Differential Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
Department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-5.
Methods for First-order ODE's - Reducible to Exact Equation.

In this video we will solve some exact equations, so let us solve some exact equations now. And also we will see how to reduce non-exact equations into exact equations and then we can integrate directly because we know the technique of how to integrate an exact equation. Okay. So this is called the method of integrating factors. So you look for certain integrating factor, so it is non-exact equation, you look for some function, you multiply it to the equation, then it becomes an exact.

So such a function which you multiply to the equation is called on integrating factor. We will see how to find these integrating factors, okay. We will solve, I will give an example of I will give an example of exact equation that can be solved, with the procedure explained you earlier. So, Example.

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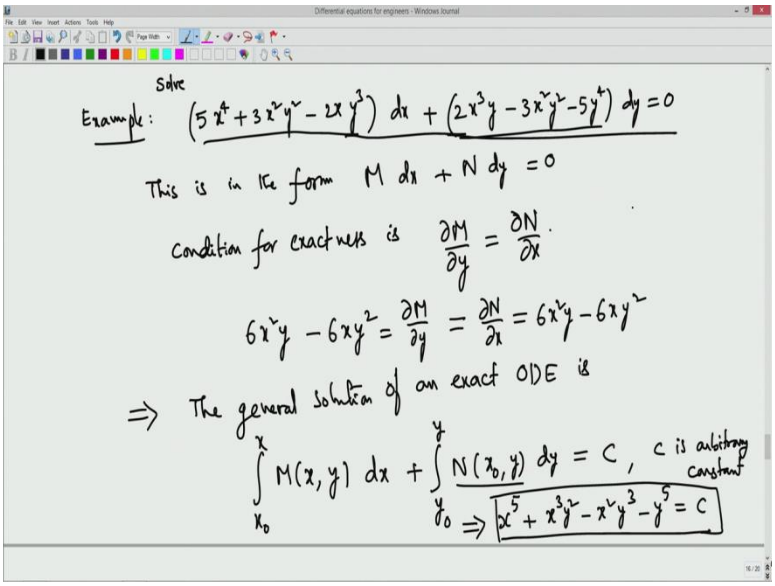

5 x power 4 + 3X square y square plus to x y cube dx plus + 2X cube y minus 3X square y square minus 5 y to the power 4 Into dy equal to 0. So how do we solve this differential equation, in the form, which is in the form M dx plus N dy. So this is in the form M of x, y dx into N of x, y into dy equal to 0. So in the procedure we just have to check whether a given

equation in exact or not. The condition for exactness is dow M by dow x, dow y is equal to dow N by dow x.

We check this condition, if this condition is satisfied, we know how to solve, how to write the solution of this equation directly, okay. So what is dow M by dow y, is 6x square y 2nd term minus 6xy square in case of 3rd come, 1st term is only function of x, so the derivative with respect to y is 0. So this is what is my dow M by dow y, I will check what is dow N by dow x, that is here also 1st term, 6x square y, 2nd term with respect to x again 6 xy square, 3rd term, function of y, if it is derivative is 0.

So clearly both are same, so the condition is satisfied. So this implies, solutions, the general solution of the equation, general solution of an exact equation, exact ODE is, we have seen this, x0 to x M of, you are doing with respect to x, okay, so keeping y as a, as it is as a variable plus you integrate y0 which is fixed number in the domain of the differential equation to y which is variable, you integrate N of y dy, x you are fixing, this is going to be x0 equals to constant. So this is the general solution.

So let us see what exactly this is, we fix that, where C is, C is arbitrary constant. So I directly write the solutions, this implies, what is the solution, M dx, this is M, this is M, whose x derivative is simply x power 5 + 3y square, so that is going to be 3X square is x cube, x cube y square minus x square y cube, that I integrated with respect to x by taking x0 is equal to 0. If I take x0 is equal to 0, this is what is the result. You can also take y0 equal to 0, if you take at 0 equal to 0 in N, you are fixing x0 which is equal to 0, this becomes only minus 5y power 4, that you are integrating with respect to y, okay.

