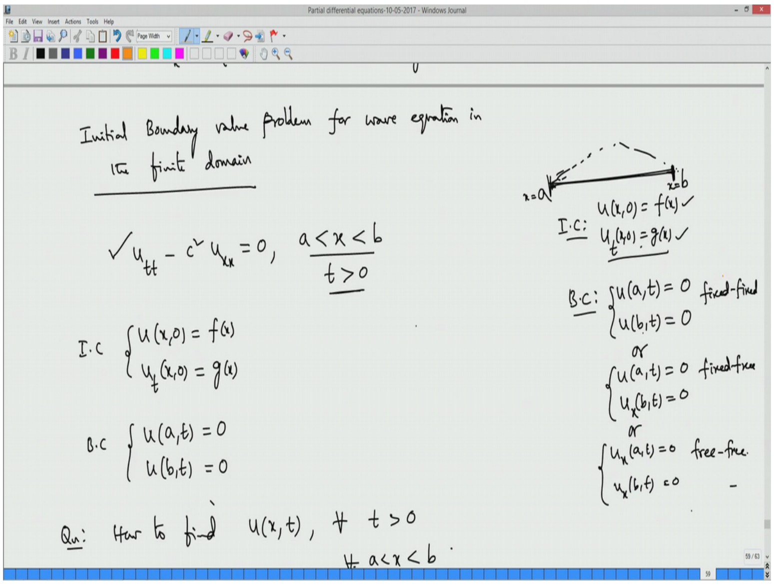
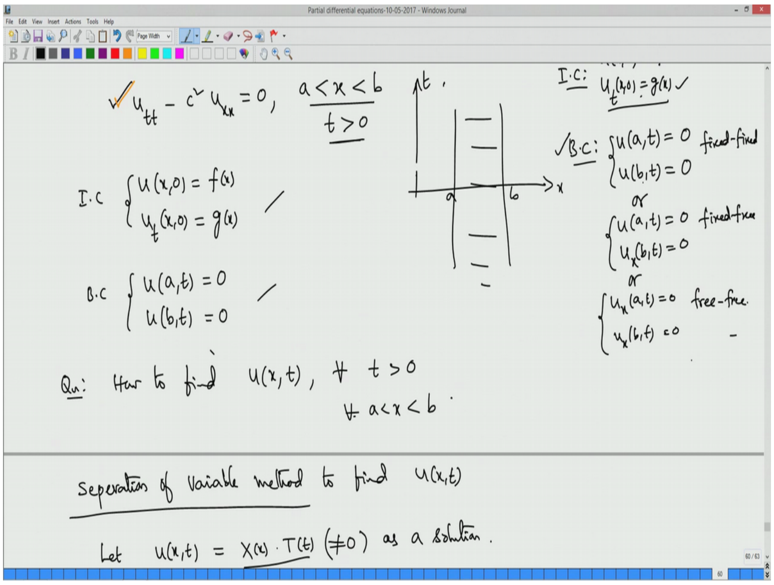
Differential Equations for Engineers
Professor Dr. Srinivasa Rao Manam
Department of Mathematics
Indian Institute of Technology Madras
Lecture 49
Finite length string vibrations

So welcome back we are looking at the initial boundary value problem for the wave equation in a finite domain. So when u consider the domain finite interval a to b so so we just we extracted a Sturm-Liouville problem in the last video. So we will try to find the eigenvalues and eigenfunctions for the Sturm-Liouville problem and corresponding to these eigenvalues we will try to solve the other ODE and we will club them together we will super impose all solutions and try to find the solution for the PDE partial differential equation. So the initial boundary value problem for which we will just get the solution. So we will try to see how we find them.

(Refer Slide Time: 1:02)

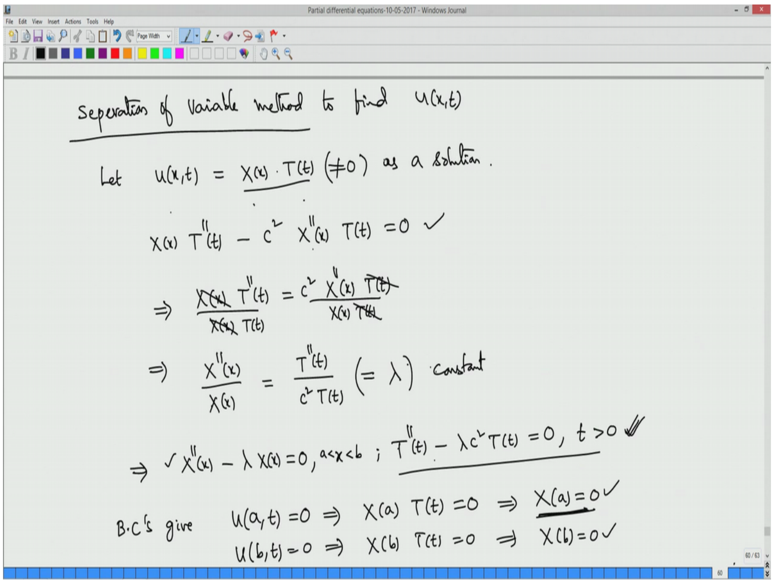




So start with the Sturm-Liouville problem so just to make u familiar again. So this is the boundary value problem we have, this is the wave equation we have this is the wave equation this is in the finite string u have a finite string attached at both ends at x equal to a and x equal to b is attached. So the boundary conditions are fixed this is the one we are looking at now and initial conditions are same anyway initial condition.

