Differential Equations for Engineers
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Lecture 48
Vibration of a finite string

Welcome back in the last video we have seen that how to solve semi wave equation in a semi-infinite domain that is 0 to infinity when the boundary condition is fixed that is u at 0, t is 0, okay. So with the general initial data we have solved we actually constructed and the solution and we have seen a new approach, so in a different approach not by extending as on a full domain and make use of the D'Alembert's solution and a new approach without doing without making use of the D'Alembert's solution we have seen how to construct a solution in a semi-infinite domain.

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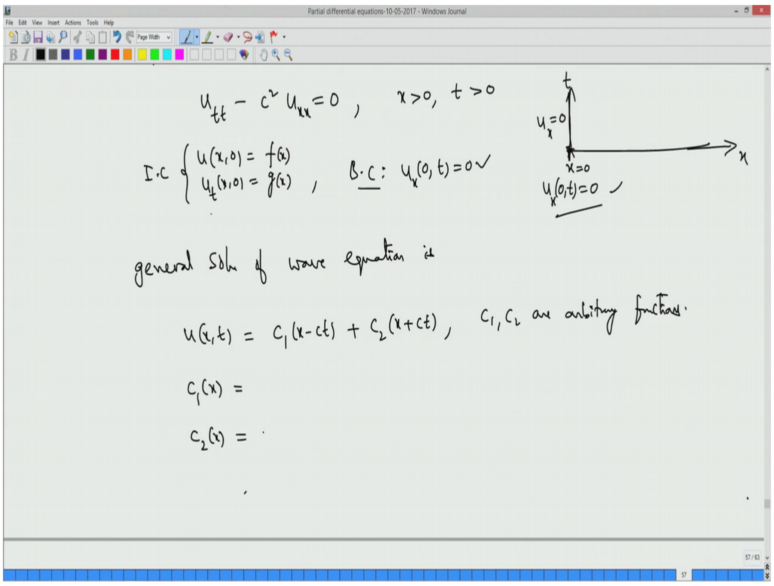

Today we will see this video we will see how to construct the same similar thing when the boundary condition is different that is free edge free edge boundary conditions. So let us write the problem so u x u tt minus c square u xx is 0 this is a wave equation for x positive and t positive the semi-infinite domain. Now initial data is as usual u x, 0 is f x, u t at x equal to 0 is g x is given this is your initial data, okay.

And the boundary data is that is you have only one boundary at 0 x equal to 0 and this is infinity, okay. So your this is t at x equal to 0 that means on this boundary you are giving that is free edge that means the slope of that is 0 at x equal to 0. So u x at 0, t equal to 0 for all times that is the meaning, okay so u x 0, t is 0 is 0 boundary condition again. So we do not really extend it as a even function here like we did earlier.

So what we do is we take the general solution of the wave equation that we already know the general solution general solution of wave equation is u x, t is let us say some c 1 of x minus ct plus c 2 of x plus ct, where c 1 and c 2 are arbitrary c 1, c 2 are arbitrary functions. And if you apply the initial data you finally see that like earlier you can actually find what is your c 1 x and what is your c 2 x, okay.

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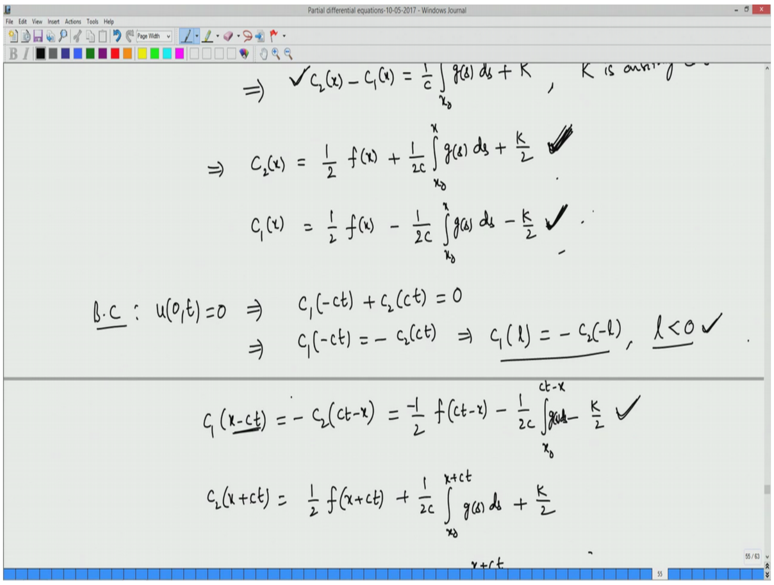

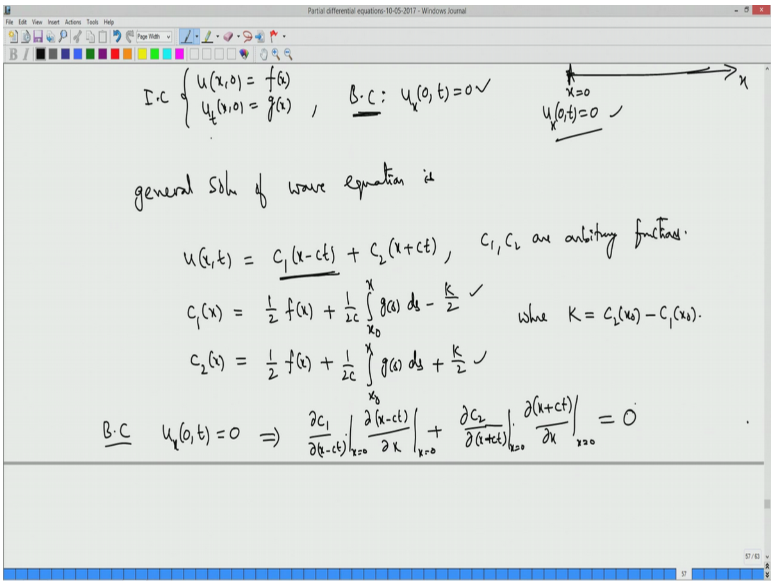

So I will just pick it up from earlier what we had derived after applying initial data so by doing this you see one is c 1 you will get minus K by 2 so K is an arbitrary constant. So you see that half f x plus 1 by 2c integral x not to x g of s ds minus K by 2 as your c 1 x, okay c 1 x you have this where K is arbitrary constant where K is actually K is actually c 1 at x not minus c 2 at x not. So that is some arbitrary constant. So another one is c 2 is half f x plus 1 by 2c x not to x g of s ds plus K by 2, this is how you will get, okay. This is what we have seen even in the earlier example so once you use this and you can see that the general solution is this.

Now again x minus ct can be negative for larger times so you need c 1 a negative function that I can get it from the boundary condition, okay. So that boundary condition so you can see that where K is where K is actually, so what is K? K is c 2 at x not minus c 1 c2 at x not minus c 1 at x not. Let me apply the boundary condition boundary condition fixed get you what is your c 1 for the negative values.