So that will give me minus y power 5 equal to constant. So this is a general solution of the given differential equation. Okay. This is the given differential equation whose general solution is given by this where C is an arbitrary constant.

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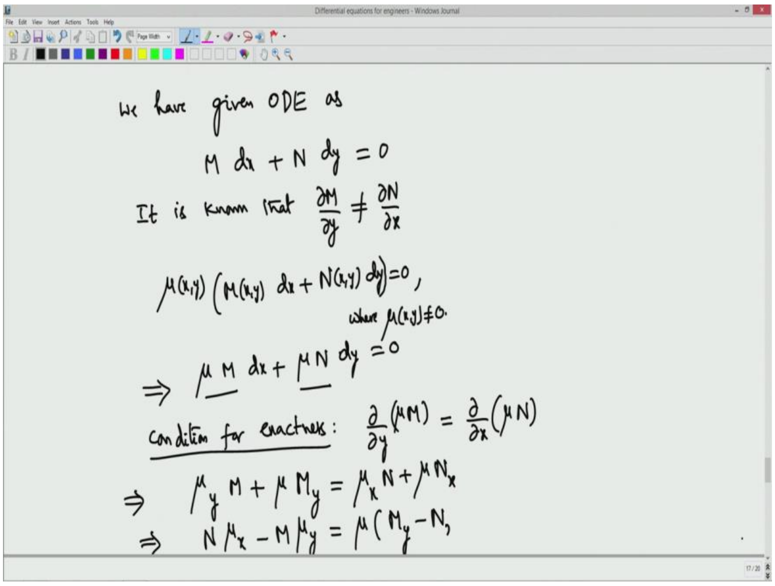

Now we will consider differential equations of first-order, ordinary differential equations of first-order which are not exact but can be made exact. So how do we make them exact? So given equations, so let us say, let us have, let us say, let us, okay. So we have given differential equation, given ODE, first-order ODE as M dx plus N dy is equal to 0, M and N are functions of x and y is not exact. That means the condition is not satisfied.

It is known that, let us say it is known that, you verify it, okay that dow M by dow y is not exactly equal to dow N by dow x, such any question you have got. So what how to make them, how to make this equation an exact equation? There is a concept called, you multiply some function to the equation. M of x, y dx plus N of x, y dy equal to 0, for this equation, left inside you multiply some function mu of x, y. So left-hand side I multiply, right-hand side is 0, so to 0.

So if I multiply like this, of course mu x, y is not 0. Where mu x, y is nonzero, if I multiply this. What function I multiply so that this equation becomes an exact equation? So that means now the equation becomes mu Mdx plus mu N dy equal to 0. So now my M and N are, this is my M and this is my N, if I multiply some function of x, y which is mu so that this equation is an exact. So what is the condition of exactness?

Condition for exactness now becomes dow dow y of M, M is now mu M, okay should be equal to dow dow x of mu of N. So that means mu y I write like this, this is the subscript denotes the partial derivative, mu is the function of x, y, so subscript, y subscription, y subscript means partial derivative of mu xY with respect to y, okay. M plus mu times MY, partial derivative of M with respect to y is equal to mu x N plus mu times N x. So this implies N mu x, I am just arranging it minus M mu y equal to mu of mu times NX, MY minus NX. Okay.