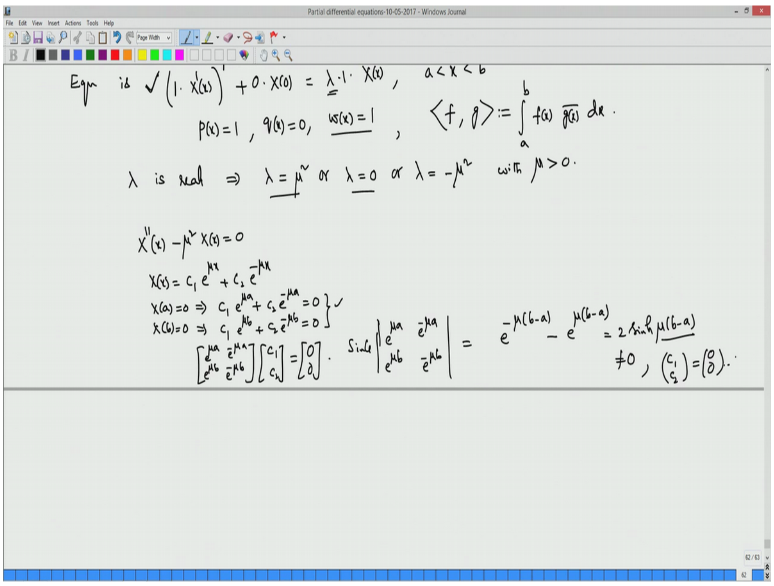
So displacement is given as f x and it is velocity of the string at time t equal to 0 is g x that is what is given. So this is what we have this is the initial boundary value problem for the wave equation in the finite domain is basically kind of strip in the strip so this is a to b and this is t so this is this is your x domain and this is your t domain so your domain is actually in this strip infinite strip if u look at only x x domain that is the finite so from which that is where u extracted a Sturm-Liouville problem.

(Refer Slide Time: 2:14)



So u look for separation of variables method u separated this method separated these variables in the solution so that u look for nonzero solution so that u substitute in the equation and divide it just to find variables are separated left hand side only functions of x, right hand side only functions of t that means it has to be constant we call that as a parameter lambda. So if u write this as a for X double dash minus lambda X that is 0 this is one problem and the other ordinary differential equation is for T that is a second order time derivative. So T double dash minus lambda c square T equal to 0.

(Refer Slide Time: 3:12)



So you apply applying you applied the boundary conditions so to get the boundary conditions for X so what we had is Sturm-Liouville problem so this is the regular Sturm-Liouville problem for this. So so we can see that p is p of x so this is in this form 1 into (X of x) X dash of x whole dash p into y dash whole dash plus q is 0 plus q is 0, right so q is 0 into X of 0 equal to lambda into w is 1 into X of x that is your y, so this is what you have in a to b.

So the equation is is of the form so this is exactly this one so this implies P of x is the Sturm-Liouville equation so P is 1, q is 0 and w is 1 so once you see this w is 1 immediately dot product of eigenfunctions you can define as a to b this is your domain f x and g x bar dx so this is what you have and so how do I find this solutions.

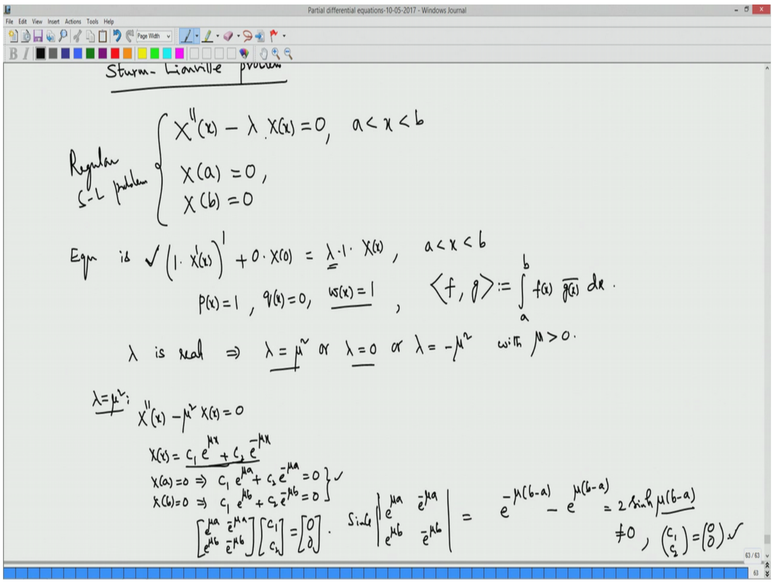
Now how to find the eigenvalues and eigenfunctions, so this is in the self adjoint form so the eigenvalues are real. So lambda should be real lambda is real because the self adjoint or head mission form already, okay so real implies lambda is Mu square, or 0, or lambda is minus Mu square with Mu is positive so that is how lambda is real. So start with lamda is Mu square let us take whether this is a eigenvalues or not.

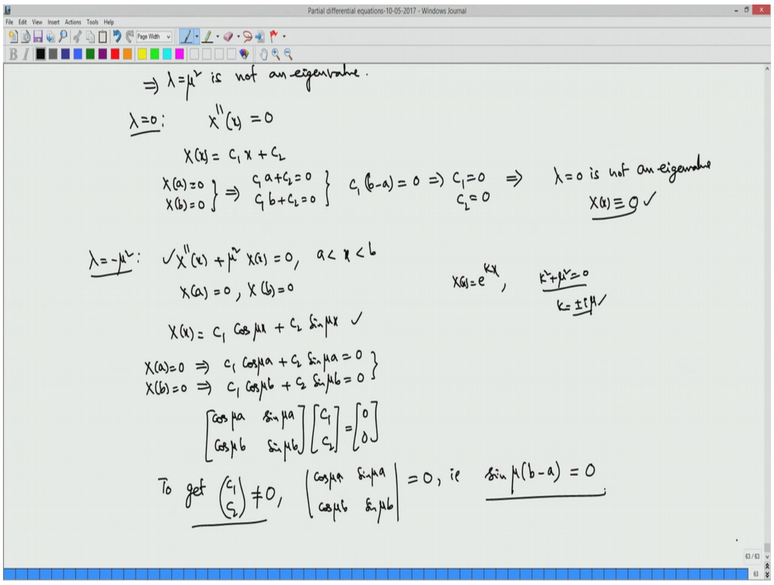
So if you call this X double dash of x minus Mu square X of x equal to 0 this is the ODE. So what are the solutions X of x is c 1 e power Mu x plus c 2 e power minus Mu x these are the solutions general solution of this ordinary differential second order homogeneous ordinary differential equation. Now you apply this boundary conditions x at a equal to 0 gives me c 1 e power Mu a plus c 2 e power minus Mu a equal to 0. Similarly x at b equal to 0 will give me c 1 e power Mu b plus c 2 e power minus Mu b equal to 0, okay.

So if you want to have a nonzero solution the system e power Mu a, e power minus Mu a, e power Mu b, e power minus Mu b, this matrix times c 1, c 2 this vector equal to 0, 0 this is how these two equations. So if you want to have a nonzero solution you should have determinant has to be 0, so what is the determinant determinant of this matrix Mu a e power minus Mu a, e power Mu b, e power minus Mu b so this determinant is actually equal to e power minus Mu b minus a, b is bigger than a so then minus e power Mu b minus a. So this is nothing but two times Sin hyperbolic Mu times b minus a, okay.