So u x at 0, t equal to 0 the boundary condition implies c 1 minus ct c 1 dash, okay so how do I do this one this is c 1 this this function you have to differentiate with respect to this is like dc1 by d x minus ct into d x minus ct divided by dx that is my dc1 by dou c 1 by dou x, okay you can dou both are all are dou because two functions are involved, okay this is what is my dou u by dou x the first term I did that is what I get. Now you see that dou c 2 by dou x minus ct and into dou of x plus x plus ct divided by dou x this is plus okay. Now this whole thing you are putting at x equal to 0, okay this whole thing at x equal to 0, okay this is equal to 0.

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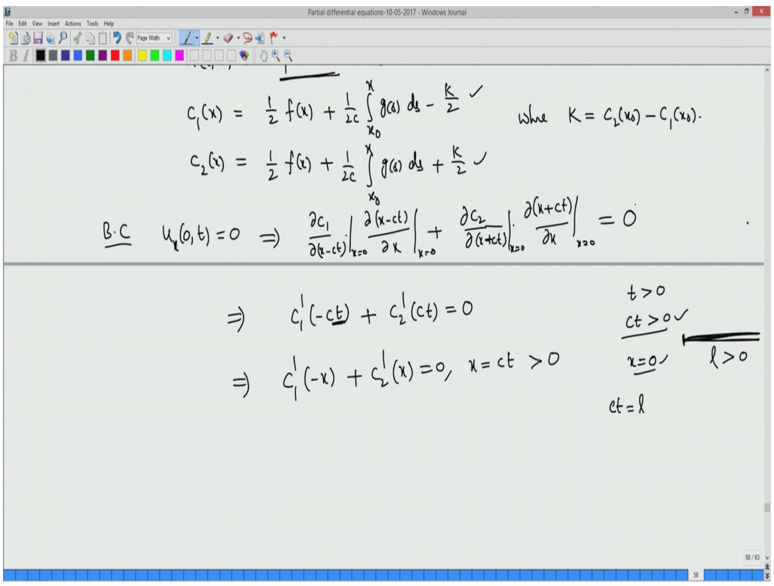

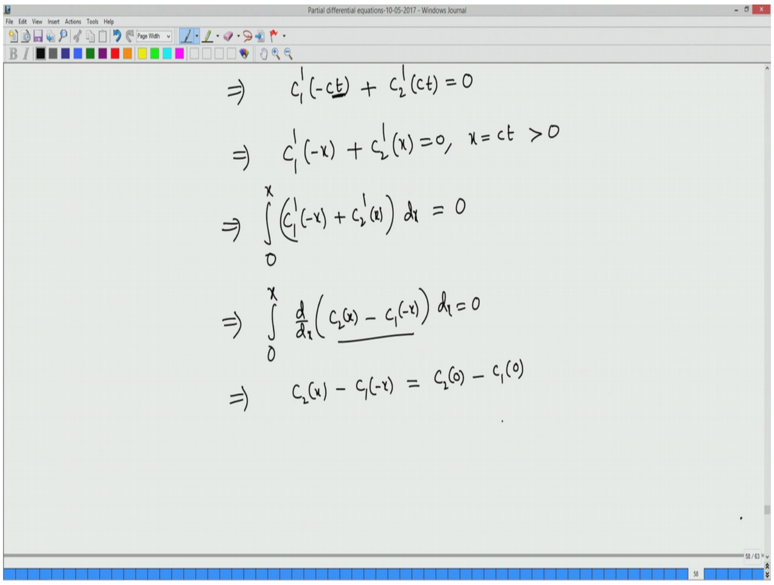

So this is exactly whatever is the left hand side is my u x at 0, t so this gives me c 1 dash of minus ct, okay when you put x equal to 0, dc1 by d of minus ct, okay of c 1 c 1 of minus ct, okay c 1 of x minus ct when you put x equal to 0 minus ct. So this is exactly c 1 dash of minus ct plus c 2 dash of ct equal to 0 this is what you will get. Now if you integrate if you simply integrate how do I integrate like earlier, if you simply integrate from so see that t is t is positive so implies ct is also positive, okay you can consider x not equal to 0 this is you have a domain is from 0 to infinity, okay.

So you can think of so this is only taking positive values, okay c 1 of minus ct minus ct, ct is positive, okay this is that you can think of ct as l so this is your l, l is always positive including 0, okay. So so you can integrate this so first you write this c 1 dash of minus x plus c 2 dash of plus x equal to 0. In a notation I am simply replacing x equal to where x is ct, which is because ct is positive this x is also positive, okay so that is what I mean so because it is positive so x is positive you can integrate this from x not to x not to x that is now I can take I can fix my x not as 0, okay so I can fix my x not as 0, 0 to x if I differentiate I can integrate c 1 dash of minus x plus c 2 dash of x into dx, okay equal to 0 right so integration anyway 0, okay.

So you have what you have is this is integral 0 to x d ds of or d dx of this is c 2 of x minus c 1 of minus x, if you differentiate this is exactly you get the integrant c 2 dash I have minus minus plus c 1 dash of x, okay to differentiate this c 1 of minus x you get minus of c 1 dash of minus x, so minus minus plus you will have this this is dx equal to 0. Now this is at x that is c 2 x minus c 1 of minus x equal to minus of at 0 so that is c 2 of 0 minus c 1 of minus x minus 0 that is 0, okay this is what is my this one.

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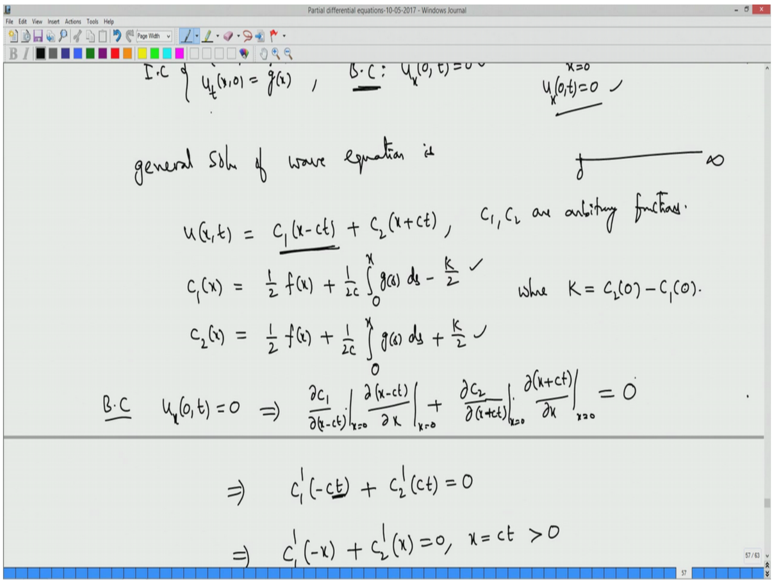

