Differential Equations for Engineers
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Lecture 51
Non-homogeneous wave equation

Welcome back last video we have seen we had a recap of what we have done we have seen the unique solution for each of this boundary value problems for the wave equations. And now we we have considered non-homogeneous wave equation with initial data for which we need to find the solution. So one can find by directly integrating out so that we will do uniqueness is also shown by the one can show this same energy argument is true even for this non-homogeneous equation with initial data. So uniqueness of the solution is guaranteed and you have only need to find out workout the solutions. So we will see that in this video.

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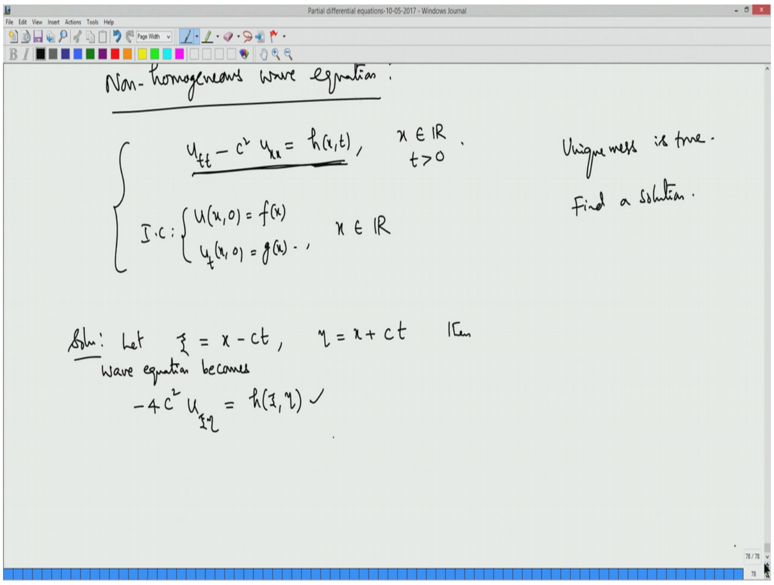

So let us take this non-homogeneous equation wave equation is right hand side you have a forcing which is nonzero h of x, t is nonzero you have a external force and you have this initial data this is for every real line on the full real line you have this one needs to find the solution. So how do I find the solution so what we do is you consider this as a partial differential equation.

So like just find the general solution how it we do, we make use of this classifications so once you try to find the normal form for this wave equation so how do we do this if you bring the new variables Xi equal to x minus ct and Eta equal to x plus ct if you consider these two new variables then you have the this equation this wave equation becomes simply you have u Xi Eta minus 4c, right if you just go back and see as (())(2:05) minus 4c square u Xi Eta equal to h of x, t x and t you replace with what happens x and t x equal to Xi plus Eta by 2 and t is Xi minus Eta by 2c so that is what is your h of h of simply x and t you can replace with Xi and Eta.

So finally you have something like h of Xi and Eta. So Xi h h is now in terms of Xi and Eta, so this is what it becomes, okay. So wave equation becomes wave equation becomes this becomes this, okay. So this will try to integrate directly so this is what we do. So we will try to see how to find how to integrate this equation so that you can find your u Xi Eta.

So what is your initial data initial data is u x, 0 equal to f x, okay so I need to so this is what your equation. So you want 0 boundary so 0 initial conditions if you want how to break this how to break this actual boundary value problem into two problems, okay so let us do this first.

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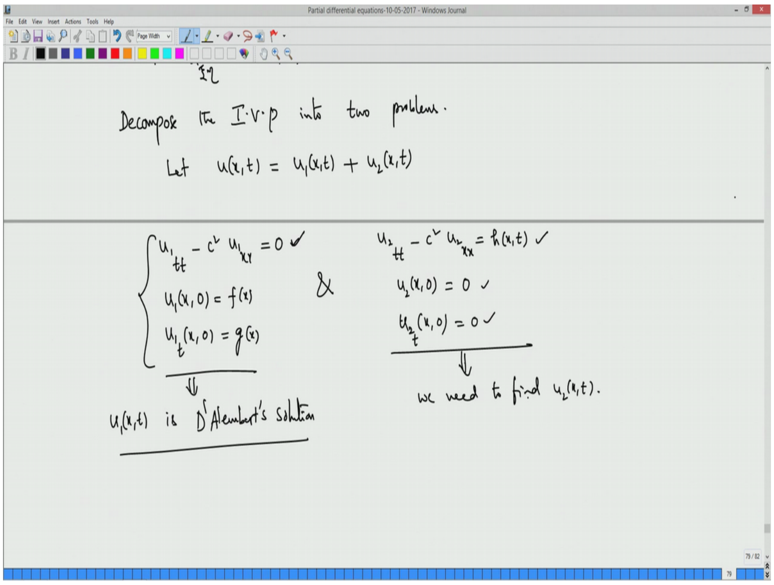

So let us break this problem into decompose decompose the boundary value problem into because this is the linear problem so you can reduce into two problems so let into two problems two problems. So let u of x, t be u 1 of x, t plus u 2 of x, t so where u 1 satisfies the homogeneous equation, okay let us choose this so that is u 1 tt minus c square u 1 xx equal to 0 and then initial values.

