Differential Equations for Engineers
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Lecture 47
Vibration of a semi-infinite string

So welcome back last (())(0:23) we have seen how to solve wave equation in the full domain as a D'Alembert's solution of course with initial data, then on a semi-infinite domain 0 to infinity with initial data on a boundary condition at x equal to 0. So you can give have given two boundary conditions one is the displacement is zero, other one is the slope is zero but these boundary data or boundary value problem initial boundary value problem is solved just by extending into extending as even or odd function to the full domain and you make use of D'Alembert's solution that is what you have seen. We also have seen for the full domain shown that the solution is unique, okay.

So before I do more general approaches suppose if you if you give the boundary condition as a different one non zero boundary condition or combination of u and u x thus for example you say u plus u x equal to 0 at x equal to 0, u at 0, t plus dou u by dou x at 0, t if this is 0 then we cannot simply extend as a even or odd function this will not work so what we do is this is the more general approach I will try to give you today but I will not do for this generalized this new boundary condition instead the approach will be given for the same boundary conditions of what I have done earlier, so that is (u at x, 0 is 0) u at 0, t is 0 at x equal to 0, okay u at dou u by dou x at 0, t is also 0.

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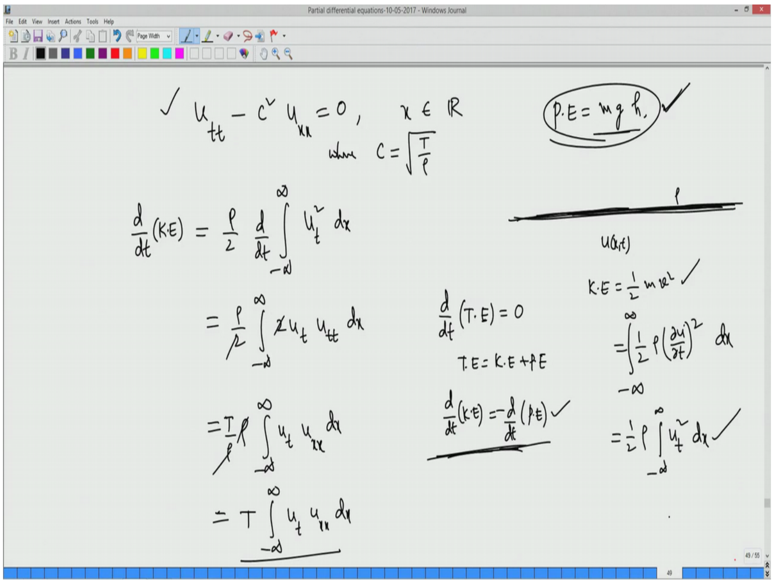

So that means at x equal to 0, you give u is 0 or ux equal to 0, okay for these two boundary conditions I give you the new approach so that this approach you can apply to any boundary initial value problem so where the boundary condition can be combination of u and u x or you can take nonzero boundary condition like u at 0, t equal to some arbitrary function you can give P of t, okay.

So that before I do this let us move on to to show that problems on semi-infinite domain is unique, okay so what we do is if you if you consider I will just explain you briefly if this is not an R what if what you do is if your wave equation is on a semi-infinite domain that is x belongs to 0 to infinity.

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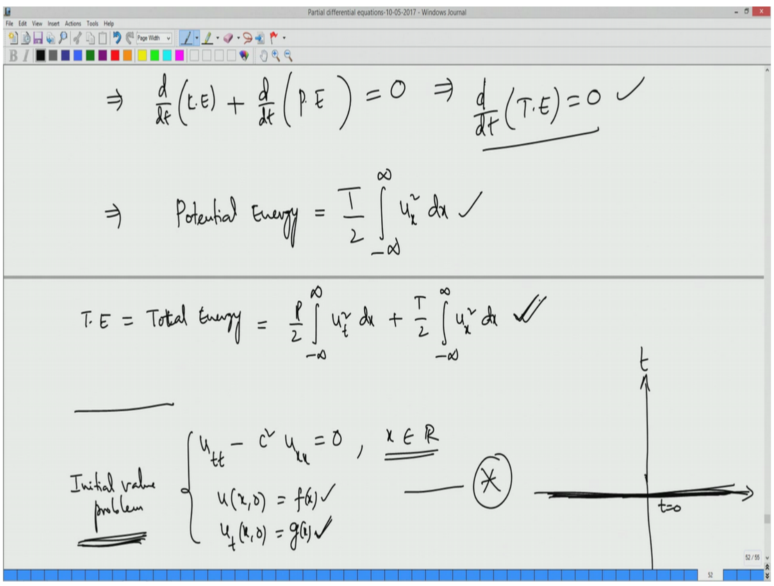