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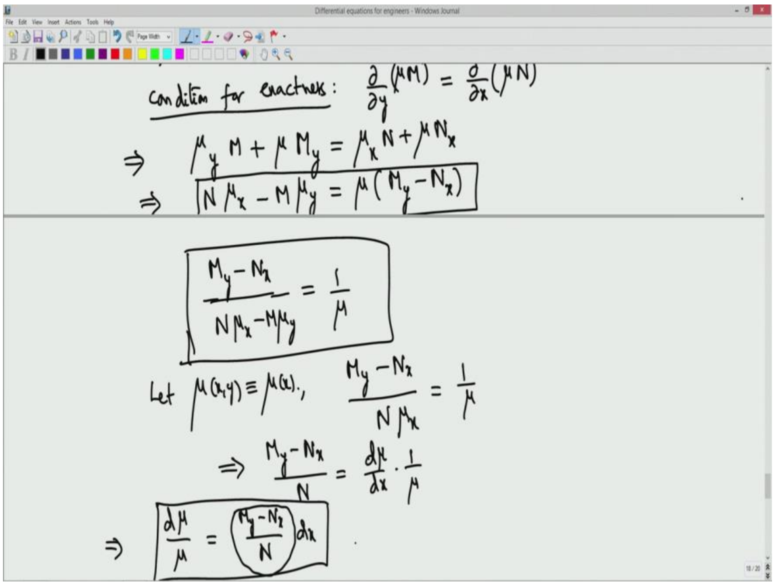

So if mu is a function of xy that will satisfy this equation. Okay, so that should satisfy this equation. So if I multiply some function mu of xy, so that is, if they satisfy this equation, I can hope to make the equation, given equation exact, okay, by multiplying this function mu. How do I do that, so how do you I find such a mu? So what you do is, you consider, so we have MY minus NX divided by N mu x minus M mu y equal to 1 by mu, okay. If mu is a function of x for example.

If you are looking for some function mu that is only function of x, then mu y is 0, okay. So this is what the above equation becomes. Mu has to satisfy which is a function of xY. Suppose sometimes instead of multiplying mu xy functions, the function of 2 variables, if you multiply only function of one variable, let us we mu x, mu x, y is only function of x, then if it becomes exact, then what happens to this equation? This becomes MY minus NX dow M by dow y minus dow N by dow x divided by, this becomes N mu x equal to 1 by mu, because mu y is 0 because mu is a function of x only. Okay.

So this implies MY minus NX divided by N equal to, what is mu x, if it is mu is a function of x alone, dow mu by dow x is nothing but d mu by dx into one by mu. If I take mu x this side, that is actually d mu by dx. Okay. So this implies, now I arrange it in such a way d mu by mu, this one I can write which is equal to MY minus NX by N into dx. If mu is a function of x which I multiply to the equation, then if the equation becomes exact, then my mu should satisfy this, okay, my mu should satisfy this differential equation.

If I want to solve such a mu, this function, dow M by dow y minus dow N by dow x divided by N should be function of x. Okay. Then only I can get such a mu, okay. So what should I check in this case? So, so the check is, what we should check is dow M by dow y minus dow N by dow x, this you see that which is not 0 divided by N. If this is rather, rather this is function of x only. If this is the case, then I can do it. Then I can then I can integrate this, how to integrate this? D mu by mu equal to, let us say function of x which you call it some phi of x.

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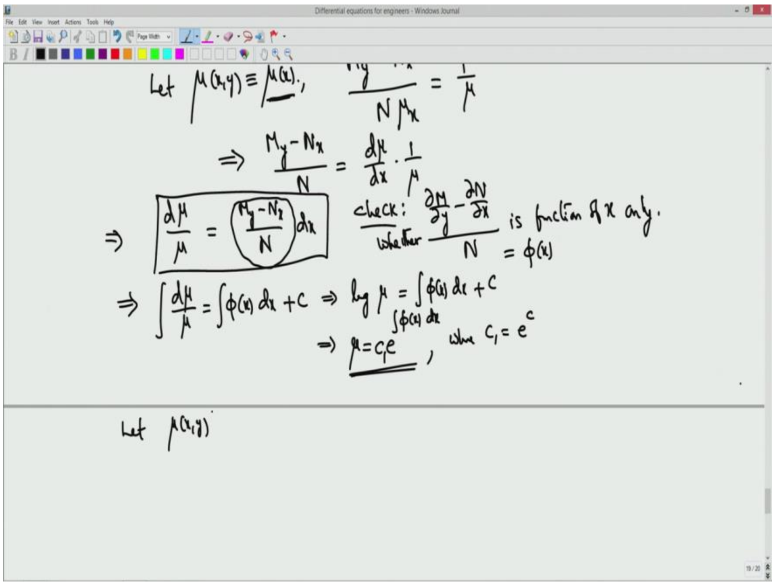