So u 1 at x, 0 equal to you take the full initial values this takes full initial values u 1 t at x, 0 is g x. So from this problem given for u 1 this initial data and the non-homogeneous part I I make it 0 for this u 1 so this is for u 1 problem and for u 2 you have u 2 tt minus c square u 2 xx equal to h of x, t. So that when you substitute u 1 plus u 2, okay you apply u tt is u 1 tt plus u 2 tt minus c square u xx that is here minus c square u 1 xx minus c square u 2 xx. So this will be 0 because of this and this will be h of x, t so finally this is satisfied equation non-homogeneous equation it becomes, okay.

And then if you now the initial data becomes u 2 at x, 0 is you have to give 0 because u at x, 0 is f x this f x is u 1 is already taken so u 2 has to be 0, okay. Similarly u 2 t at x, 0 has to be 0. So I made I decompose the actually non-homogeneous boundary value problem initial value problem, okay. So actually it is a initial value problem non-homogeneous initial value problem into two decompose into two problems one is homogeneous wave equation with initial data this is what you have D'Alembert's solution. So u 1 of x, t is D'Alembert's solution D'Alembert's solution.

So you know that you know this solution is D'Alembert's solution, this is what I should find out. So we need to find we need to find u 2 of x, t this I do not know, okay this we have not seen so far. So to do this what we made use of this new variables Xi and Eta, So Xi and Eta so this is what is your problem now so your problem is not actually non-homogeneous now non-homogeneous with nonzero initial data what you have is this non-homogeneous equation with 0 initial data is actually your problem that is how you decompose.

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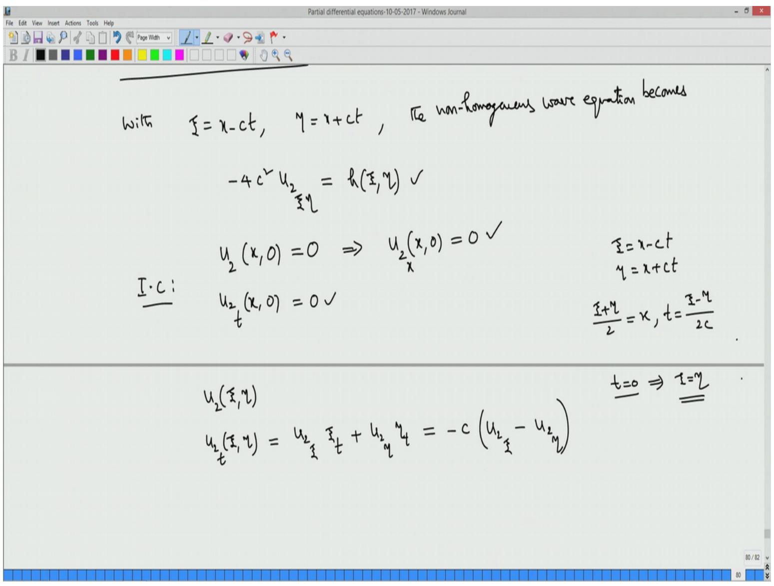

So with this new Xi equal to x minus ct, Eta equal to x plus ct with Xi is this, Eta is this we can see that u 2 of Xi Eta u 2 Xi Eta derivatives minus 4c square equal to h of Xi Eta this is what it becomes for this non-homogeneous wave equation becomes this, okay the non-homogeneous non-homogeneous wave equation becomes with this new variables. So this is simply normal one putting them in the normal form in the new variable Xi and Eta we have seen. So we will do in the classification we have already seen that how to make this one so this is what you will get so this is what you have to integrate so you want to integrate to find this u 2.

How do we do this so you have this you now you consider this initial data so initial data is u 2 at x equal to 0 u 2 at x, 0 equal to 0 that is the initial data, right u 2 at x equal to 0 is the initial data. So this implies u 2 x at x, 0 is also 0, okay so you simply differentiate with respect to x u 2 so you will see that this is true. Now what is the other initial data initial data initial conditions are this u 2 t at x, 0 equal to 0, okay so this is the other boundary condition.

So now u 2 is function of Xi and Eta, okay so what is the u 2 t of Xi and Eta so u 2 t is u 2 Xi dou Xi by dou t Xi t plus u 2 Eta Eta t, okay so this is equal to what happens Xi t is minus c minus c u 2 Xi and here plus c Eta t is plus c that is u 2 this is minus minus plus, okay so you have c u 2 Eta so that is what your u 2 t Xi Eta, okay. Now from this Xi and Eta one can find x Xi is x minus ct, Eta is x plus ct. So you have Xi plus Eta by 2 is x, t is Xi minus Eta by 2c. So t equal to 0 means Xi equal to t equal to 0 means Xi equal to Eta, okay. So that is what you have to find out so when you want put t equal to 0.