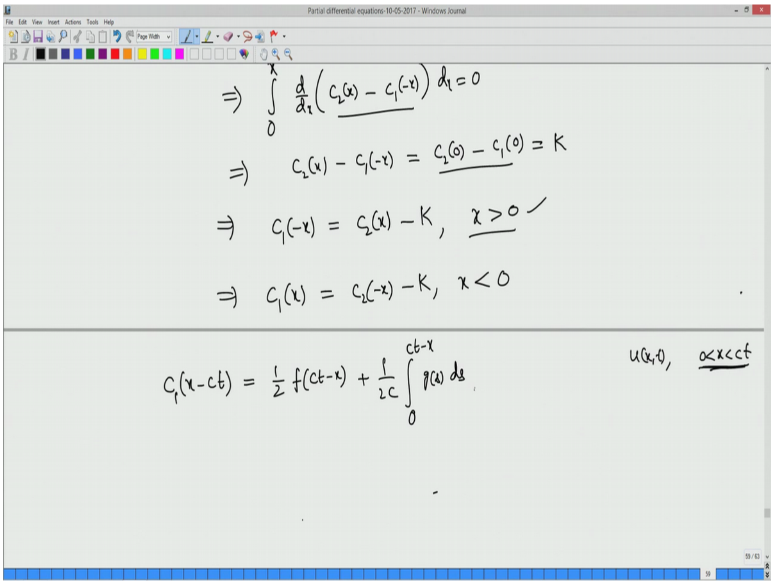

So what is this one this is equal to now when you are applying the initial conditions you integrate it from x not to x there also you have a domain is from 0 to infinity x is from 0 to infinity you can choose your x not as 0, if you do that your K will be simply c 2 at 0 minus c 1 at 0. So C 2 is 0 c minus c 1 is nothing but your K same K. So this will give me c 1 at minus x, x is positive, okay this is equal to c 2 x minus K, this is for x positive, okay.

So this means c 1 x equal to c 2 of minus x minus K for x negative. So this this is what I got. So I I now find what I want c 1 function for the negative values, I know what is my c 2, c 2 is this I make use of this to write for the negative values, okay. So what I want is c 1 at x minus ct so this is where when I will get it this is (())(10:38) I know that when I require u of x, t for x is between 0 to ct that is where c 1 of x minus ct in the general solution u have a general solution here okay so that can be negative so that negative function is actually equal to in this when x is between 0 to ct c 1 of x minus ct x minus ct is negative like here.

And this is nothing but c 2 of minus x what is c 2, c 2 is half f x that is half f of minus x f of minus x means that is c 2 x c 2 x is minus x is now x minus ct. So you have ct minus x, okay then plus 1 by 2c integral 0 to x g s ds 0 to x is how to write minus x that is ct minus x, x minus ct minus of that is ct minus x, how g s ds, okay.

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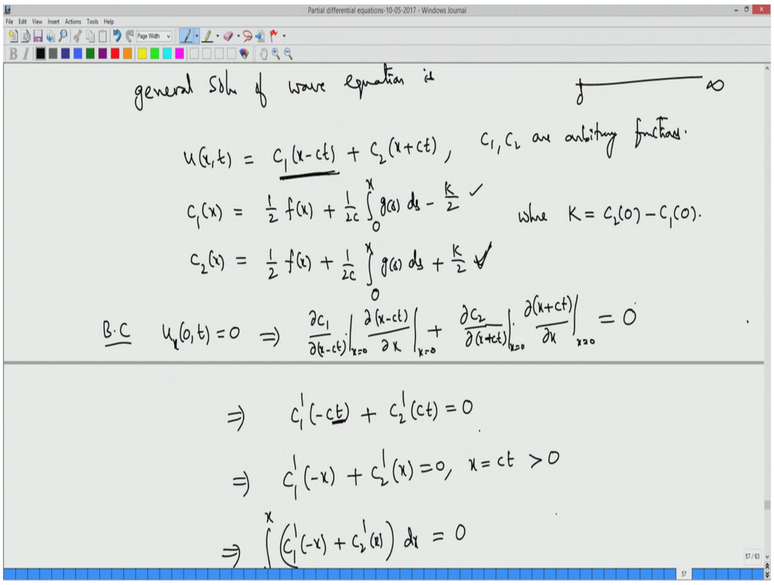

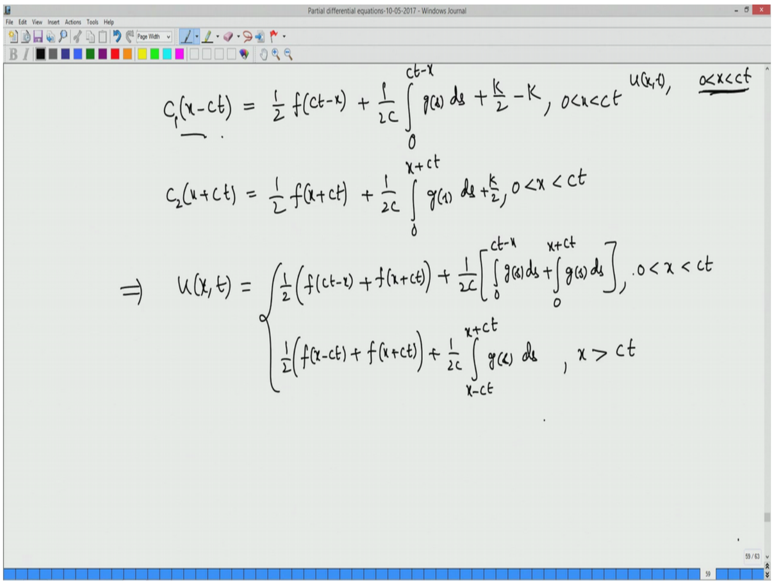

And this one if you add with c 2 c 2 of x plus ct, c 2 of x plus ct is simply half of f of x plus ct plus 1 by 2c integral 0 to x plus ct g of s ds so this sum is nothing but so this is what is x minus ct so this is from 0 less than x less than ct, okay. So for the same x is between 0 to less than ct your c 2 is anyway whatever may be x plus ct is anyway still positive so you have to use the same ct.

So this implies u of x, t when x is between 0 to x is between 0 to ct you have you have this summation that summation is half of f of ct minus x plus f of x plus ct and now this one you can simply add add together and of course c 1 have minus K, okay so I miss something here so you have a minus K, right so you have a minus K here and here c 2 is plus K by so sorry c 1 so you are replacing c 2 so that is plus K by 2 that is plus K by 2 what is that I am missing something so c 2.

So this is you have what you have written is c 1 this equal to c 2 of minus x minus K minus K and what you have is what I have missed is c 2 is plus K by 2 is missing, so plus K by 2 is also there, okay. So you have a c 2 of minus K you have plus K by 2 and you have a minus K here, okay. So together this is minus K by 2 and c 2 is you have here plus K by 2 and if you add this two these two gets cancel, okay this is what it means this is for x less than ct.

So what you have is when you add it what you are left with is simply this into 0 to ct minus x g s ds plus 0 to x plus ct g s ds for this domain. What happens when x is greater than ct both c 1 and c 2 both are positive c 1 x minus ct x plus ct both are positive you simply take from here their addition. So this K by 2 K by 2 goes and simply have usual D'Alembert's solution for c greater than ct what you have is simply D'Alembert's solution that is half times f of x minus ct plus f of x plus ct plus 1 by 2c integral x minus ct to x plus ct g s ds.