So this becomes phi x dx. So this is simply separation of variables. This implies, if I integrate both sides, this is what it becomes, so I can integrate, C is the integration constant. That will give me log mu equal to integral phi x dx because phi we do not know exactly what it is. So we do not know. So this gives me what exactly my form of, form of mu, mu equal to exponential of integral phi x dx plus e power C, so into e power C. C is the arbitrary constant, e power C is also the arbitrary constant, so I say some arbitrary constant C1 where C1 is e power C.

So I multiply such a function e power integral phi x dx where phi x is this, which I check whether this function is function of x alone. In such a case I can get my function mu as function of alone, only function of x alone. So I have got my mu as a function of x. If I multiply this, we can make the equation exact, so that I know how to solve the exact equation. You know how to solve an exact equation, so I follow the procedure to get the solution of non-exact equation which I made it exact by multiplying mu.

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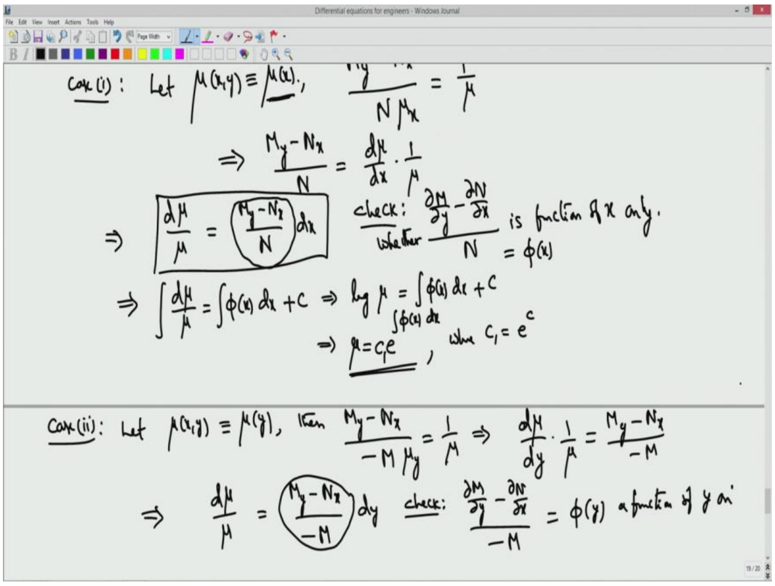

How do I know my mu? I check this condition. So that it is only function of x. If it is only function of x, then I solve this equation, I get this form. Okay. So now let us see other situation where such a mu, we, suppose we look for mu as a function of y alone. This is case 1, this is case 2, okay. So you can think this as case one and you have a case 2. So you can get many cases like this. General case I will discuss. 1st we will see when, when in case 2, we will see mu xY is only function of y.

If I look for such a function of y, what happens to your equation? So this is the original equation. So this becomes dow, this becomes what? MY, then we get MY minus N x divided by N mu x, because mu is a function of y alone, N mu x will be 0. Divided by, so what we get, denominator is only minus M mu y, minus M mu y equal to 1 by mu. So this implies, you get, mu is function of y, mu y is dow M by dow y which is nothing but d mu by dy into 1 by mu equal to dow M by dow y minus dow N by dow x divided by minus M.

So this I can integrate if I write like this. If I separate the variables D mu by mu equal to MY minus N x divided by minus M into dy. So when can I solve this, when can I integrate this equation? If, if this function is only function of y, then I can integrate both sides, okay. So what is the check now? So we will check whether the function is dow M by dow y minus dow N by dow x divided by minus M is function of y alone. Okay, a function of y only.