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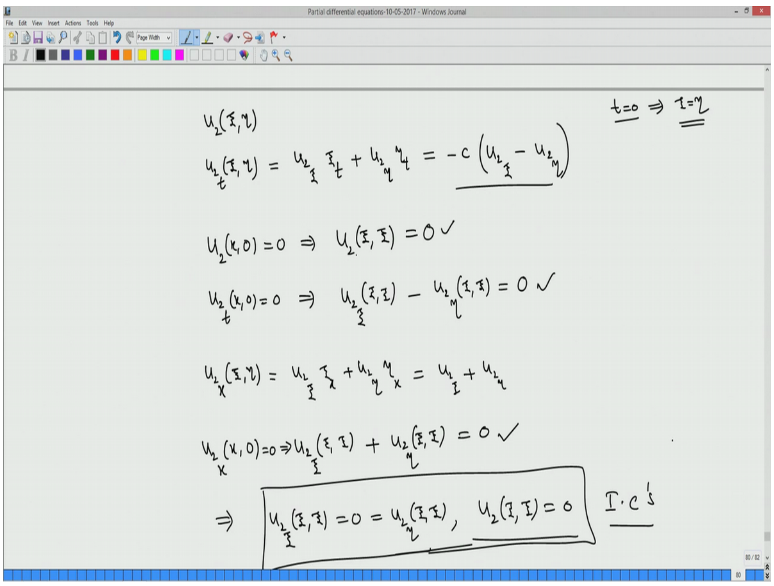

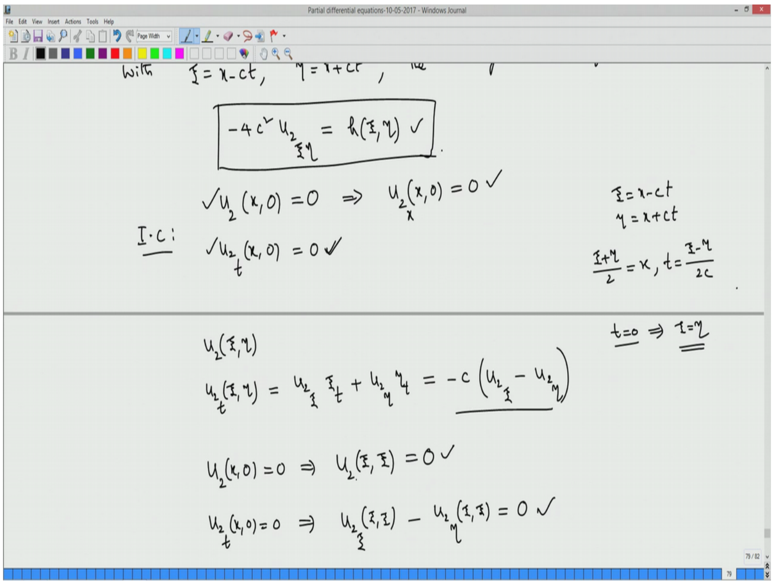

So u 2 at t equal to 0 u 2 at x, 0 equal to 0 is nothing but u 2 at Xi, Xi equal to 0, okay because I put Eta equal to Xi so this is what is the condition in the new variables u 2 Xi, Xi is 0 so that is what is this first initial condition. And second one so the second one second one is this second one will give you second boundary condition that is u 2 t time derivative of x, 0 equal to 0 will give me that means this is equal to 0.

So you have u 2 Xi minus u 2 Xi at x, 0 means you have to write Xi, Xi because Eta is equal to Xi at t equal to 0 minus u 2 Eta at Xi, Xi equal to 0 that is what is this initial data. Now you also know this u 2 x at x equal to 0 what is u 2 x we do the same way like u 2 t if you do for u 2 x u 2 x derivative at x, 0 or Xi Eta Xi Eta is u 2 Xi Xi x now you have to take Xi x plus u 2 Eta Eta x. So Eta x Xi x is one so you have u 2 Xi plus u 2 Eta.

So this means u 2 x at x, 0 is nothing but u 2 Xi the derivative at Xi, Xi plus u 2 Eta at Xi, Xi equal to 0, okay. So this is equal to 0 means this is equal to 0, okay. So consider this and this their sum is 0 and difference is 0 that means each of them is 0. So u 2 Xi Xi, Xi equal to 0 is also true for u 2 Eta Xi, Xi is also 0, okay and you also know that u 2 at Xi, Xi is also 0. So you have these are the initial data initial data becomes this this is your initial data initial conditions in the new variables, okay. What is the equation becomes? Equation becomes this, equation is this and your initial data is this now try to integrate.