So this is your this is how you constructed your solution when your boundary condition is free free edge boundary condition, okay at x equal to 0 the string is like a free so you have the slope is 0.

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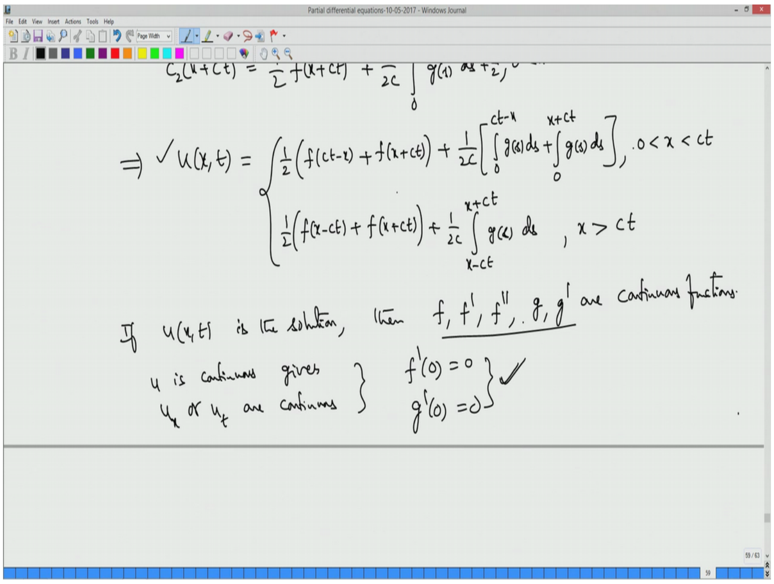

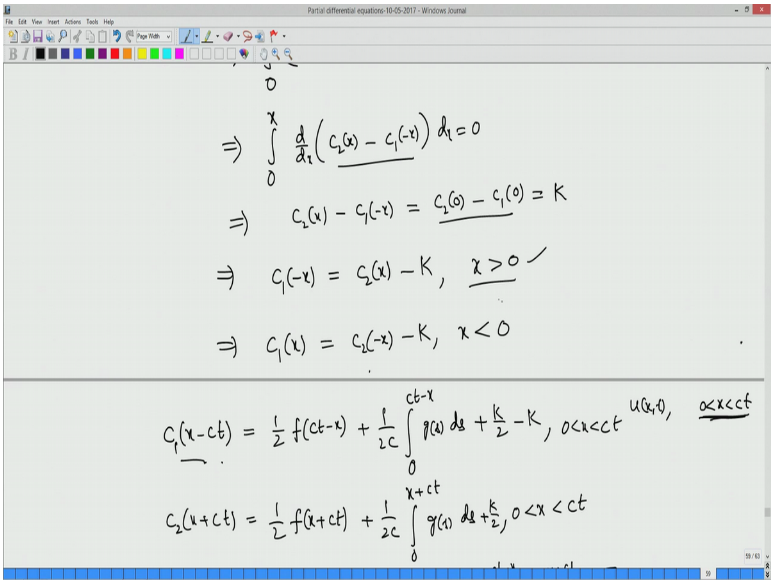

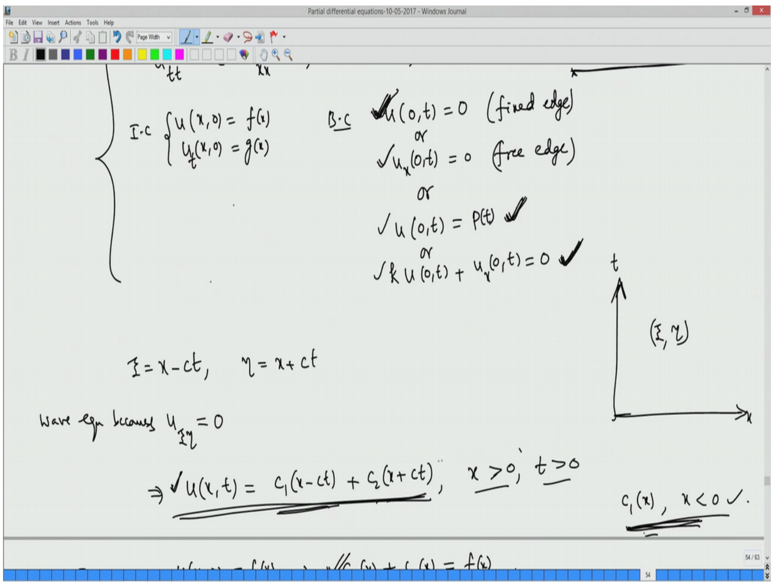

If u want this to be a solution if u is a solution u of x, t is a solution is the solution because is a unique solution is the solution then again it has to satisfy wave equation so f is twice differentiable f, f dash, f double dash and g g is g, g dash or continuous exist and continuous or continuous functions, okay and immediately by making use of u continuity u is continuous gives gives something okay u or u x u x or u t u t or continuous if these are all continuous in fact u xx also u xx, u tt all continuity will give you finally f dash of 0 is 0, g dash of 0 is 0.

So these are the conditions we used by extending the domain 0 to infinity to as a even extension you have taken so you extended it the domain to minus infinity and made use of D'Alembert's solution for the full real line. In that case when you extend as an even function these are satisfied automatically.

So by the initial data and here also by just substituting this and into the if you this has to be a solution necessarily the initial data that is by f and g functions should satisfy this, okay along with this differentiability and continuity of this functions f and g. This is how you can construct your solution when the boundary data is given like this when the boundary data is free edge and the boundary data is one of this other edge, you approach approach is same you first have by apply the initial data get your c 1 x and c 2 x.

Now because of u 0, t because x minus ct because your x is positive x minus ct can be negative so apply the boundary conditions either this or this and try to get your c 1 for next values of x next ordinate, okay c 1 of x when x is negative that is what I have to find out that you can get it from this one just by applying to the general solution here of the wave equation you can get that because when you put x equal to 0 what you are getting c 1 of minus ct.

So obviously you are getting you are able to get c 1 of minus ct that you call it some x, so you have a negative values of x you can get your function c 1 and that you substitute that you can get in terms of c 2 from the boundary condition that make use of that and finally you substitute into the c 1 and c 2 you will get your solution.