So we know that Sin hyperbolic never be 0 (())(6:53) it is 0, okay Mu is positive b minus a is positive Sin hyperbolic which is nonzero so Sin hyperbolic of Mu into b minus which is nonzero that means this is cannot be 0 so since determinant is this nonzero c 1, c 2, c 1, c 2 vector should be 0 this is the only solution you will get, okay simply invert it invert this matrix you can see that c 1, c 2 you will get it as 0, 0 is the only solution.

(Refer Slide Time: 7:28)





Implies implies Mu square lambda equal to Mu square is not an eigenvalue you do not have a nonzero solution once you get c 1, c 2 is nonzero 0 then your solution general solution becomes 0 completely 0, okay. So similarly you can try for lambda equal to 0 this is corresponds to lambda equal to Mu square now you have lambda equal to 0, in this case X double dash of x equal to 0 that is how the equation becomes if you put lambda equal to 0 here it will become to the second derivative of x is 0.

So the general solution is c 1 x plus c 2 now you apply the boundary condition X at a, X at b equal to 0 together will give me c 1 a plus c 2 equal to 0, c 1 b plus c 2 equal to 0. So you take the difference you solve this you take the difference you see that c 1 into b minus a or a minus b will be 0 if you take the difference, so b minus a cannot be 0, c 1 equal to 0 once c 1 is 0 you put it here any of these equations to see that c 2 is also 0 implies lambda equal to 0 is not an eigenvalue because X of x will be identically 0 here when lambda equal to 0 you do not have nonzero solution.

So the only option you have is lambda is minus Mu square so in this case lambda equal to minus Mu square if we put so it will be plus so x double dash equation becomes plus Mu square X of x equal to 0, x is a to b and you have X at a equal to 0, X at b equal to 0 or boundary condition this is your problem now. So the solution of this is X of x is c 1 general solution of this is c 1 Cos Mu x plus c 2 Sin Mu x, so general solution and you have Mu is positive this is how we get.

Simply look for the solutions of the form e power Mu x e power p x rather if you look for if you look for X of x as e power kx what you see is k square plus Mu square equal to 0. So k is plus or minus i Mu, so e power i Mu x is one solution, e power minus i Mu x is another linearly independent solution.

So if you can also take the sum and difference they are also linearly independent they nothing but Cos Mu x and Sin Mu x of solution. So together we super pose and get the general solution as c 1 Cos Mu x, c 2 Sin Mu x. Now you apply the boundary conditions so x at a equal to 0 will give me c 1 Cos Mu a plus c 2 Sin Mu a equal to 0 and you apply X b equal to 0 will give me c 1 Cos Mu b plus c 2 Sin Mu b equal to 0, again this system you can write it like a system system of in a matrix form Mu a, Sin Mu a, Cos Mu b, Sin Mu b times c 1, c 2 equal to 0, 0, okay.

So c 1, c 2 to get to get nonzero c 1, c 2 vector determinant has to be 0, this is Cos Mu a, Sin Mu a, Cos Mu b, Sin Mu b has to be 0 so that is what is this one see this determinant is nothing but Cos Mu a, Sin Mu b, minus Sin Mu a, Cos Mu b is nothing but Sin Mu b minus a equal to 0, okay. So to get a nonzero solution this has to be satisfied.

(Refer Slide Time: 11:58)

$X(a) = 0$, $X(b) = 0$ $\qquad X(x) = e^{kx}$, $\underline{k^2 + \mu^2 = 0}$, $\underline{k = \pm i\mu}$ ✓

$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$ ✓

$$\left.\begin{aligned} X(a) = 0 &\Rightarrow c_1 \cos \mu a + c_2 \sin \mu a = 0 \\ X(b) = 0 &\Rightarrow c_1 \cos \mu b + c_2 \sin \mu b = 0 \end{aligned}\right\} ✓$$

$$\begin{bmatrix} \cos \mu a & \sin \mu a \\ \cos \mu b & \sin \mu b \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To get $\underline{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq 0}$, $\begin{vmatrix} \cos \mu a & \sin \mu a \\ \cos \mu b & \sin \mu b \end{vmatrix} = 0$, ie $\underline{\sin \mu (b-a) = 0}$

$\mu (b-a) = n\pi$, $n = 1, 2, 3, \ldots$

$\Rightarrow \underline{\underline{\mu = \frac{n\pi}{b-a}}}$, $n = 1, 2, 3 \ldots$; $\lambda = -\frac{n^2\pi^2}{(b-a)^2}$ non-zero solution exists.

$c_1 = -c_2 \frac{\sin \mu a}{\cos \mu a}$

$\underline{c_1 = -c_2 \frac{\sin \mu a}{\cos \mu a}}$ ✓ put this into general soln. to get

$$X(x) = -c_2 \frac{\sin \mu a}{\cos \mu a} \cos \mu x + c_2 \sin \mu x$$

$$= \left(-\frac{c_2}{\cos \mu a} \right) \left[\sin \mu (x-a) \right] ✓$$

$\Rightarrow \sin \mu (x-a)$ is a non-zero solution

From 2nd eqn, $c_1 = -c_2 \frac{\sin \mu b}{\cos \mu b} \Rightarrow X(x) = -\frac{c_2}{\cos \mu b} \sin \mu (x-b)$

$\Rightarrow \sin \mu (x-a)$ is a non-zero solution

From 2nd eqn, $c_1 = -c_2 \frac{\sin \mu b}{\cos \mu b} \Rightarrow X(x) = -\frac{c_2}{\cos \mu b} \sin \mu (x-b)$

$\Rightarrow \sin \mu (x-b)$ is also a non-zero solution.

$\underline{\sin \mu (b-a) = 0} \Rightarrow \underline{\sin \mu b \cos \mu a - \cos \mu b \sin \mu a = 0}$ ✓ $\Rightarrow \frac{\sin \mu b}{\cos \mu b} = \frac{\sin \mu a}{\cos \mu a}$ ✓

$$\begin{aligned} \sin \mu (x-b) &= \sin \mu x \cos \mu b - \cos \mu x \sin \mu b \\ &= \cos \mu b \left(\sin \mu x - \cos \mu x \cdot \frac{\sin \mu b}{\cos \mu b} \right) \\ &= \cos \mu b \left(\sin \mu x - \cos \mu x \cdot \frac{\sin \mu a}{\cos \mu a} \right). \end{aligned}$$

So so what happens here so it implies you can see that Mu will be Mu is actually Mu into b minus a should be equal to Sin so it is n n is running from 1, 2, 3 because you put n equal to 0 Mu is 0 Mu is 0 we have seen that it is not an eigenvalue. So that is corresponds to lambda equal to 0. So we have seen that it is not an eigenvalue so it should starts from 1, 2, 3 onwards.