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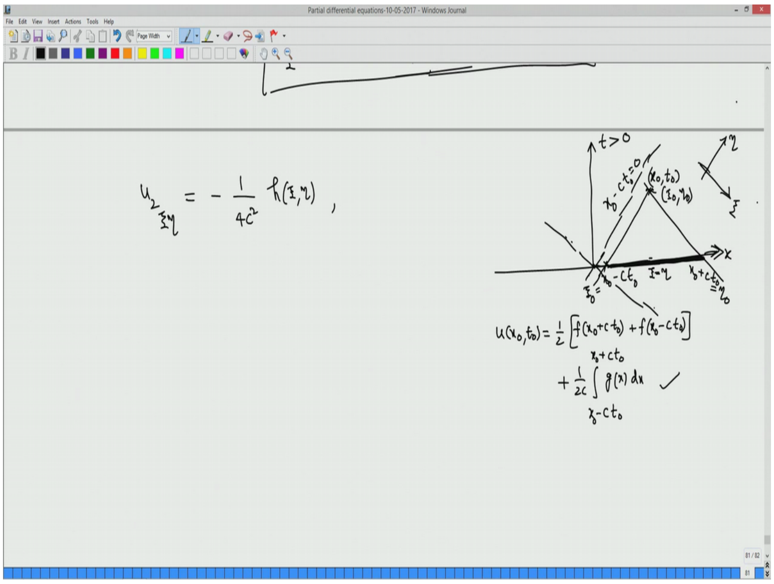

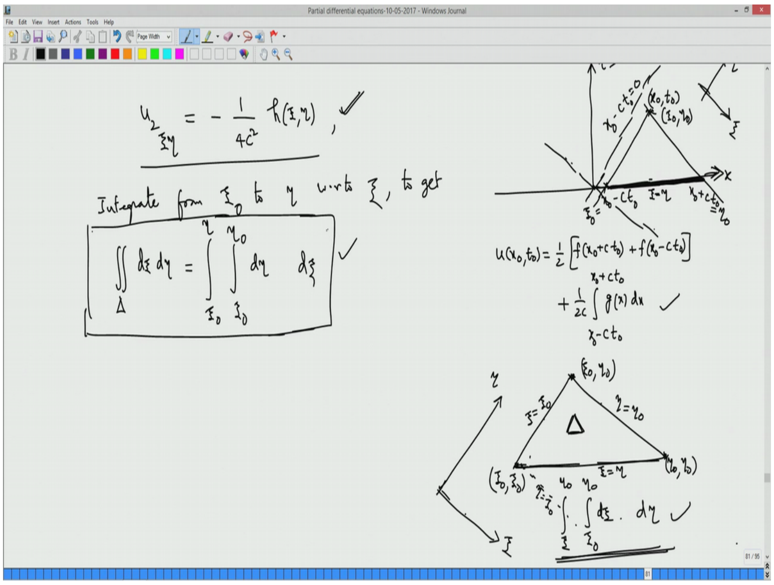

Now if you try to integrate this one simply get u 2 Xi Eta is u 2 Xi Eta equal to minus 1 by 4c square h of Xi, Eta this is what is this this you have to integrate. So your full x is full real line and t is positive, this is your x, okay. So if you want if you want solution at some point let us say some x not, t not really want this one, okay. So we have to define something like domain of dependence domain of independence.

So if you look at D'Alembert's solution (u x, t) u x not, t not if I write u x not, t not that is actually dependent half of initial data that is f at x not plus ct not Xi is whichever okay x plus c t not plus f at x not minus c t not, okay this is half of this plus 1 by 2c integral x not minus ct not plus x not plus ct not into the other initial data it is velocity that is g of x dx. So this is what you have this is what is the D'Alembert's solution in the full domain.

So if you want this solution and what is this x minus ct, x plus ct, x not minus ct not so this is x not minus ct not is see this is the line this is the line if you take this cuts at x not minus ct not if it is 0 there is a parallel line cuts here this is the line that passes through origin that is x not minus ct not equal to 0 and this is something else some nonzero and something similar parallel lines will go even this side this is x not plus ct not line.

So these are your characteristics, this is your Xi so Xi, Xi not equal to so if you fix your Xi Xi Xi equal to constant, okay Xi equal to Xi not this is your Xi not in Xi t in Xi Eta plane so if you take this is your this is actually this is your Xi and this is your Eta, okay. So this is the line Xi is constant so you have Xi is constant means Xi is here, okay so this is the line Xi equal to constant so Xi equal to constant is this line.

So you have Xi equal to constant means so this is your Eta axis, okay so this is your Eta, this is your Xi. So this is how it looks so this is this is the Xi, Eta so you have this this is your Eta so this is equal to some Eta not. So X not, t not this point in the new variables Xi not and Eta not, okay and you have this Xi not and Eta not, okay and what is this line this line so parallelly if you look at this x minus x plus ct equal to 0 will give me this line, okay so this is this line is exactly Xi equal to Eta this line is actually Xi equal to Eta.

So if you want a solution here at x not, t not from this D'Alembert's solution you can see that it depends on the initial data here for that matter any point inside this triangle actually it depends on this data. So this is the this is the influence solution for the solution this is the domain of influency is the domain of influency as a triangle area and this is the domain of dependence for the solution so this is actually depending on initial data only here if you want here or any data inside any solution at any point inside you need only a data maximum here, okay.

And this is because it you get every solution within this based on this this is the influence of for the solution this is the domain of influency. So this is the domain you consider as delta for which you are going to integrate try to get this solution. So you will see how we do this. So we are going to integrate over this domain in the Xi, Eta variables, okay. So if you do this you will try to redraw this picture so if you do that so what you see is this is the triangle what you have over which you want to integrate.