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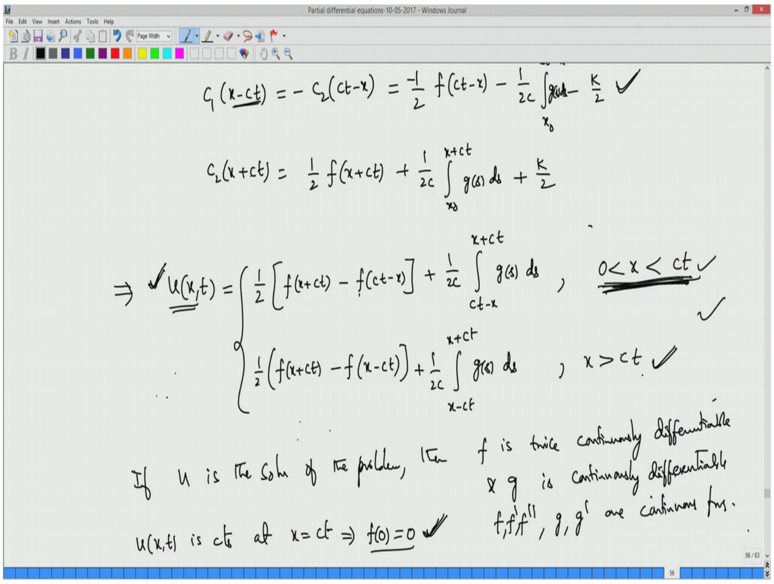

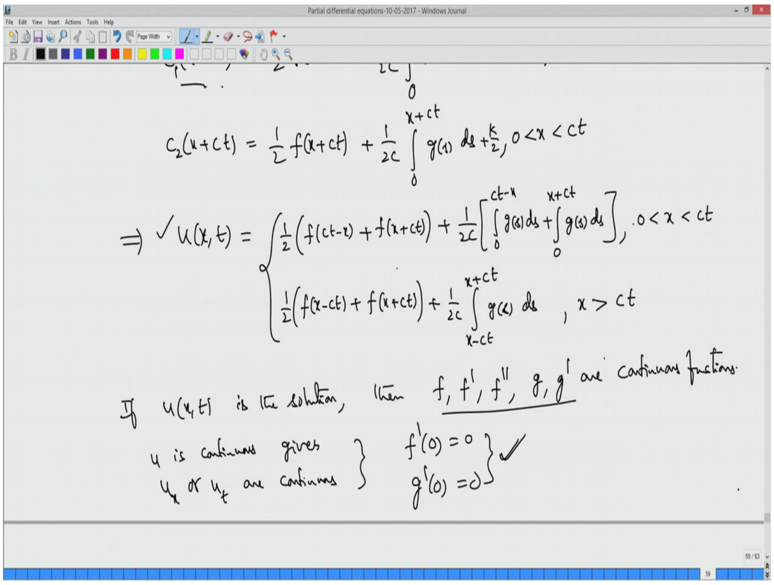

In the in this domain that is the only thing extra otherwise other part is simply D'Alembert's solution, okay. So this is how you can apply this approach this is the most general approach so that you can apply you can even use for other boundary conditions. So this is how so far we have seen how to solve wave equation with the initial data in the full domain minus infinity infinity or a wave equation with initial data and on a semi-infinite domain that is 0 to infinity. So that you have a boundary on the boundary you can give any boundary data, okay.

So any type of boundary data whatever is given you can actually construct your solution in this fashion, okay.

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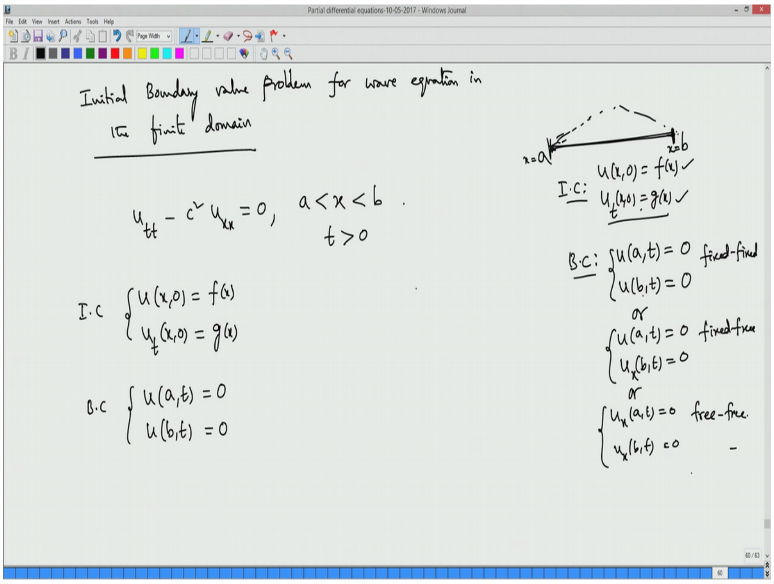

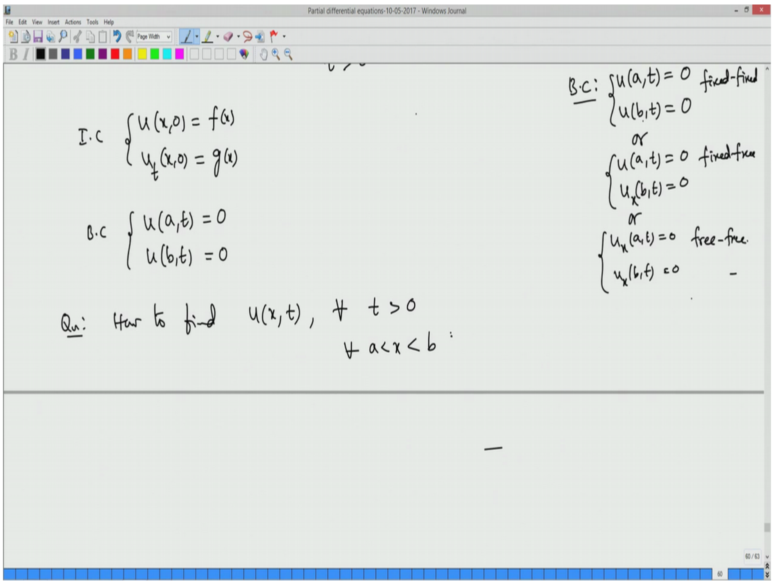

So now let us move on to move on to wave equation solutions what are the problems we can have if you have a finite domain. So that means consider a finite string if you have a finite string vibrating string so initial data is like u so it is the displacement of the string is given initially initially at t equal to 0 you can have u at x, 0 equal to f x that is given x is between a to b, a to b is a finite string, okay.

So your finite string is string of length b minus a that is so let us say at x equal to a is one end and x equal to b is other end and other initial data is u t at x equal to 0 is g of x that is the slope of this string is at every point is given as not not slope sorry this is actually the vibrate so initially your displacement of the string is velocity of the string at initial time at t equal to

0 is given as g of x that you can show graphically so these are given this is what is the initial data.

Boundary data now you have to fix because you have a boundary here a and b so that is u at a for all times u at b for all times you have to provide this boundary data because you have a boundary involved here, okay. So because you have this boundary involved so you have to provide this boundary data for the solution so that is how you make the problem well defined, okay.

So how do I what do I do so you can fix physically you can fix your string as fixed string or allow it to be free just like you have seen earlier. So if you make this these two boundary condition you can give 0 or boundary conditions you can change it so you can have u you can fix it one end you fix it, one end I can fix it, other end I can simply take the I can make it to be free that is x derivative at b, t equal to 0.

So this is like one end fixed is like fixed fixed free okay, fixed free, this is fixed fixed fixed, okay both end fixed. Like that you can have a combination u x at a, t is also 0, u x b, t is also 0, so this is like free free free-free, free-free, okay at both the ends. So like this you can have so a string is satisfying the wave equation so you can have you can now write the boundary value problem initial boundary value problem for wave equation in the domain in the finite domain rather, okay in the finite domain one dimensional wave equation in the finite domain.