So this implies Mu is n Pi by b minus a, so for these Mu values you have lambda which is minus Mu square so n square Pi square by b minus a whole square. So for this lambda you have a nonzero solution exists because this is how the determinant gives because the determinant is 0. So what is that nonzero solution you have to find so simply to solve these two equations to get the nonzero solution X of x so from this equation first equation you can see that c 1 I can write c 1 as minus c 2 Sin Mu a by Cos Mu a, okay.

So for these Mu values Mu now I am fixing so this is the Mu value I fixing anyone of these values okay Mu is this for n 1, 2, 3 onwards, okay. So for this Mu values c 1 so once you see that Sin for this Mu values Sin Mu into b minus a is 0 that means Sin Mu a, Sin Mu b never be 0 Sin Mu a or Cos Mu a this is never be 0. Similarly Sin Mu b and Cos Mu b never be 0 for these Mu values, okay for any n any of this n so n is from 1, 2, 3 onwards.

So once you from the first equation you get this put this into the second equation to see that c 1 that is minus c 2 put this into second equation then get minus c 2 Sin Mu a by Cos Mu a Cos Mu b plus c 2 Sin Mu b equal to 0. So this implies c 2 (no you do not put) so once you get this c 1 I can put it into, okay. So you take first equation you get this c 1, okay put this into a general solution general solution to get X of x what is X of x?

So this is the c 1, c 1 is this Cos Mu x and here Sin Mu x so this is the general solution. So what happens to this general solution this is minus c 2 is common by Cos Mu a if you take it out what you get is Sin Mu x minus a so this is what you get. So this is a constant, okay constant times this is your solution this is the solution it does become. So this implies a nonzero solution is Sin Mu times x minus a is a nonzero solution, okay constant times this is also a solution that is how you got this, okay.

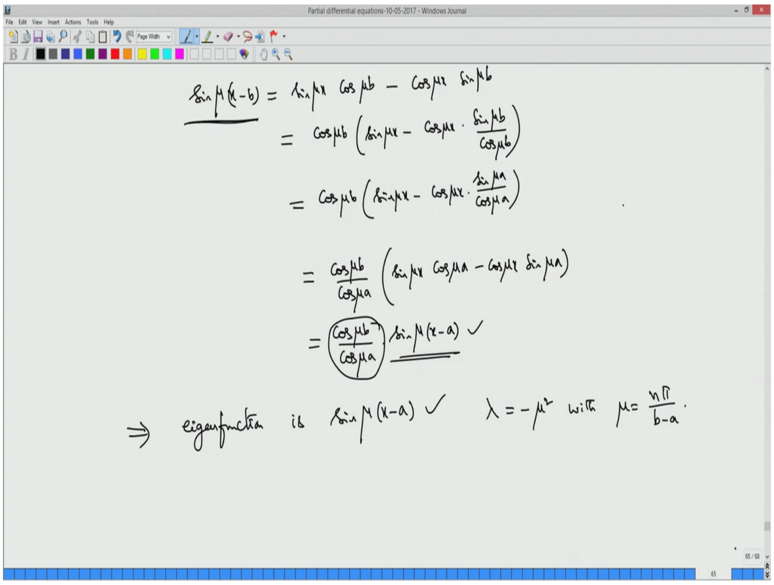
Now if you consider a second equation you take the second equation gets c 2 or you can also take the second equation and write c 1 equal to minus c 2 Sin Mu b by Cos Mu b that also you can see that you will have the same equation, okay. If you substitute or you can see that this will be this is one solution if you take second equation from the second equation if you

you can if you write c 1 equal to minus c 2 Sin Mu b divided by Cos Mu b this is how you get the second from second equation from second equation you can get this.

Now if you put this this will give me X of x will be minus c 2 divided by Cos Mu b again you see that Sin Mu times x minus b so this implies Cos again see that this will be Sin Mu times x minus b is also a solution is also a nonzero solution. But then we know that Sin Mu b minus a is equal to 0 so that means Sin Mu b (Cos Mu b) Cos Mu a minus Cos Mu b Sin Mu a equal to 0, okay because of this.

So this implies if you make use of this you can see that Sin b so what you do so if you make use of this into this Sin Mu times x minus b this is Sin Mu x Cos Mu b minus Cos Mu x Sin Mu b, so this you try to write this as take this Cos Mu b out to write this as Sin Mu x minus Cos Mu x into Sin Mu b by Cos Mu b. Now from this you can get Sin Mu b by Cos Mu b is same as (Sin Mu b) Sin Mu a divided by Cos Mu b. So you can replace this so you can see that Cos Mu b times Sin Mu x minus Cos Mu x times this I am replacing with this, so Sin Mu a divided by Cos Mu a.

(Refer Slide Time: 18:50)



So this means Cos Mu b by Cos Mu a times if you now multiply cross multiply you see that Sin Mu x Cos Mu a minus Cos Mu x Sin Mu a, so this is nothing but Cos Mu b by Cos Mu a into this is nothing but Sin Mu x minus a. So you see that even though you have a solution from the second equation if you write c 1 in terms of Sin Mu b Cos Mu b what you get is Sin Mu x minus b Mu into x minus b as a solution that is simply a constant multiple of this is a constant multiple of the earlier solution.