So this is your Xi not, Eta not x not, t not correspondingly that takes you to the new variables Xi not, Eta not and this is your Xi not Xi not Eta is also same so Xi not, Xi not, okay here this will be Eta not and here this is the line on which Xi equal to Eta so Eta not is same as Eta not. So this is what is your solution this is what is the in this what is your your coordinate axis is this, okay so this is parallel to this is your Eta line and this is your this is parallel to this line so this is your Xi line.

So within this you want to integrate this one so first you so to do that so you integrate from Xi not to Eta not as your Xi not to Eta not as your d Xi, okay and you integrate from this is your Xi equal to Eta. So we do it from Xi to Eta not with your d Eta then actually you are doing integration over this triangle, okay. So closely observed you see this this is from so if you actually see this one, okay.

So Xi is this is your Eta let us say this is your Eta line Eta is from Xi not to Eta, okay. So Xi is this is your Xi Xi line Xi is from Xi not to Eta not so I integrate from Xi not to Eta not and what is your Eta? Eta goes from Eta is from Xi from this line to upto here that is Eta not, okay. So if you try to integrate that so first you integrate first Xi to Eta not so integrate integrate from Xi to Eta not to get so x not, t not in the new variables Xi not, Eta not and this line x minus ct not equal to Xi not that is Xi equal to Xi not and this is Xi equal to Eta line and this line that is x plus ct not equal to Eta not that is Eta equal to Eta not because your Xi Eta or these are the variables Xi is x minus ct, Eta is x plus ct.

So once you have this you just have to integrate this equation you integrate. So what is your Xi? Xi is you are in this coordinate system Xi and Eta, Xi is Xi you have to integrate Xi is actually varying from so this is the line this is the line this is the Xi line Xi equal to Xi line that means this is your Eta axis. So Eta is running from Xi not to Xi not to Eta not, so Eta you have to do it from Xi not to Eta not and Xi is running between this line to this line.

So this line is Xi not to this line is Xi equal to Eta, okay. So for that reason so first you integrate so this is double integral. So double integral over this delta so double integral over this delta is basically any double integral. So this is d Xi, d Eta is actually one can see that this is the line so this is your Eta axis, Eta axis which is varying so this is from Xi not to Eta not. So Eta is running from d Eta running from Xi not to Eta not, okay this is constant.

Now Xi is between this line to this line so that you go upto here so (Xi) Eta Eta is actually Xi not means this line this line, So this line to this line so this line is Eta equal to Eta not so Eta equal to Xi not is this line, okay that is parallel to the Xi axis. So Eta equal to Xi not to Eta equal to Eta not that is d Eta so that is integration from this and then if you want integral over this delta so you should take between this line to this line that is Xi.

So the Xi is d Xi, so d Xi Xi integral should be from Xi not to Xi equal to Eta so Xi not to Eta so this is what you have to do. So we will try to so any integration goes like this integration over delta is as an itinerary integral is this. So now we try to integrate this this equation now starting with Xi. So integrate from Xi not to Eta with respect to Xi to get.

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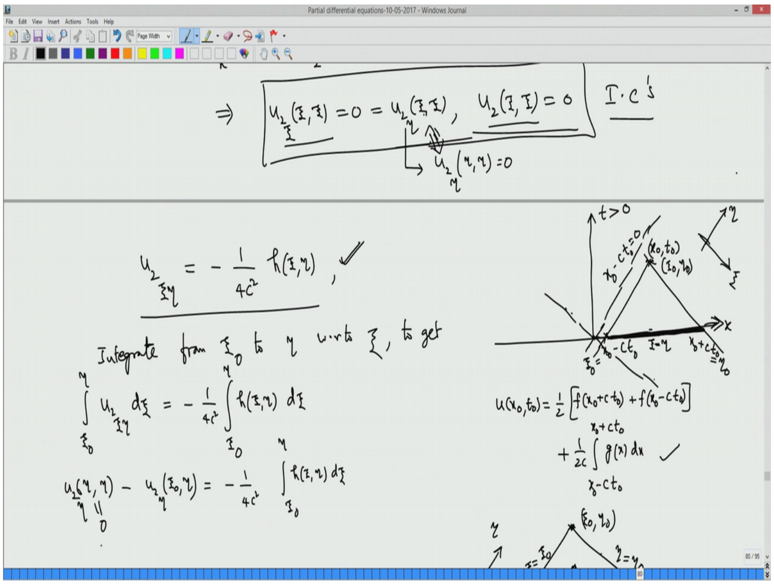