So you have u x u tt minus c square u xx equal to 0, x is now not full real line not semi-infinite so it is simply from a to b you have a to b and t is positive, okay. So these are this is your domain this is your domain now give the initial data that what you have there so you have u at x, 0 like earlier f x, u t at x equal to 0 velocity of the string at x equal to 0 this is your initial conditions, okay.

Now give the boundary data or the boundary conditions you can give one of them I start with this fixed fixed-fixed string that is u at a, t equal to 0, u at b, t equal to 0. So this is the problem now, how do I solve how do I find my u of x, t for all t. So this is initially you know something boundary data you know. So the question is how to find how to solve, how to find the displacement of the string for all times for every for all t positive and for every x everywhere in the all along the string how to find the displacement this is the question, okay.

So this is where we use we can actually use something similar arguments like earlier arguments but let us what because we have learned ordinary differential equations we try to when we used we have learned Sturm Liouville theory, we try to extract Sturm Liouville theory, okay.

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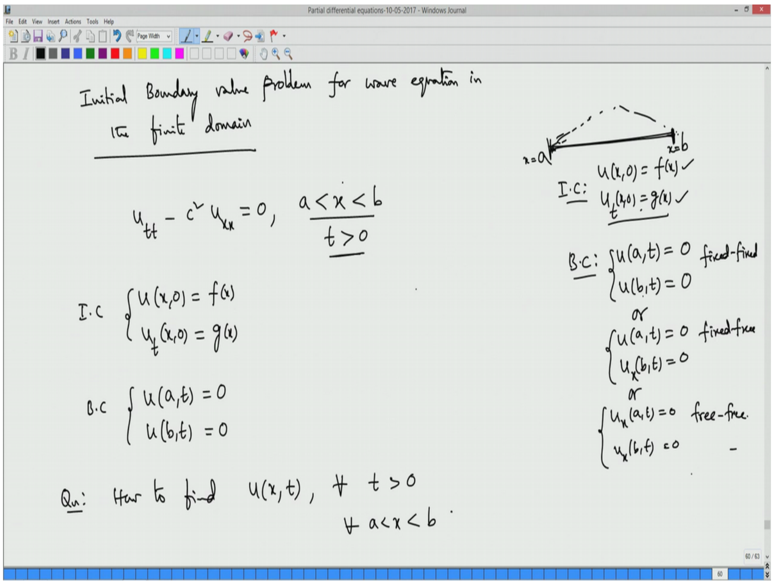

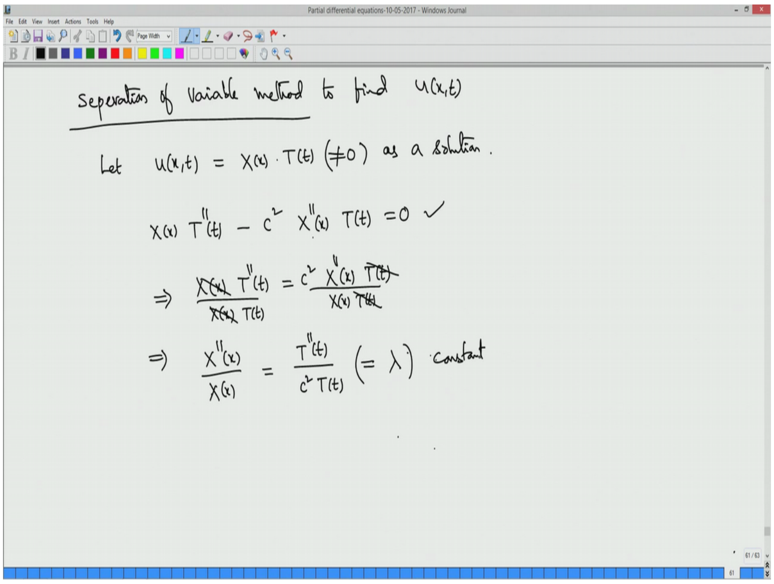

So when the special domain one of the special domain is finite, okay you have a t t and x variables t is positive, it is a infinite domain but x domain is finite, okay you can see this when you have such domains and you can actually extract Sturm Liouville value problem out of this PDE by a technique called separation of variables, if you apply if you look for solutions so this is called separation of variables variable technique whether to find u x, t okay. So this is what we do to find this u of x, t, okay.

So separation of variables I am applying to just to find the solution of the wave equation let us see, okay. So wave equation is homogeneous equation so you have a 0 is actually a solution because look at the equation 0 also satisfies if you replace u of x, t as 0, if you take u of x, t as 0 0 satisfies, okay. So we are looking for nonzero solution look for let u of x, t be some X of x I where a separation I separate this variables X of x into T of t which is nonzero, okay as a solution as a solution of solution of the wave equation.

You look for in this form look for solution in this form, okay so if you do that substitute that into the this form substitute into the equation if you do that what you get is u tt that means u tt means now this is X of x is only function of x and function of t. So what you get is X of x T double dash of t with respect to this is like d square t by dt square minus c square that is anyway constant and now here u xx that is x double dash of x T of t equal to 0. So if this is how you look for a solution this is what it becomes, okay.

Then this implies you can write (X of) X double dash of x, okay by X of x equal to okay you can make it equal you take it to the right hand side and divide with because this is nonzero you can divide both sides with X of x and T of t, okay we can simply divide because it is nonzero that is how we are looking for solution. So if you do that you get this the one side, okay if you first time so let us do this so X of x T double dash of t equal to c square X dash of x T of t. You divide now both sides with X of x and T of t because this is nonzero, okay this T of t goes, X of x goes.

So what you are left is I will write like this X dash of x divided by X of x equal to T dash of t divided by c square T of t, okay this is what you get. So now the variables are separated X of x and T of t, this one side is functions of x, other side is functions of t. So if you have two independent variables x and t, one function is same left one side is function of x, another side is function of t and x and t are independent variables then it has to be constant so this is some constant call this lambda, okay it has to be constant that is call it you call it lambda is a parameter.