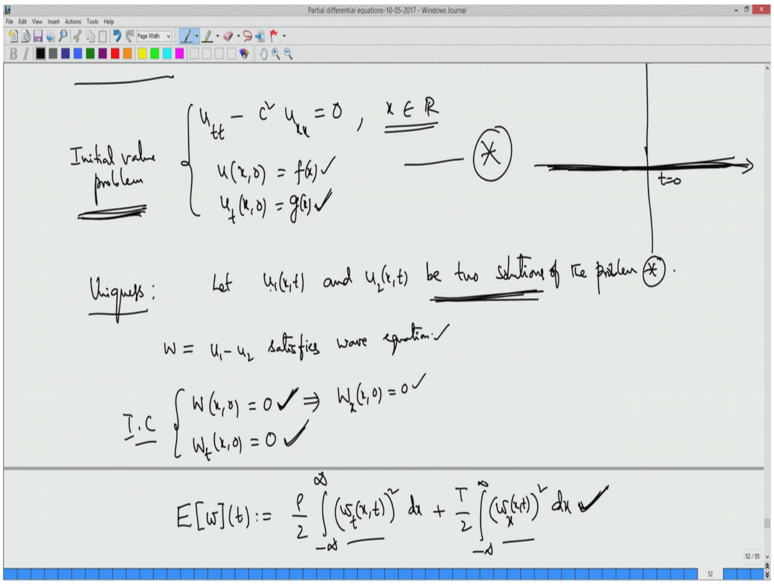

In that case the energy is same so energy total energy is the same expression. And this the initial value problem you have to change so only change here is x belongs to R, R you have to replace with 0 to infinity and we give the boundary condition, okay.

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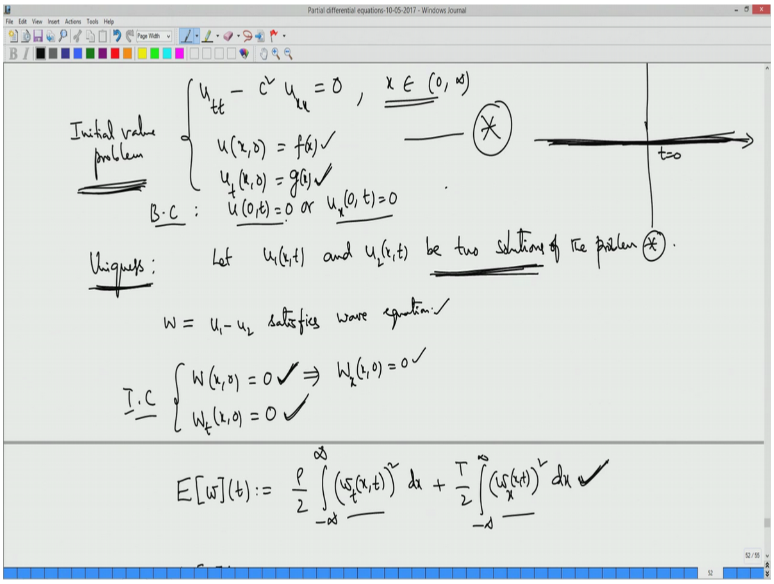

So the boundary condition when you give what is the change so the proof is to show the uniqueness you assume that u 1 and u 2 are two solutions of the problem, okay not this problem now now you consider simply x belongs to 0 to infinity and you give the boundary condition boundary condition is u at 0, t is 0 or u x at 0, t equal to 0 so either one of them, okay.