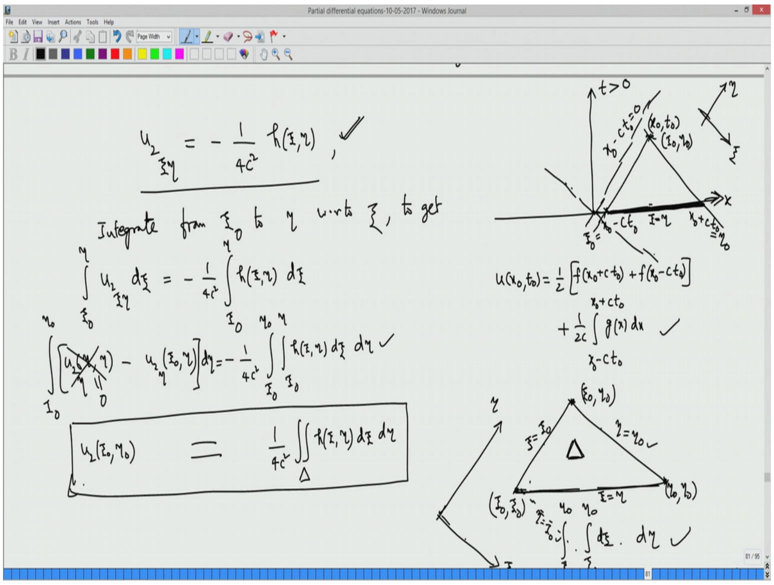

So I will just remove this so if you integrate the above equation from Xi not to integrate integrate integrate from Xi not to Eta with respect to Xi. So what you get is integral Xi not to Eta d Xi so u 2 Xi Eta of d Xi equal to minus 1 by 4c square integral Xi 0 to Eta h of Xi, Eta d Xi doing with respect to Xi. So this becomes u 2 u 2 of Xi Eta so u 2 of u 2 Eta of Xi, Eta so that is Xi you have to replace with Eta Eta, Xi minus Eta Eta Xi Eta in the place of Xi I put Eta so you have already have Eta.

So u 2 Eta of Eta, Eta minus u 2 Eta now it is at at Xi I have Xi not so Xi not Eta. So this is what after integration so this will be dou dou Xi of u 2 Eta so the u 2 Eta at Xi equal to Eta is this and u 2 Eta at Xi equal to Xi not is this this is equal to right hand side. So this 4c square Xi not to Eta h of Xi Eta d Xi. Now if this one I can integrate so now I know that u 2 Eta of Eta, Eta is also 0 this is what you get u 2 Eta of this is same that is, okay that is the condition. So u 2 Eta of Xi, Xi or Eta, Eta both are same so, okay so that is 0.

So Xi, Xi equal to Eta so that is Eta, Eta means Xi, Xi equal to Eta. So that is this is also 0 so this is actually 0. So this means one can see that this is same as u 2 Eta of Xi Xi, Xi this is Xi, Eta so both are same so when both are same this is also 0, okay so both are same these are equivalent, okay. So if you use this this is what it becomes this is 0 so now you integrate this from now.

So now your integration from Eta so Eta is if you integrate Eta variable from Eta equal to Xi not to Eta equal to Eta not so this between these two lines if you do that is the triangle. So if you now do this from Xi not Eta is from Xi not to Eta not this is with for this full integral that is 0 d Eta equal to now if you do this from this side also Xi not to Eta not d Eta. So this is the integral.

So this is anyway 0, so this quantity is 0 so you can remove this so you have a minus now you can write u 2 this is now this is dou u 2 by dou u 2 by dou Eta integral so you have finally what remains is u 2 at Xi not, Eta not. So u 2 at Xi not, Eta not minus minus plus u 2 at Xi not and you have a Xi not here. So this also 0 this we know that is 0 so this is equal to 1 by 4c square this double integral this iterate integral right hand side is nothing but the integral over this delta region h of Xi Eta d Xi d Eta, okay so this is what you get.

So u Xi is finally so we make it plus both sides cancel minus you get plus plus so finally you end you so this is 0 so we can remove this so this is nothing but simply this is what you get this is the u 2 solution at Xi not, Eta, okay.

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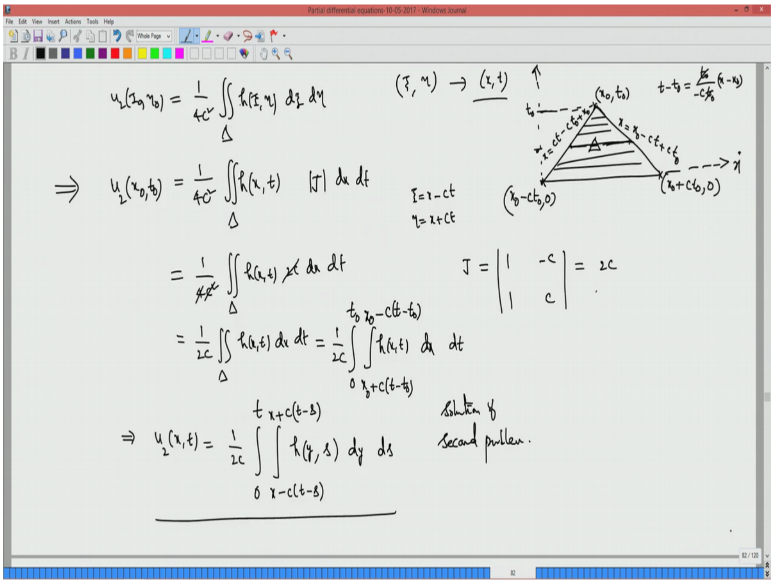

Second solution is u 2 of Xi not, Eta not is 1 by 4c square over this delta region h of Xi, Eta d Xi d Eta. So what is the delta region the domain of influency that is the domain of influence that is this is the domain. In the old variables you want to convert this into old variables so that is Xi not, Eta not becomes x not, t not and then this part becomes x not minus ct not, 0 and this becomes x not plus ct not and then 0 this this this points, okay these are the three points.