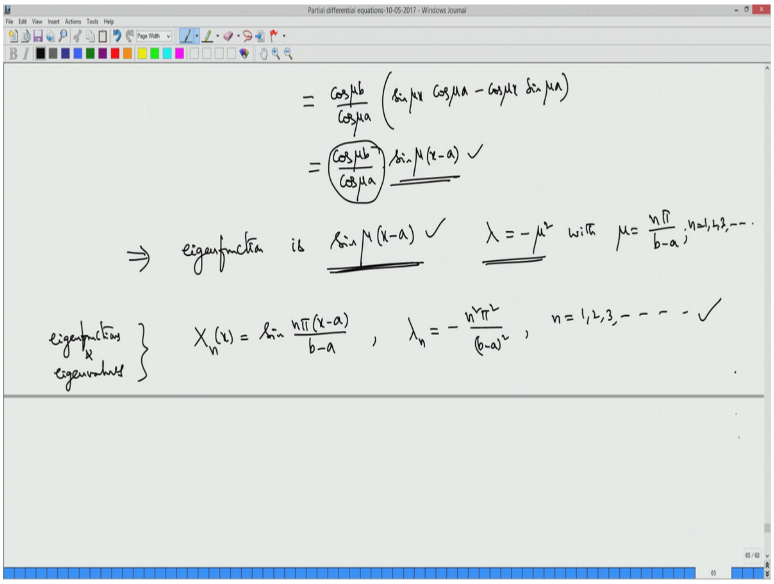
So this implies the only solution both the solutions are simply constant multiple of each other. So that means so either of this equations from which you take c 1, from either of these equations you take c 1 you write in terms of c 2 and substitute into the general solution you get actually constant multiple of Sin Mu into x minus a, okay. So that is what you see so that implies eigenfunction is eigenfunction is actually is Sin Mu x minus a, okay corresponding to minus Mu square lambda is minus Mu square minus Mu square is Mu value is minus Mu square with Mu is n Pi by b minus a, so this is what you found.

Now see this is what if I use c 1 in terms of c 2 if you write c 2 in terms of c 1 from each of this two equations c 2 in terms of c 1 and try to put it into the general solution again you will see that you get the same you will see that Sin Mu x minus a is a solution or Sin Mu into x minus b as another solution but they are actually constant multiple of each other that is what we have shown by making use of this this (())(20:58) relation as also called this is how it is giving eigenvalues that is Sin Mu into b minus a equal to 0.

So so casually I wrote this I just have to point out something here so I actually wrote casually this problem in the domain a to b so we actually want finite string a to b, a to b is a finite string a is finite number, b is a finite number. So the finite string is string of finite length length is b minus a instead you can actually make it a as 0 and b equal to some l, so that this unnecessary calculations you can avoid but anyway just to you can get a good inside if you actually work with the general a and b so that how to see that actually you end up the same only one eigenfunction that is Sin Mu into x minus a, okay.

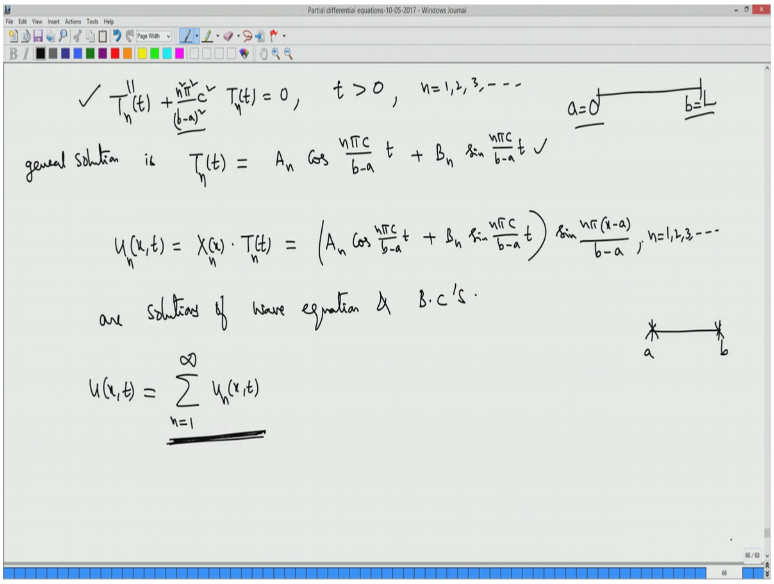
Even though you get a Sin Mu into x minus b as another solution but that is also actually constant multiple of Sin Mu into x minus a so finally you will get corresponding to this lambda equal to minus Mu square when Mu is this for n is from 1, 2, 3 onwards you get one eigenfunction that is Sin Mu into x minus a.

(Refer Slide Time: 22:14)



So you call this some X n's X n of x which is Sin n Pi into x minus a by b minus a, so this is what your eigenfunction, eigenvalues or lambda n's which are minus Mu square that is n square Pi square by b minus a whole square, so n is running from 1, 2, 3 onwards. So these are your eigenvalues and eigenfunctions, okay eigenfunctions and eigenvalues, okay. So this is how you get for this thing.

(Refer Slide Time: 22:58)



So if you so you can actually you can similarly work with instead of a to b you work with 0 to L string, so you take a equal to 0, b equal to L. So then the calculations will be easier. So in many case we have solve this for general a to b so for your understanding it will be better to to have this calculations (())(23:17). So you have found what is your x n's corresponding to n is from 1, 2, 3 x n is this and what happens to so this is why this is my x n solution, I have x n solution so I know what is my so corresponding to those lambda values that is lambda is minus Mu square if you put this will be plus Mu square.

So you try to solve this t equation, so T double dash minus lambda c square, what is other ODE? T double dash of t minus lambda minus lambda is plus Mu square c square into T of t equal to 0. Now t is positive, okay t is the time is actually positive. So what is the general solution of this general solution of this is of this ODE is T of t corresponding to Mu is Mu square is n square Pi square, okay Mu square is n square Pi square by b minus a whole square.

So T of t is what you get as c 1 call this for each n so you can rewrite Mu as I can put n square Pi square by b minus a whole square, I replaced, now this is now you have n is running from 1, 2, 3 onwards. So this is how you get a system of ODE for each n for each n you have a second order ODE for T of t.

So then you call this because it depends on n you call them as T n, T n so I have a T n for each n I have x n and T n, so T n of t will be once you (())(25:07) as you fixed n then this will be T n so you have we call this some A n is a arbitrary constant times because this is the positive number. So you have a Cos n Pi c by b minus a t, okay plus B n another arbitrary constant Sin n Pi c by b minus a into t. So this is the general solution of this equation.

Now you know that so once you fix your n you have X n and T n so you know that u of x, t is X of x and T of t. So for each n you fix you call this you are calling this X n and T n so you call the corresponding solution as this one. So the product is A n Cos n Pi c by b minus a t plus B n Sin n Pi c by b minus a into t. So this into this is my T n of t, X n of t is you have already Sin n Pi x minus a by b minus a. So Sin n Pi x minus a by b minus a, so this is what you have as one solution.