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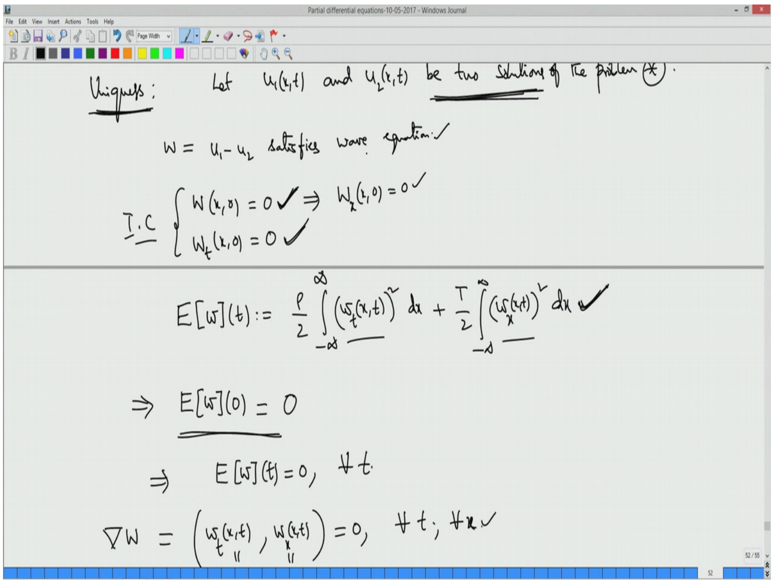

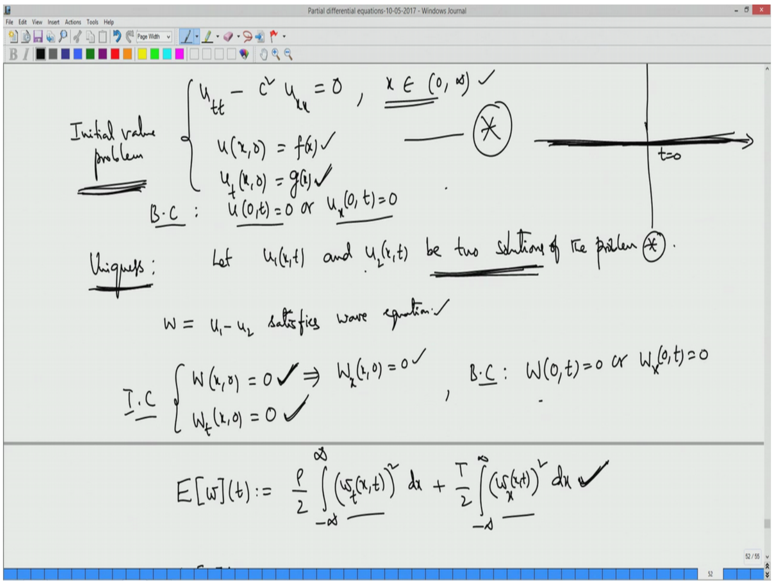

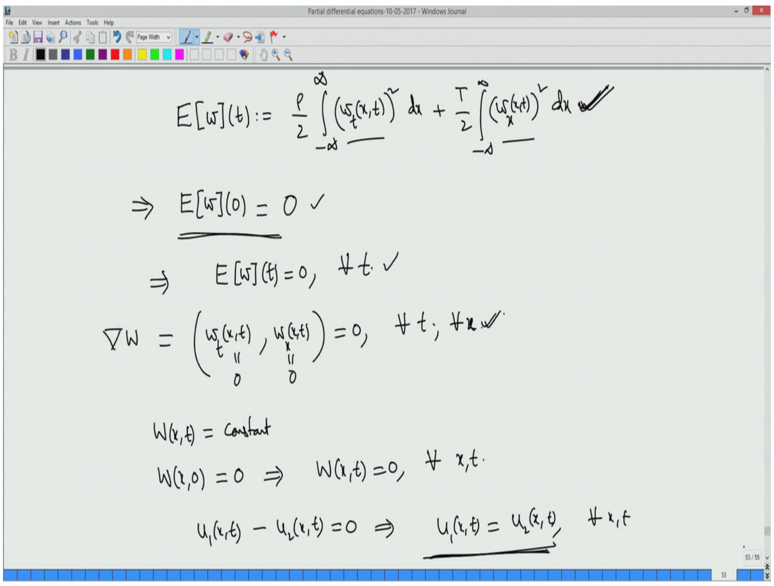

So assume that u 1 and u 2 are two solutions of this problem now then you can see that the difference is also satisfying the wave equation this equation in this domain and the difference u w at x, 0 is 0 w t at x, 0 is also 0 because it is a same same f, u 1 at x, 0 is f x u 2 to x, 0 is also f x, so the difference is 0. So this is same only thing is boundary condition will change it will not be change here also it is same so W at 0, t is 0 or W x at 0, t is also 0, okay.

So this is what becomes you have a W that satisfy the difference between two solutions the difference of two solutions if they are distinct their difference satisfying the wave equation these are the initial data and this is the boundary data, okay. So you consider the same way so everything is same so you consider the same total energy as the energy function of t and you know that initially at 0 t equal to 0 because of the initial data this is 0, okay that implies because energy is conserved so you have for all times this has to be 0 and this is also same, so this implies this is true.