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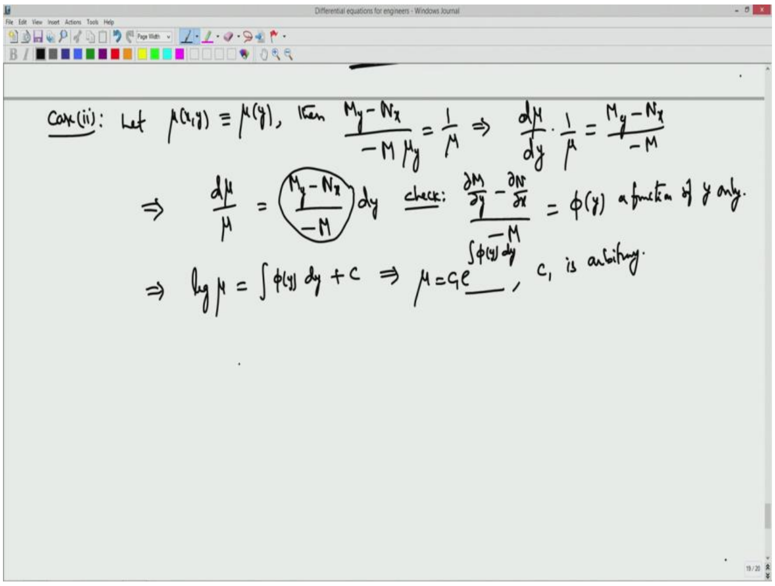

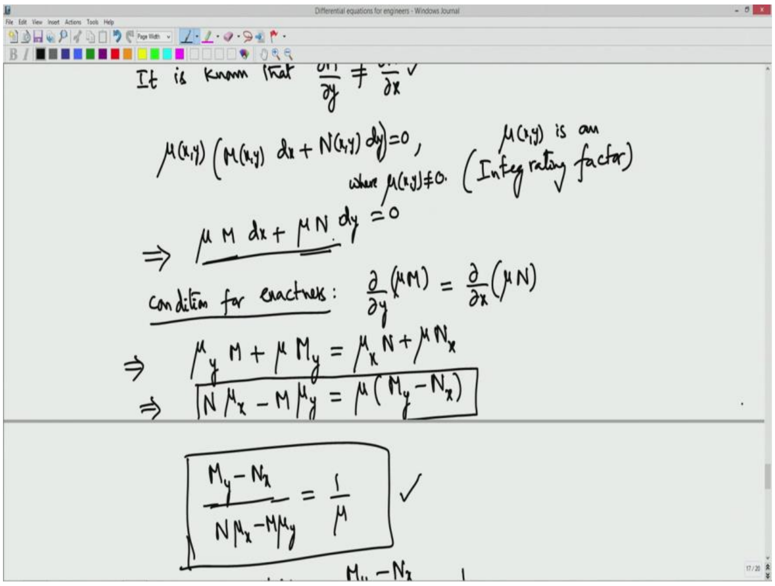

So given differential equation which you know M and N are known, so you calculate this. If it is only function of y, then you solve this, so by integrating this equation, integration will give me log mu which is equal to integral phi y dy plus C which is an arbitrary constant, here implies, we get mu as e power integral phi y dy into e power C that you can say an arbitrary constant, C1 is also an arbitrary. But I need only function, if I multiply this, I make the equation exact. If I multiply with some constant, it is also fine.

If I choose mu e power integral Phi y dy, if I multiply this to your equation, that you, that can also make exact. I can choose any value for C1, so without loss of general, we can choose C1 as 1. Okay. So this way you can get integrating factor. So this is what is called integrating factor. This is called integrating factor. So what we do is I make the equation, given equation, you know that, priorly, you know that given equation is not an exact equation because this condition is not satisfied.