So this is the solution of the wave equation in the domain a to b what you applied is only boundary conditions but not the initial conditions, initial conditions are yet to apply this is true for every n, n is from 1, 2, 3 onwards or or solutions of solutions of wave equation and boundary conditions. Now you super pose all these solutions you take a finite combination of them you can sum them, okay u n of x to t n is running from 1 to some in actually infinity you can take so finite finite number of them n is from 1 to big N let us say is also a solution, these are all solutions for the wave equation homogeneous equation linear combination is also a solution, okay.

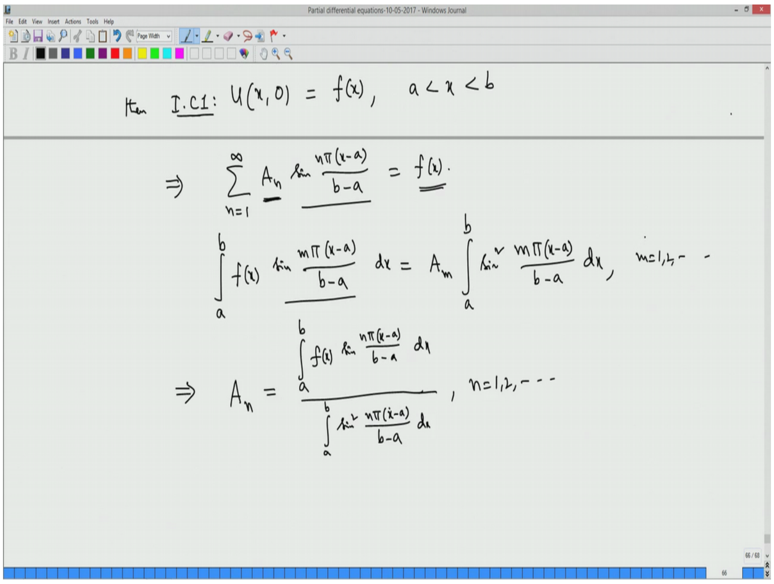
So if you take this their sum (sum as a linear) sum as a solution is also solution and you see that n is from 1 to infinity is also a solution this this is little tricky only thing is you can actually show that this series is actually uniformly convergent and by imposing certain conditions on the initial data, okay. So the data preciously the conditions are f your f and g your initial data f should be twice differentiable so f third derivative of f that is f triple dash should be piece wise continuous and g g double dash is piece wise continuous under these conditions when f triple dash and g double dash are piece wise continuous functions I can actually show that this sum is uniformly convergent this infinite sum, okay.

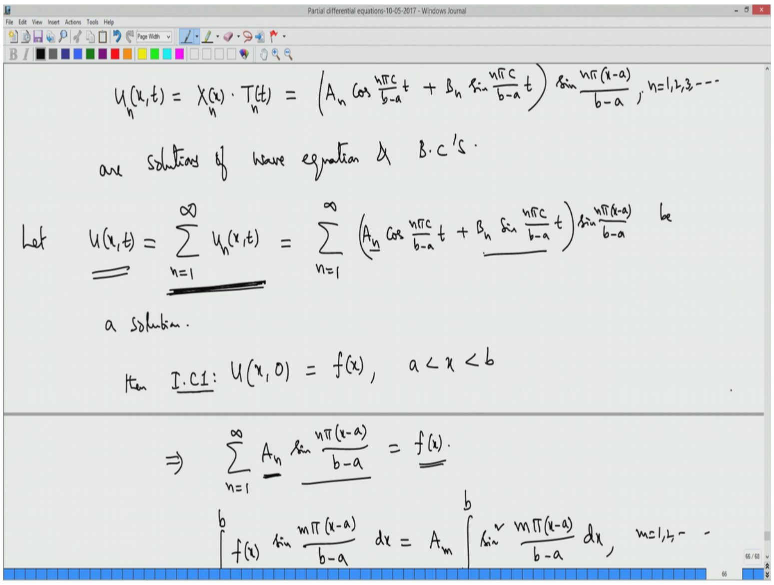
So we will not do this we can assume that you will always chose the super position of all this solutions even though they are infinite one one can actually this will be a solution only if it is this series is uniformly convergent, okay that is possible under certain conditions on the initial data that is f f triple dash should be piece wise continuous and g double dash is piece wise continuous then I can actually show that this series is uniformly convergent, okay that is known that is the theorem with a proof let us you can assume that you can always do this as a super position of all this solutions assume that is also solution, okay.

So this let let b what is this sum sum is from n is from 1 to infinity A n this is my u n so A n Cos n Pi c by b minus a into t Cos t plus B n Sin n Pi c by b minus a into t this into Sin n Pi x minus a by b minus a so this is what is your solution this is the so proportion of solution be a solution. So when this is actually solution this is a solution actually only if this series is uniformly convergent solution of the wave equation means it has to be differentiable and you can differentiate if you differentiate u t for example u t or u x you can differentiate only term by term that means series should be uniformly convergent in both the variables X and T, okay that you can actually show.

So mathematically it is known that uniformly convergent and certain conditions on the initial data that is why I am just assuming that is this super position this is called the super position principle super position of all the solution that means the sum of all this u n is also you can assume that it is a solution, okay. Then so that you can find using your dot product you can find your A n’s and B n’s we will see how it is done by using initial conditions so be a solution.

(Refer Slide Time: 30:32)





Then then initial data gives what is the initial data u at t equal to 0 is f x, okay this is x is between a to b, okay so what is that u at x is 0 that means this sum this series is from 1 to infinity A n this is so when you put t equal to 0 this part will not be there so you have Cos Cos 0 is 1, so you have A n this will be entirely 0 and you have Sin n Pi x minus a by b minus a equal to f x.

Now I can apply both sides I can make a dot product with the eigenfunctions, eigenfunctions are Sin n Pi into x minus right what is your Sin Sin n Pi x minus a by b minus a. So these are your eigenfunctions, okay. So you have basically you got for this function you got a Fourier series, so for inverse transform is nothing but so you can think of Fourier transforms are your A n’s so that you can get it from your dot product.