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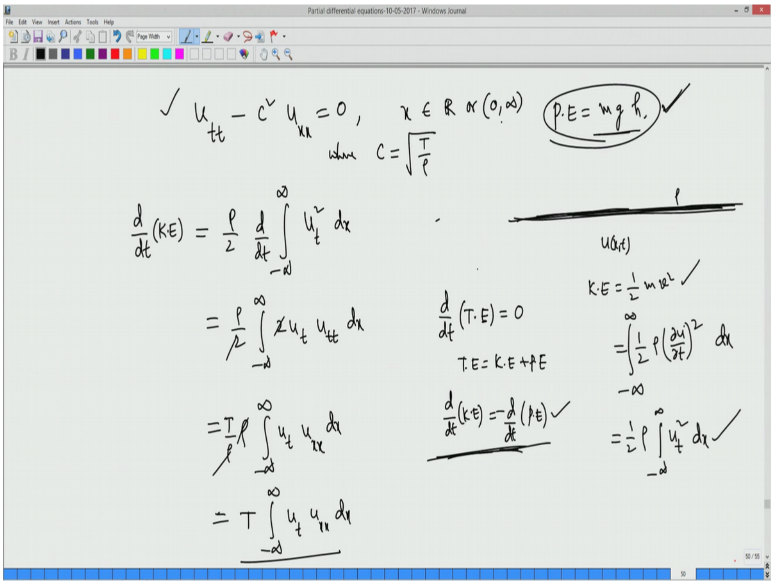

I think not not there but here onwards you have to do. So we will start from here so this is the boundary this is the you start with so what is the energy here we have to see, okay. So we will, show first show that the energy is same even when the wave equation is given on a semi-infinite domain. So x belongs to instead of R now you have from 0 to infinity, okay. So earlier you have R okay I do not want to remove this R or 0 to infinity. So let us say 0 to infinity what happens here.

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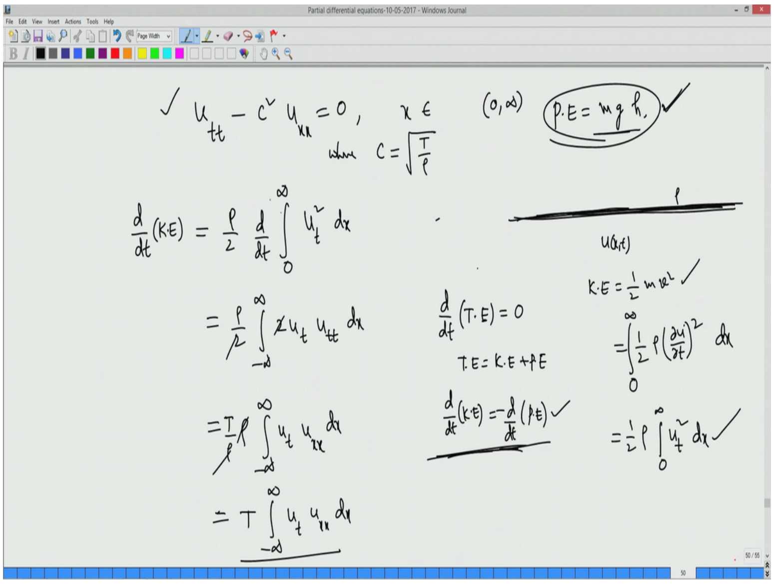