So if you write this 4c square double integral over this delta now I just changed I go from Xi, Eta to x, t x, t variables, okay. So if I do this so d Xi, d Eta thus the area element here is Jacobean times dx dt so that this becomes h of x, t, okay this is what it becomes and so that this is actually u 2 of x not, t not. So in the (new) in the old variables this is what it becomes what is Jacobean? Jacobean is modulus of the Jacobean is first of all Jacobean is the determinant of Xi x Xi x Xi is actually x minus ct, Eta is x plus ct so Xi x is 1, Xi t is minus c, Eta x is 1, Eta t is c.

So you get what you get is c plus we have 2c, so what you are left with is 1 by 4c square so 2c if you replace this is over the delta in the x, t plane this is actually t, this is actually in the x, in the x, t plane this is what you have so this is your domain d, this is your delta d, okay h of x, t now this is 2c dx dt. So this is still the double integral so 2c, 2c goes what you are left with is 4c square.

So you have 1 by 2c double integral over this delta this double integral or delta you can have you have to write as a iterative integral so that we will see how to do this. So how do we do this? So you need so if you want to do this t is running from 0 to t not so you fix your t equal to t not, okay between 0 to t not you fix your t then your x should go from this point to this point so that means from this line to this line, okay.

So this is how you are doing you fix this point and you are going from here so for every fix point you just fill this domain, okay so for that you need equation of this line. So equation of this line is you can find this equation of this line x minus x not actually y minus y not that is t minus t not equal to so t not divided by x not minus x not plus c. So you have a ct not into x minus x not.

So equation of a line passing through x not t not and you have a two points are given so from that you can calculate the slope, okay. So you what you are left with is the equation of this line is x equal to ct in the x, t plane. So you have ct plus c times minus ct not, so ct, ct minus ct not and then you bring this x not the other side so you have plus x not, okay x not plus c times ct minus ct not.

So if you do the same thing here so the equation of this line is if you do the same thing so t not and what you are left with is this difference the slope is t not minus t not divided by x not minus x not minus ct not so only difference is minus here, okay for this line. So in this case you have x equal to minus x not x not first of all minus ct plus ct not so this is are the were this is are the equation.

So if I write this as an iterative integral what you have is 2c now this is now I have 0 to t not dt now this integral other iterative integral is running from x is from x not plus c times t minus t not to to x plus x not minus x not minus ct minus t not, okay. So difference am I doing right properly so you have t 0 minus this that is fine, okay that is your h of x, t dx dx x is running from this to this, okay.

So what you have is now you can change the dummy variables x not and t not you can write it as x of t so you can write as x of t so that your dummy variable you can replace with something else. So we have 1 by 2c 0 to t t I can write it as d so let us say t I am replacing with t not, okay t not I am replacing with t not I replace with t and t not's I am replacing with t not, okay.

So you have x x not I replace with x so x plus c times (c not) t not minus t, okay so that is minus t minus t not to x minus c times so it is going to be plus because the t is t not and t not is t. So we have finally t minus t not, okay h of x is x is simply x not so I am replacing x by x not, x not by x, comma t not dx not. So x not and t not are dummy variables. So if I change this to say t not by some s what you have is t minus s, here t minus s, okay you do not want x not so you write it as some y dy x not you replace with y that is all that is what you have, okay x not you replace with y so that is only dummy variable so that is what you have, okay.

So nowhere else you have x not so you do not need to replace. So this is your second solution second solution of the second second problem, okay this is the this is the solution of second problem second problem that is for u of x 2 u 2 of x, t. So what is the solution, solution is now so one is D'Alembert's solution that is u 1 u 1 plus u 2. So what is u 1 u 1 is the wave equation with the initial data f and g.

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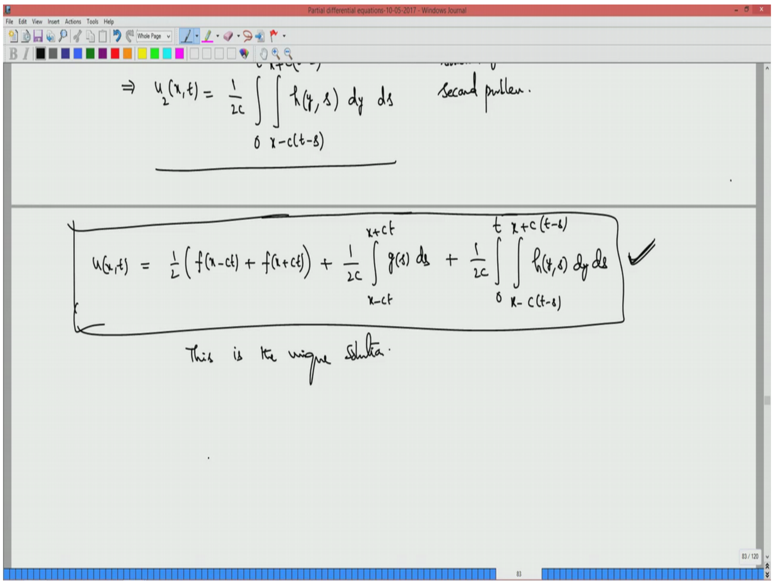