So you multiply mu which is called an integrating factor, so this is called mu, mu is called an integrating factor, so if I look for, if I know that, if I multiply the equation mu of x, y, this becomes this in general, then I hope that the equation becomes exact for such a mu, then that means it has to satisfy this condition of exactness, that actually equal to this differential equation, for mu which is function, actually this is a differential equation of neither, it is not ordinary differential equation because mu is a function of x, y, you have a partial derivative, it is that partial differential equations x y. Okay.

So mu is unknown, mu is the dependent variable which is and xy are independent variable, capital M and capital N are known. So this is actually PDE, you can see actually PDE here. So you do not know how to solve this because this is partial differential equation. Because you have 2 linearly independent variables x and y, if I view this as a differential equation. So you have 2 cases, when you look for mu is the function of x alone, if you want, this has to be, this has to be function of x alone. That with your project, you can check this condition, then you can find such a mu.

So you can take C1 as 1, that gives you e power integral phi x which is this as your integrating factor. Similarly if you look for, if you want only the function of y, if you check this condition, if you verify that, this is a function of y alone, then you can find your mu, and integrating factor as function of y alone. So we will do some examples in these 2 cases before we go for general case. Okay.

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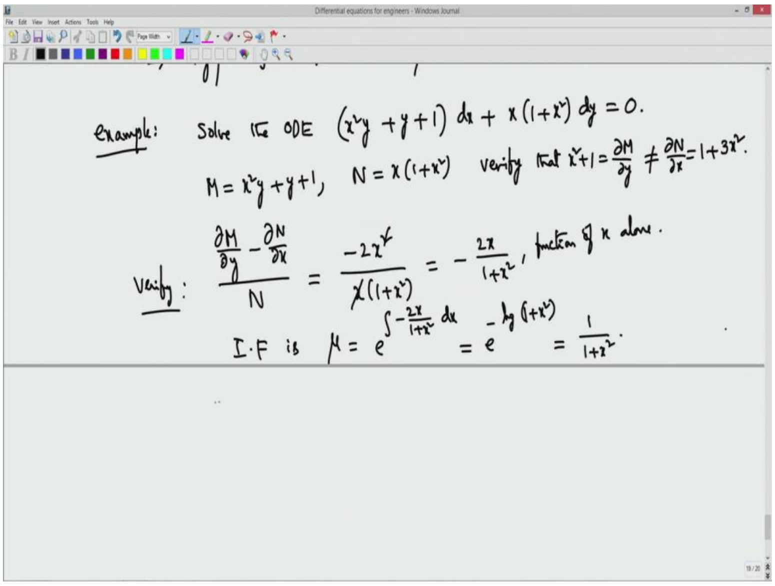

For example, let us consider, try to solve the ODE x square y plus y + 1 dx plus x into 1+ x square dy equal to 0. This is what we want to solve. And you can see clearly M is this x square y plus y Plus 1, N is x into 1+ x square, you can see the condition that, you can verify that dow M by dow y, that is x square which is equal to dow M by dow y, dow N by dow x which is equal to 1 + 3X square. This is not same. So my equation is not an exact, we cannot apply our method for exact equation to solve the exact equation.

So what we do, we know dow M by dow y, dow N by dow x, so we can see, we can verify your MY. So dow M by dow y minus dow N by dow x is equal to, not is equal to, this divided by either M or N. If I divide by N, it should be function of x alone, I will verify this, okay. Verify this one. Which is equal to x square minus, so 1 minus, what do you get, so dow M by dow y minus dow N by dow x, sorry this is not x square, so this should be x square +1, dow M by dow y, okay, so which is not same.