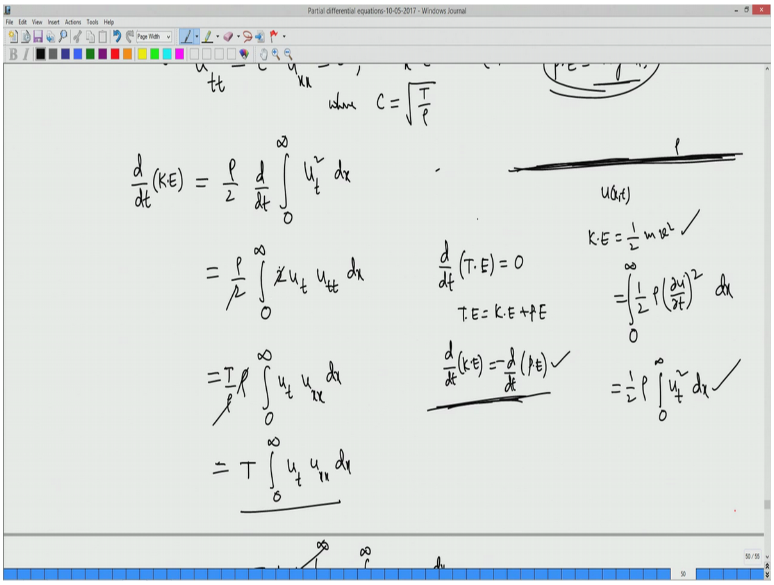

So you have a kinetic energy is same so instead of if you consider this one if you consider the x belongs to 0 to infinity so you have to consider this kinetic energy is half mv square instead of minus infinity to infinity you have to do it from 0 to infinity, so this is your kinetic energy so this is your kinetic energy. So you take that derivative and you have this one so you have instead of minus 0 to infinity you have this one, okay this is the this is the change of kinetic energy with time means the derivative of kinetic energy is this.

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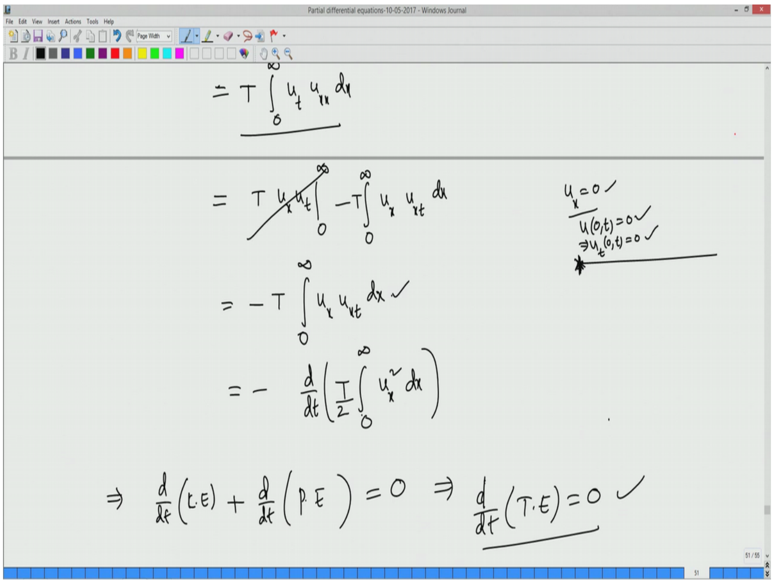

Now by now you do the integration by parts is what you have done earlier also so if you do this this is what you get. So what you see that u x u x at infinity is because displacement or slope has to be 0 at infinity so this is 0 but at 0 u x at 0 if your boundary condition is u x is 0 then so for the semi-infinite domain if your boundary condition here is u x is 0 then anyways there is no issue even at x equal to 0.

Suppose u is 0 u at 0, t is 0 here if this is given clearly u t of 0, t is also 0 because you take this this is the function of t which is 0 whose derivative with respect to t is also 0. Now this implies if this is your condition this implies u t at 0, t is 0 also 0 so that makes it 0 so in any case this will not be there. So in this case for the semi-infinite domain you have this is your total energy, okay so this is your total energy.

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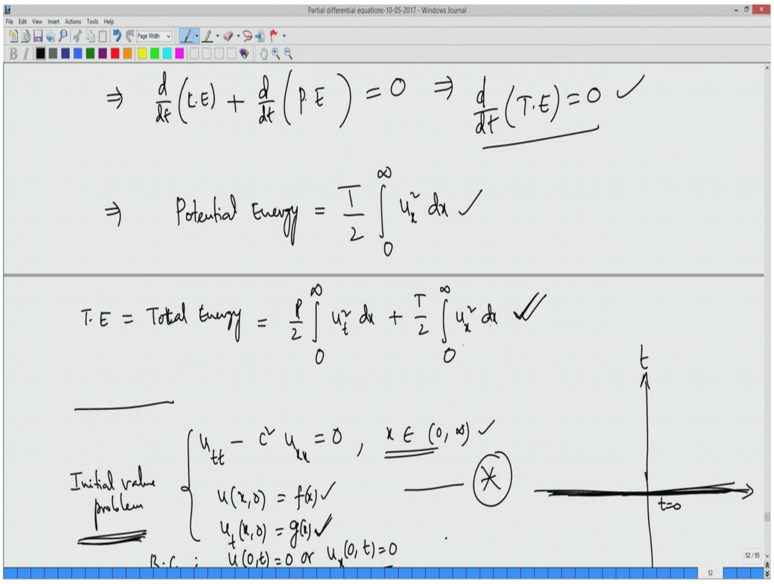