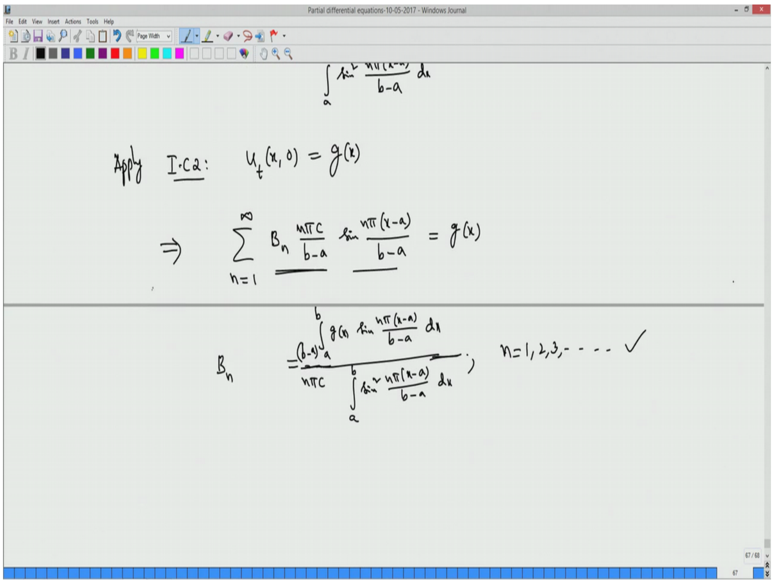
So just like what we had seen in the Sturm-Liouville theory. So you make a dot product so you can see if you make a dot product that means you both sides you integrate if the right hand side if you take f x into this anyone of them so let us take m Pi x minus a by b minus a with some m dx should be equal to if I multiply Sin m Pi into x minus a by b minus this eigenfunction, if I multiply both sides and integrate from a to b what you end up is only A m all other terms will be 0 because they are orthogonal to each other.

This eigenfunctions form complete orthogonal set that is what we have seen, okay that is what we have seen in the Sturm-Liouville during while doing Sturm-Liouville theory. So if you do that so A m so this side you get a to b Sin square n n equal to m so you have m Pi x minus a by b minus a into dx of dx, okay. So this is what you get so this that means I can I know what is my A n that means A n so m you can write it as A n so this is true for every m m is from 1, 2, 3 onwards.

So you can write now I know what is my A n, A n is simply f x this is given data of this sin (m Pi) n Pi x minus a by b minus a dx divided by this you can actually integrate and find the values so that is Sin square (m Pi) n Pi x minus a divided by b minus a of dx. So this is your this is the number if you know f x so this is true n’s are known now n is from 1, 2, 3 onwards. So the general solution actual solution is in this solution this is the solution of now the wave equation, boundary condition and first initial data, okay initial condition 1, okay.

Now this using the initial condition 1 now I know it is my A n. Now use the initial data second initial data to find B n so that so that I know everything about A n's, B n's that means u x u x, t.

(Refer Slide Time: 34:04)

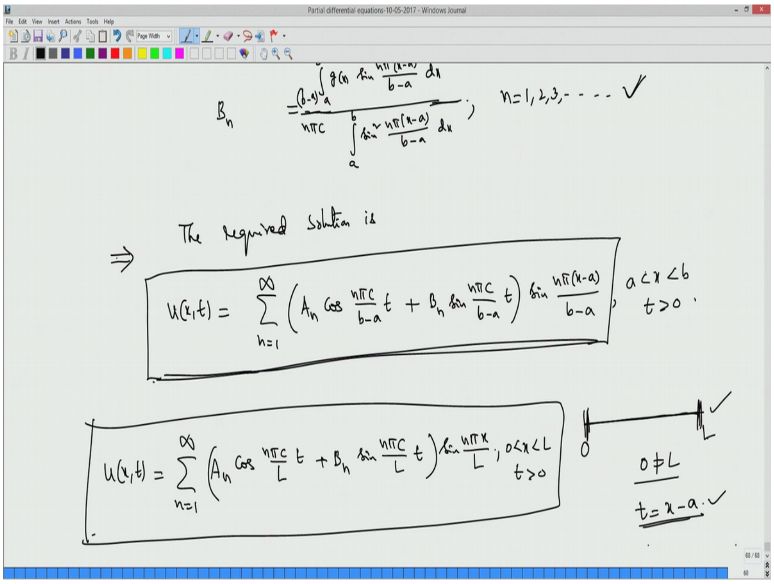


Apply apply initial condition 2 that is u t at x, 0 is g x this is the initial data, okay. So if you apply that so what you see is you differentiate this with respect to t u t is this will become minus Sin so you okay. So once you differentiate this with respect to t this will become Sin t and when you put t equal to 0 that will be 0. So what you end up is simply this summation n is from 1 to infinity B n this will become Cos n Pi c by b minus a into t into this constant will come out because of the differentiation.

So I will write directly see that sigma n is from 1 to infinity you see that B n B n into n Pi c by b minus a into Cos so Cos n Pi c by b minus a into t that is 1, okay. So I do not write I do not get t equal to 0 that is 1 into Sin n Pi Sin n Pi x minus a by b minus a. So this is what you get as g x, so this is what if you apply. So again same idea you apply both sides with this eigenfunctions and integrate to see that B n will be B n into n Pi c this is the number, this is the number unknown number so B minus a is equal to so like earlier a to b instead of f x now you have g x g x Sin n Pi into x minus a by b minus a dx divided by integral a to b Sin square n Pi x minus a by b minus a into dx, okay this is what you get.

So so that means if you if you actually write take this b minus a other sides so you can write b minus a here and n Pi c you can write here, okay. So you can remove here. So now I know what is my B n, n is from 1, 2, 3 and so on, okay. So this is my A n and this is my B n in this u so this is my general solution. So this is the solution now it satisfies everything, okay. So we can write again.

(Refer Slide Time: 36:37)



So this implies with this A n, B n, u x, t the general solution is the now the required solution is u x, t which is equal to sigma n is from 1 to infinity A n A n is known so A n Cos n Pi c by b minus a t plus B n Sin n Pi c by b minus a t into Sin n Pi x minus a by b minus a. So this is the solution with this A n's and B n's with this A n's and B n's so these are the two, okay, so these are the two A n's and B n's.

So A n's and B n's are given in as a integral involving initial data f x and g x. So if you impose the conditions on f x and g x you actually show that this series is this series is actually uniformly convergent that is the this solution, if this solution is uniformly convergent that is same as that is (())(37:48) you can assume that so this is the general solution this is the solution that you are looking for so that satisfies wave equation and the boundary conditions and A n's, B n’s are calculate from the initial data that implies this is the solution that satisfy all the conditions. So this is the solution of initial boundary value problem for the wave equation on a finite domain.