So you calculate dow M by dow y minus dow N by dow x, so you have minus 2 x square divided by N, what is N, x into 1+ x square. So clearly we have 2 x divided by 1+ x square is, it is function of x alone. So my integrating factor, I already calculated is, mu is e power integral whatever I get here, this is my phi of x, there is one is to x divided by 1+ x square dx. Of course with constant, that I can take it at 1, so this is equal to, what is integration of this?

e power minus log 1+ x square because for, for any x, x positive or negative, 1+ x square is always positive for which log is defined, so this is exactly you log of this is actually antiderivative of your 2X by 1+ x square. So this is the integration, this is equal to 1 by 1 plus x square. So this is your integrating factor.

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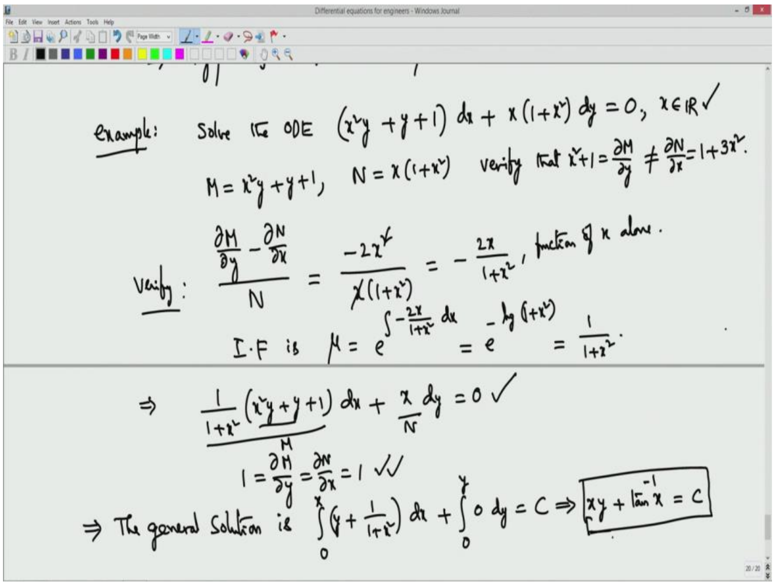

So if I multiply this to the given equation, so this implies, integrating factor is known, so you multiply this to the your equation. So there are 1 by 1 plus x square into M, that is x square y plus y plus 1 dx plus x into 1+ x square divided by, so I multiply this 2nd term, so that goes, dy equal to 0. Now this equation in exact with this M, this N. Okay, you can verify that. So you can easily say that, now, this is my M, this is my N, dow M by dow y equal to dow N by dow x which is 1, which is 1. So verify.

Now the solution, the general solution, that implies, the general solution is, solution is, what you have to do, integrate from x0, y0, x0 to x, which x0 I always take it at 0 because 0 is also part of the domain. So here x is full real number. So I can choose x0 as 0, y also there is no issue, so y is also can be full real number, y also, y0 also I can take as 0. So 0 to x, x square + 1, so that is going to be y. These 2 terms if I do, divided, this will have y + 1 by 1 + x square dx, okay.

Plus y0 is 0, if I can choose my 0, y0 as 0 into y, x0, that is x0, if I put x by x0 which is 0, so it is 0. So 0 dy equal to constant. This is my solution, this is anyways 0. So this implies what is the solution, you can see. xy, if you integrate the 1st term, it is going to be xy plus tan inverse x, tan inverse 0 is 0, minus tan inverse 0, it is 0, okay, equal to, this is 0, so constant. So this is your general solution of the given non-exact equation which we made, which we made as an exact equation like this, then we solved with the procedure that we know already. Okay.

So in this, so far we have solved some exact equations and we explained the mother how to reduce certain non-exact equation to an exact equation and then we can, so that we can integrate the equation to get the general solution of the given differential equation, first-order ordinary differential equation. This is a method called integrating factor method, so when, so we have seen, we have seen integrating factors that are only functions of x or y, okay.

So we have seen, we have explained the method what to check, what expression to check to make the given non-exact equation into an exact equation so that the integrating factor will be only function of x or function of y only. So we will see the general method in the next video.