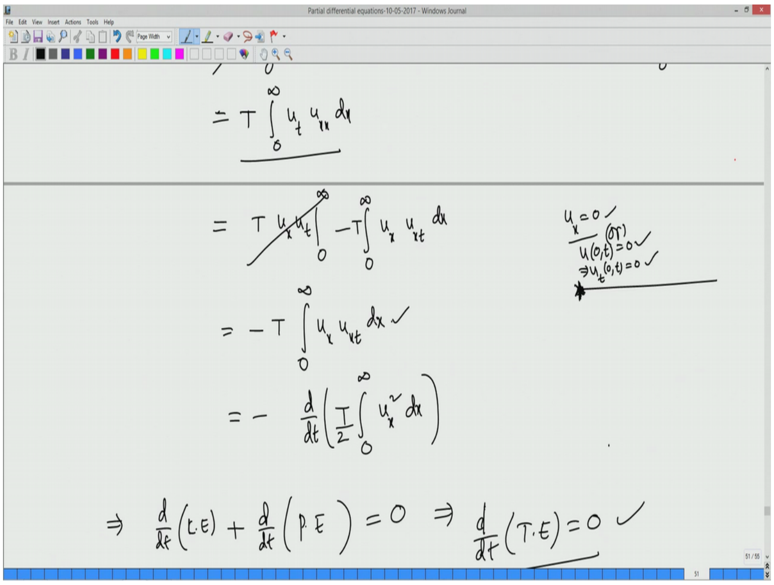

And total energy now is potential energy now 0 to infinity you can define so that your total energy is simply from 0 to infinity 0 to infinity. So whatever may be your boundary condition either this or this if this is the case directly this is becoming 0 there is no issue but suppose displacement is 0 is given as a boundary condition and you can see that time derivative of it is also 0 because u at 0, t as a function of t 0 if you differentiate with respect to t is also 0. So the because of that you have u t here so that it will also be 0 at x equal to 0. So there is no issue even here, okay.

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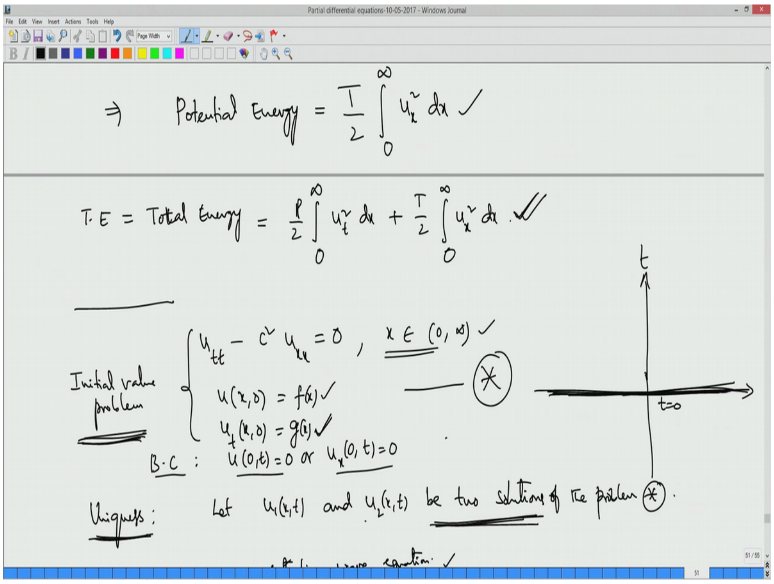