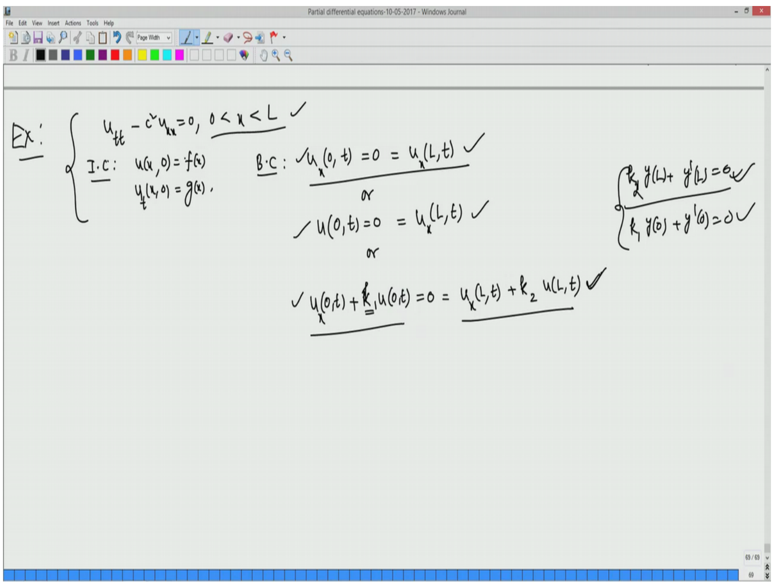
So I casually took I just want to make a remark here that if I when I took casually that domain is a to b why a to b, a is positive, b is positive, okay or any a, b a belongs to r and b belongs any a, b a is not equal to b because of this all these calculations little involved. So instead we only have to this is the model for a finite string. So that means I can choose simply for simplification I can choose a equal to 0, b equal to L so that still 0 is not equal to L, L is not equal to 0.

So if you work with this nicely you will see that the solution of this string this equation will be n is again you get the same thing you get A n's Cos n Pi c b minus a is L so you get n Pi c by L t plus B n Sin n Pi c by L t into Sin (n Pi x minus) n Pi x by L that is what you get so this is your solution this is for x belongs to 0 to L and t positive. So here this is for x is between a to b t is positive.

So that is the only difference you can also make a transmission from a to b to 0 to L just by taking a transmission x minus a x x is between a to b and you can think of t as the domain that is from x minus a so with this transmission on the x domain so you can see that your domain will be transformed to 0 to L, 0 to b minus a that L is actually b minus a, okay. So you can also do that way or you can just think directly you take it as 0 to L and work you can see that this is what you get where A n A n will be something is where you just put in this A n, B n you put A equal to 0, B equal to b minus a equal to L, okay.

So that way you can simplify as a solution here, okay. So you work with this just go through this video and you work on this 0 to L that is enough so now onwards so finite string means I will take only 0 to L, so that calculations will be simpler, okay. So this is how you solve wave equation on a finite domain. So only difference is initial data is same we can only give f x and g x arbitrarily initial data only boundary data will be we fixed it both ends now you can actually allow it to be free both sides or you make a combination of it.

(Refer Slide Time: 41:05)



So whatever may be this you can have a different problems I will just define so each what problems you have so you have a u what kind of problems you can have u tt minus c square u xx equal to 0, x is now between 0 to L I choose from 0 to L and initial data is same initial conditions are simply u x, 0 equal to u x, t equal to 0 is f x, u t at x, 0 equal to g x. Now boundary data this is the boundary conditions these are the things earlier we have chosen fixed conditions now we can allow it to be free.

So that is so you have u x at 0, t for all times should be 0 and also at at L other end you can allow it to be free, okay or you can you can make a one end one end you can fix it that is this, other end you can just allow it to be free or you can make a combination. So that means you can make absorbing conditions, so u plus u at 0, t plus some constant K this is the spring constant K times u x or you say u x plus K times u at 0, t equal to 0 at one end and other end you can also make it you can also attach this spring spring of stiffness K, if you do that 0, t this is now L, t plus K times same springiness you can also have different springs of stiffness K 1 and K 2 u at L, t.

So that these are absorbing boundary conditions so whatever the energy so whatever these waves are absorbed at that end, okay. So this is where you this is how you can make whatever may be the problems so this is this will be this will be coming as your this will be coming in your Sturm-Liouville problem the boundary conditions in your Sturm-Liouville problem. So you can see that Sturm-Liouville y at a and y dash at a so that is what you will get as a combination c 1 plus c 2 not both are 0.

So if you choose here you will get c 1 as 1 and c 2 as c 2 as 1 and c 1 as K 1. Similarly K 2 you will get for the other one at L a is 0 so L, okay so at L you get this one at 0 you will get K 1 at y 0 plus y dash of 0 equal to 0, so this is how you get your Sturm-Liouville boundary conditions if you choose this absorbing boundary conditions. So like this you can make a form so we can form a problem either with this, with this, or with this, okay.

So you can work out all this three problems, okay only idea is simply extract the Sturm-Liouville problem using the boundary conditions use the boundary conditions get the extract extract the Sturm-Liouville problem get the eigenvalues and eigenfunctions. Corresponding to this eigenvalues and eigenvalues you solve other ODE by the separation of variables the T T n's and finally make a product for each n for each of this discrete eigenvalues you form a product that is also a solution of the wave equation and the boundary conditions, make a super position that is also solution of the assume that is also solution of the wave equation with the boundary conditions.

Now apply the boundary data simply use the dot product of this eigenfunctions which you get it from the Sturm-Liouville problem and to get the unknowns involved in the infinite sum so that you completely solve your initial boundary value problem for the wave equation on a finite domain 0 to L with any of this boundary conditions, okay. So I will just leave this exercise so I will eventually give you in a assignment.

So you can work out you can take it as anyone of this boundary conditions and work out this problem similarly similar to what I have done in the earlier example for the fixed ends, okay. So we will move on in the next video we will move on to the other remaining problems on wave equation and looking to the two dimensional wave equation and certain initial data on a circular domains, okay that is where we will see Bessel equation comes into picture, Sturm-Liouville problem for the Bessel equation. So we will see that in the next video, thank you very much.