So you can write together the solution of the problem is solution of the problem is now so you can write u of x, t original problem is u 1 that is half times f of x minus ct plus f of x plus ct is the D'Alembert's solution so plus 1 by 2c integral x minus ct to x plus ct g of s ds this is my u 1. Now plus u 2 is this part so that is 1 by 2c integral 0 to t x minus c times t minus s to x plus c times t minus s this is the forcing term h of y, s dy ds so this is your this is the required solution this is the required solution of the problem non-homogeneous problem, okay.

So far we have constructed this is the solution this is the solution we have constructed where just splitting into two problems one is the D'Alembert's problem, other one is non-homogeneous problem just by integrating the equation we just found this u 2. So this is how we constructed the solution and by the same energy argument we can see that the solution of this problem the original problem for u is having a unique solution. So that means this is the solution, okay.

So this is the unique solution this is the unique solution solution for the problem, okay same energy argument that we have seen earlier we can also apply here. Consider two solutions and take the difference and it satisfies you take the difference it will satisfy some other boundary value problem. Now we use the energy argument to show that the w 1 is equal to w equal to 0. So this is the this is how you solve the non-homogeneous equation, okay.

So in the next video we will so far we have seen only 2-dimensional 1-dimensional wave equation and its problems, okay initial boundary value problems for the 1-dimensional wave equation, okay. So this is how we solve non-homogeneous equation for the 1-dimensional wave equation, okay so this is what this is the solution that is the sum of both we split the problem into two problems one is the D'Alembert's problem, other one is non-homogeneous problem with the what is exactly we have is the second solution is non-homogeneous wave equation with initial 0 initial data, okay that is what we find just by integrating over this Xi, Eta domain so over this domain of influence region.

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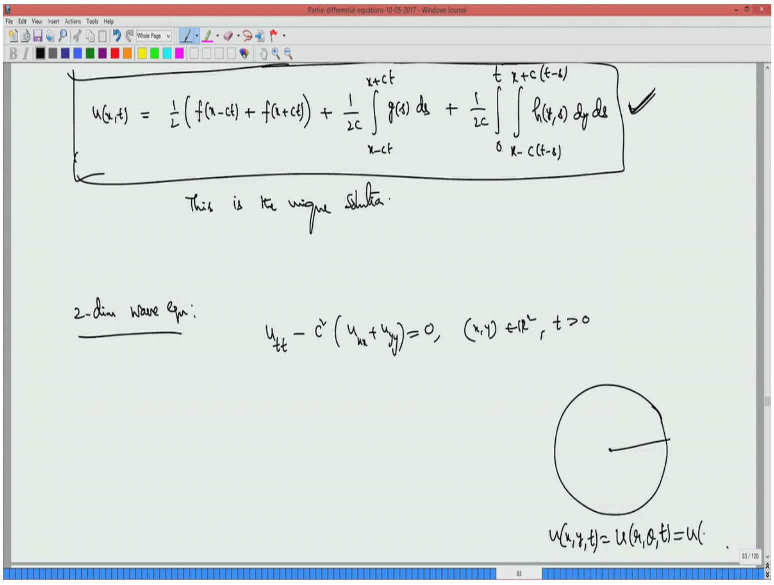

So far we are solving only wave equation in 1-dimensional, so we can also find the we can also solve 2-dimensional wave equation, okay so that we will see what is the 2-dimensional wave equation in the next video 2-dimensional wave equation it is actually same equation only thing is special domain will take the plane region. So that is u tt earlier minus c square u xx now you have u xx plus u yy equal to 0, so x, y belongs to R 2 and t positive, what we do is we do not solve in the general full domain.

So what we do is we consider we can solve certain problems in some domains that means let us consider some circular domain. So 0 to r so because the circular domain you can you can reduce into polar coordinates, we can reduce this equation into polar coordinates r and theta variables and look for solutions radial symmetric solutions that means the solution u of x, y, t that is in the polar coordinates this will be u of r, theta, t you look for solutions only radially symmetry that means should depend on it does not depend on theta that means solutions are only u of r of t, okay.

You look for solutions like that whatever solution is known if you are vibrating if whatever is the vibration along at theta equal to 0 it is repeated all along theta between 0 to 2 Pi. So that means radially symmetric vibrations you can if you look for as a solution we can actually solve this 2-dimensional wave equation using separation of variables. So we can actually bring in Bessel equation, okay that you have learned in earlier videos, okay so using the solutions of the Bessel equation we can find the solution of this vibration of the drum so for all times, okay this is what we will see in the next video, thank you very much.