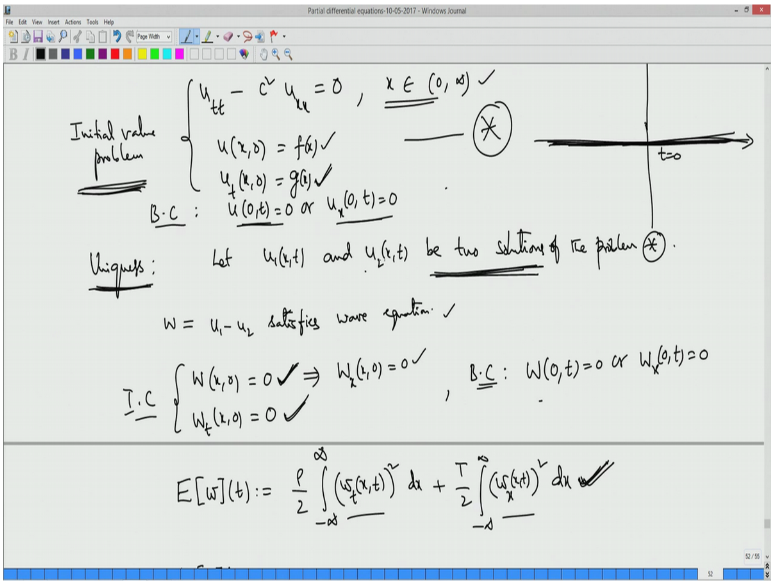

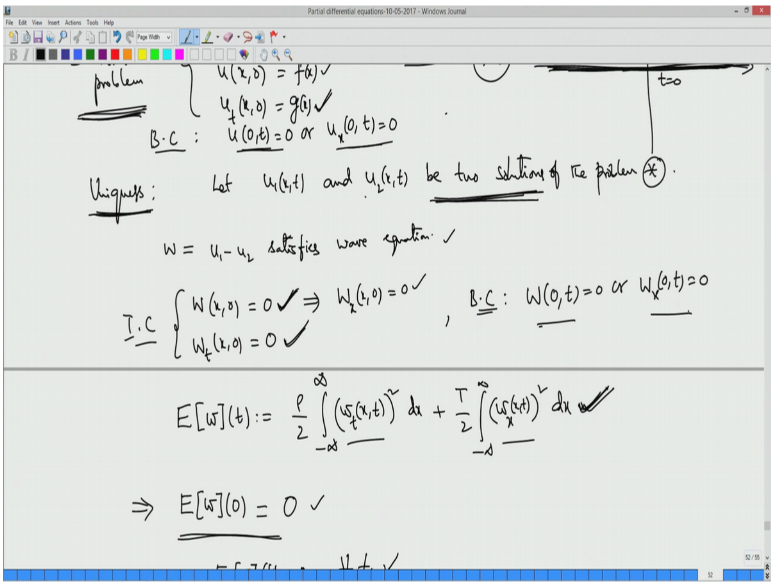

So this is what so the total energy is like this. Now u consider this initial value problem on a semi-infinite domain and now u is 0 either this boundary condition or this boundary condition you consider and u 1 and u 2 be two solutions of the problem if you consider the difference of that is W we call it W and W satisfying the initial condition 0 initial conditions and the boundary condition is either of that.

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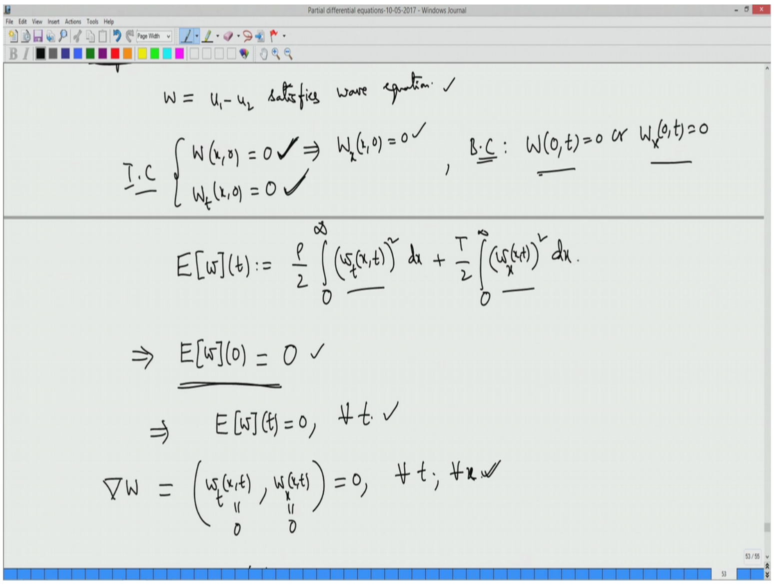