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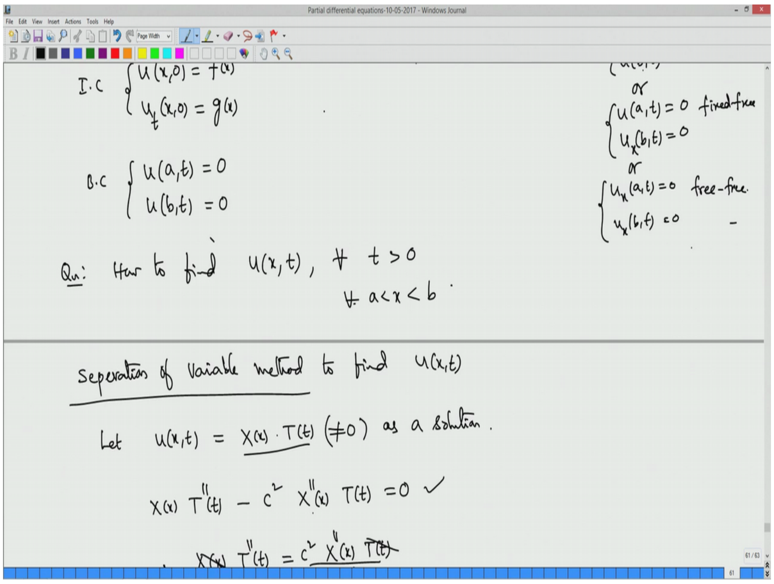

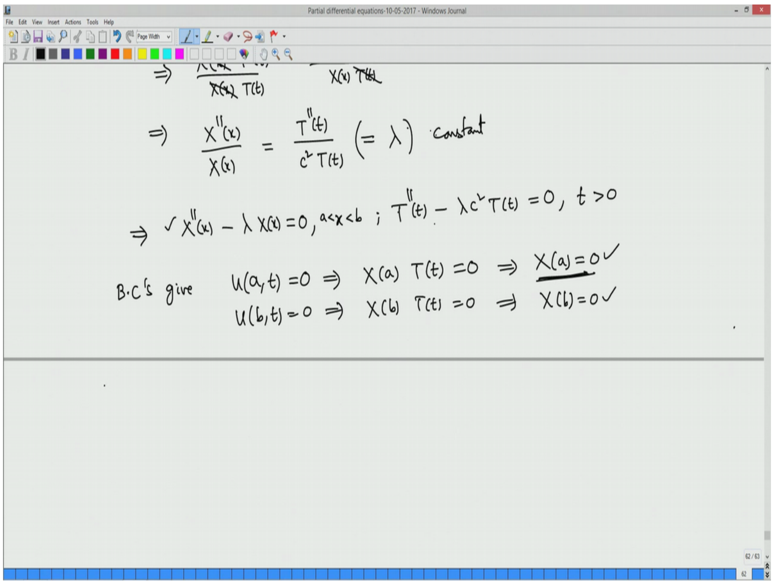

So this implies X double dash of x minus X of x minus lambda into X of x equal to 0, okay this is one problem for X of x, other one is T double dash of t minus lambda c square T of t equal to 0 that is another problem for T T of t. Now we consider this one, okay now your u of x, t is product of x and t so apply the boundary conditions what you have the boundary u at a, t is 0.

Boundary conditions give u at a, t is 0 gives this means X of a T of t has to be 0, T of t cannot be 0, okay because it is a function of t. If this is 0 then this is 0 u of but we know that we started with nonzero solution. So this implies X at a equal to 0, so you got a boundary you have a boundary condition for x this is what is your domain for x, x is between a to b when you are separating your write like that and here t is positive.

Now for the this problem you have found a boundary data x at a. Similarly you have u at b, t equal to 0 will give me X at b T t is 0 so this because T t cannot be 0, so that is X at b equal to 0, okay. So this is your boundary data boundary conditions will give you if you apply the boundary conditions they give you that x of a equal to 0, x of b equal to 0 this is the Sturm-Liouville problem.

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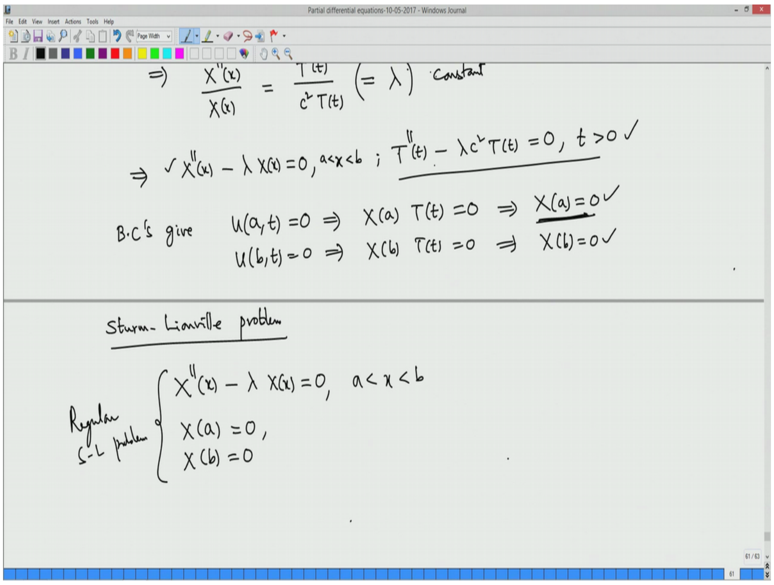

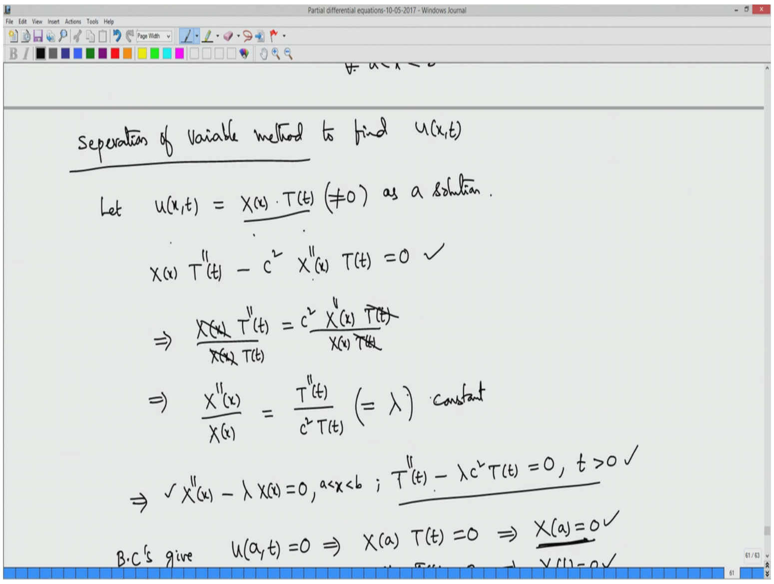

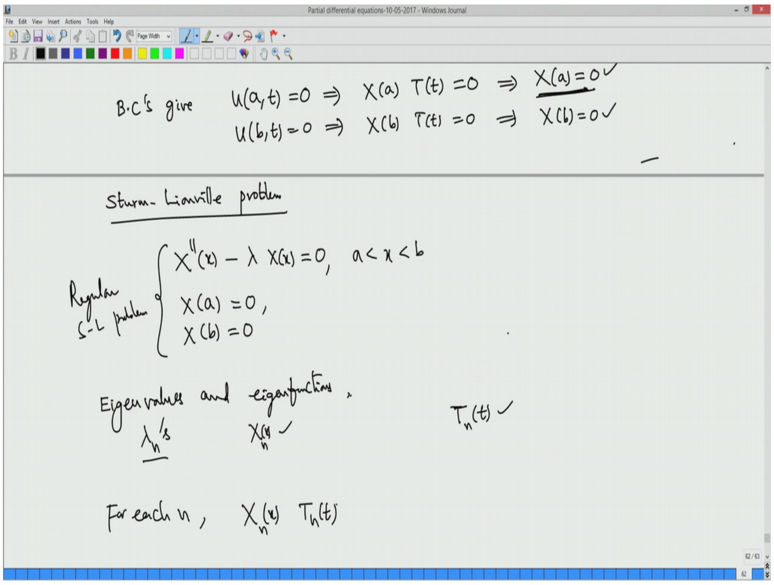

So now we have extracted Sturm-Liouville problem a system a system regular Sturm-Liouville system you extract it what you have is mostly regular usual problems but when you are working in the Cartesian coordinates you always extract regular Sturm-Liouville system, okay when you are working on the problems for the Laplace wave equation or some other equation for the equations when you are working on a circular domains for example cylindrical coordinates, polar coordinates you may end up you may end up getting let us say legendary equation that is not regular Sturm-Liouville problem so that is singular Sturm-Liouville system so that is what you get, okay.

