H of x, y is 0 in this case H of x, y is 1. So if you split that integral like that so what you get is minus infinity to x f dash of dy that is where it is 1 otherwise it is 0.

So what you see is finally f at minus infinity plus now if you can integrate now fx minus f at minus infinity so this is 0 what you get is fx it gets canceled what you see finally is u x, 0 is actually satisfying fx what is my ux at 0 u of x, t what I choose is this function, okay. So this is the function I have chosen so this is nothing but is this one so this is the solution now I know what is everything here f is known f is the given initial data and this is the integral that makes sense e power minus so with respect to y it makes sense e power minus y square, okay.

So this is the solution I constructed that satisfy this initial data so implies u x, t that is 1 by 2 alpha 2 alpha root pi t times minus infinity infinity e power minus x minus y whole square by 4 alpha square t f of y dy is the required solution of the initial value problem, what is the initial value problem? Ut minus alpha square uxx is 0 for x belongs to r and u at x, 0 equal to initial rod temperature that is ux, okay.

So this is what is required so this is how we constructed this solution that we have seen that a solution is unique this problem this initial value problem we have unique solution and I construct this is one solution so implies that satisfies the initial data that means this is this solution, okay. So this is how we can construct a solution just like you can construct the solution for the 1 dimensional heat equation with the initial data, so you have a infinite rod that is having a initial temperature as smooth function fx, okay.

So just based on the properties of the solutions of the heat equation and what else we used we used a solution of another uniqueness of the solution of the initial value problem what we constructed is this solution finally it gives you the required solutions so we just constructed so because the solution is unique for the initial value problems implies this is the required solution. So this is how you have constructed the solution so you solve the heat equation and the infinite line x minus infinity infinity, okay when your domain is minus infinity infinity you found a solution we have nicely constructed based on the properties of the solutions of the heat equation.

So how to construct when you have a semi-infinite rod with the initial data on it and the boundary condition at x equal to 0 because semi-infinite means at x equal to 0 is the boundary at x equal to 0 so is a rod with initial heat so initially initial temperature is given so what you have is the at x equal to 0 either you maintain say 0 temperature or flux make the insulate it insulate at x equal to 0 end finite, okay and that at one end, then you can actually get the solution just make use of particular solution this solution this solution for the full real line, okay.

Just the idea is just to reduce just to extend this domain to minus infinity infinity make use of this solution whatever we have derived in this video. So we will see those semi-infinite (())(48:26) and then go to finite rod so finite rod for the 1 dimensional heat equation in the next video, thank you very much.