So here mostly if you working in the semi if you are working on the Cartesian coordinate system you will always extract regular Sturm-Liouville. So you have this problem ordinary differential equations X of x minus lambda is the arbitrary number so this is the parameter X of x equal to 0, in the domain x is a to b and X at a equal to 0, X at b equal to 0. So this is your Sturm-Liouville problem regular Sturm-Liouville problem so regular type, okay okay.

So you know how to find eigenvalues and eigenvectors, eigenvectors are all those for those values of lambda for which you have nonzero solutions satisfying these boundary value conditions, boundary data they eigenfunctions and corresponding lambda values or eigenvalues. So you find this eigenvalues and eigenfunctions and then you corresponding to those eigenvalues you come here for pickup this problem and those eigenvalues you have you can solve this second order equation and get the solution. And once you know this t and X of x you go on substitute into this x and t, I will give you a solution, kind of solution with some constant, okay.

So we will see that so first get the eigenvalues and eigenfunctions here, so because we have done earlier so eigenvalues and eigenfunctions. So because I have done earlier so I am just giving you directly so here (30) already so we can I can give you directly eigenvalues and eigenfunctions so anyway so may be because of time so we cannot finish in this video so we will have a we will try to find the eigenvalues and eigenfunctions and once you get this eigenvalues and eigenfunctions so you will see that you will have a lambda n's you can get and you have some call this some f n so eigenvalues what did I use we use Phi n's okay so we have Phi n of x this is what we used.

So try to get this eigenvalues and eigenfunctions. Now substitute this lambda n's into this corresponding to if you put it in this ODE we will get we will try to get your (T n of x) T n of t for each n, okay discrete number of eigenvalues you will get you will get discrete number of eigenfunctions you will get, okay as your X n of x eigenfunctions or solutions you will get these are these are your solutions, okay let us call this only X n of x as eigenfunctions, okay.

Here T n of t they are corresponding solutions of other ODE which we have here. So because for these are nonzero eigenvalues their product for each n you have X n of x T n of t which is nonzero which is a solution because X n of x is nonzero and correspondingly you have T n of t is also nonzero if you solve it, okay. So this solution this is a each is a solution for each n for each n this is a solution, okay.

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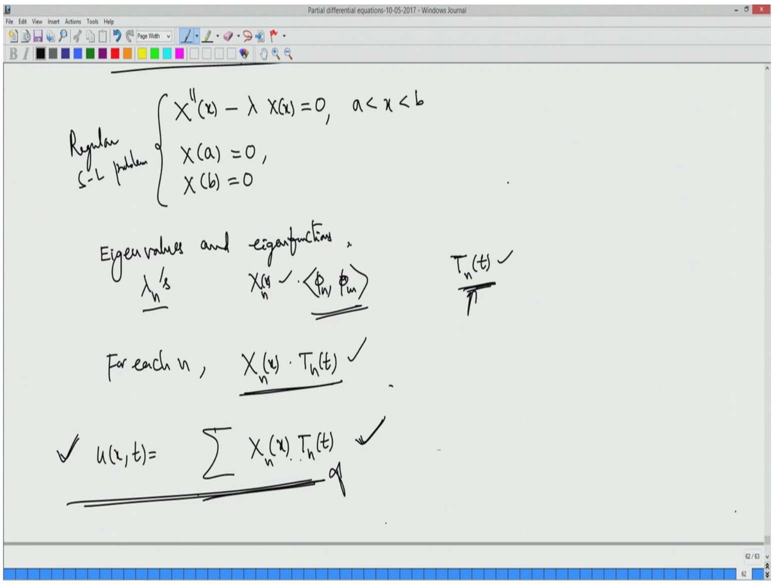

And then because n is discrete you take a make a sum may be n is may be sometimes 0 to infinity or 1 to infinity whatever, okay let us say from 0 to infinity on a finite sum let us not write here because we have not seen what are eigenvalues. So you make a sum over the discrete index n X n x T n of t, this sum you call it u of x, t as okay this you call it this is the general solution this is the general solution because this if this is a for each n this is a product is a solution any constant c n is also a solution, okay or may be that constant so this arbitrary constant may be involved for each n inside here so you have arbitrary constant inside here, okay for each n you have arbitrary constant their sum actually one can show the sum is uniformly convergent so that you can differentiate term by term if you differentiate u x if you want you simply differentiate term by term that is how you can get u x, u xx all that, okay.

So finite sum if it is a finite sum actually you can differentiate because all are differentiable function we put it into the wave equation and try to this is actually satisfying because each term each term is satisfying the wave equation the linear super position of finite terms is also a solution of the wave equation because it is a linear linear equation you are using here super position principles these are all the solutions you are making a super position sum of finite number of them, okay are also solution.

But if you do if you take this infinite sum the sum infinite one can actually show that but we do not show here one can actually show that this series is uniformly convergent in both the variables x and t that means we just think of imagine that uniformly convergent series means you simply can do term by term differentiation, term by term integration, okay basically what we need term by term differentiation.

So it is uniformly convergent that makes that means the finite sum that infinite sum it makes sense and actually you can if you want the derivative of that full sum is actually is equal to that sum of you just term by term differentiation you can do and finally when you substitute that also satisfies wave equation. So if you make this super position, okay you assume that this is a solution because because it is uniformly convergent you can do that is a solution you look for solution in this form so far we have not used initial data, okay. So once you see that this is the solution that satisfies the wave equation and the boundary data so far.

Now on this general solution you apply the initial data and get the arbitrary constant involved in this T n that means it is involved here, okay that is where we use the dot product of this eigenfunctions so the dot product of the eigenfunctions you can use here Phi n, Phi m, okay Phi n Phi m this dot product you have to use so make use of that we can get those eigenvalues of the arbitrary constant involved once you know the arbitrary constants you know that is the final solution this is what we will see in the next video, okay we will continue from here try to get the eigenvalues and eigenfunctions and we will try to get this T n of x for each discrete value of n and we try to make the super position and finally apply the initial conditions and get the arbitrary constants finally that will be the solution of your initial boundary value problem in the finite domain, okay finite string if you want to know the vibration of the string, if the initial displacement in initial velocity of the string is known and with the boundaries are fixed this we can find the displacement of the string for all times, okay from this solution this is what we will see in the next video, thank you very much.