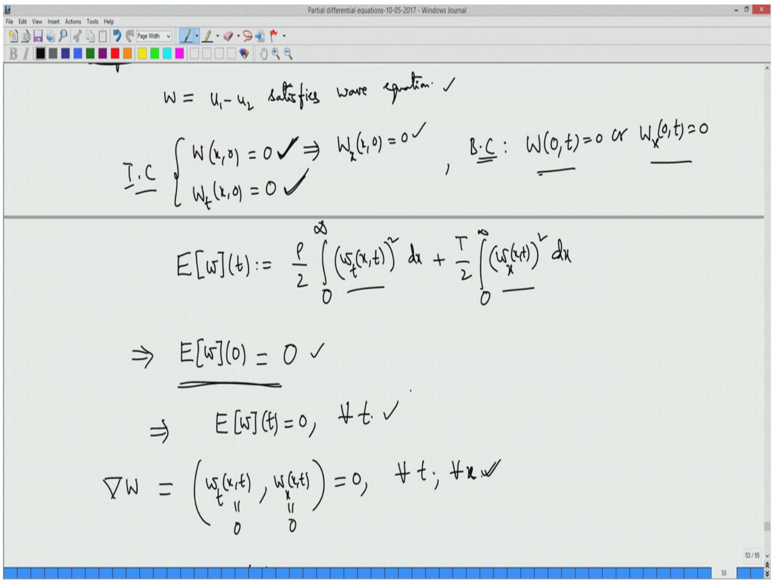

Now u consider this this is same now only thing you have to write as total derivative so energy function of t is now from 0 to infinity to this integral is the kinetic energy and this is the potential energy. So plus the total energy because of this initial data this energy is at t equal to time t equal to 0 is 0 because of conservation of energy has to be 0 for every t.

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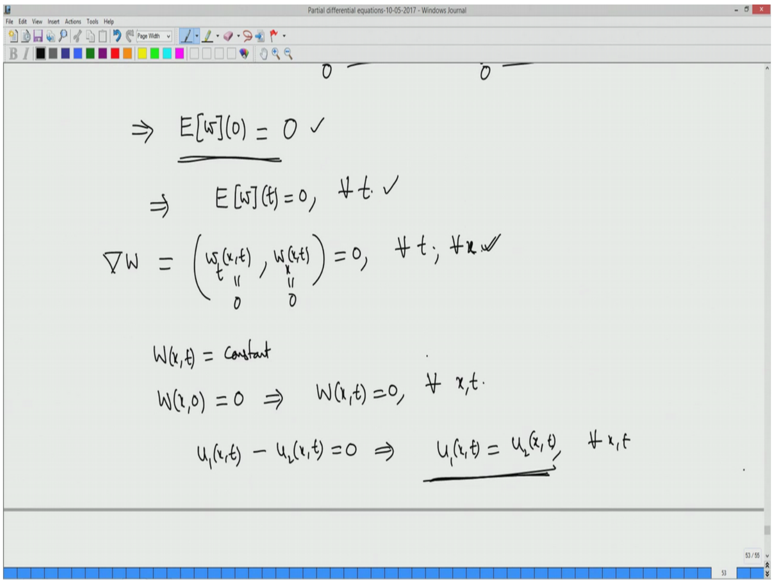

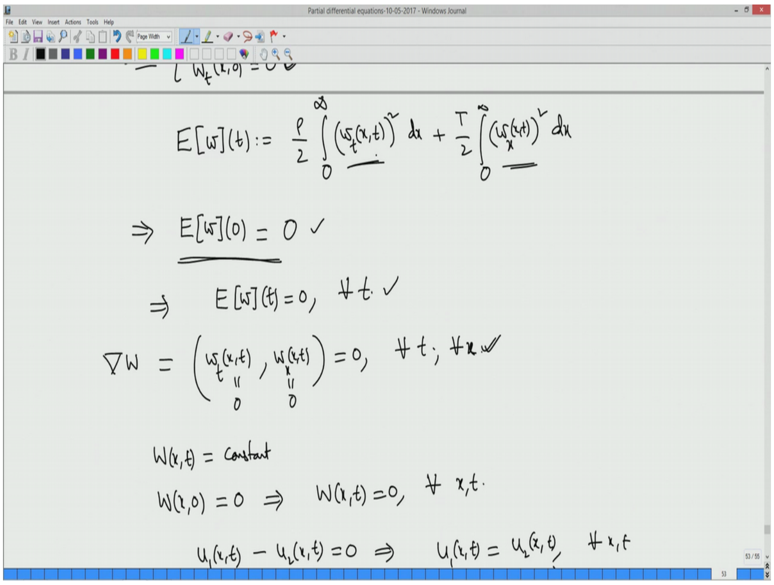

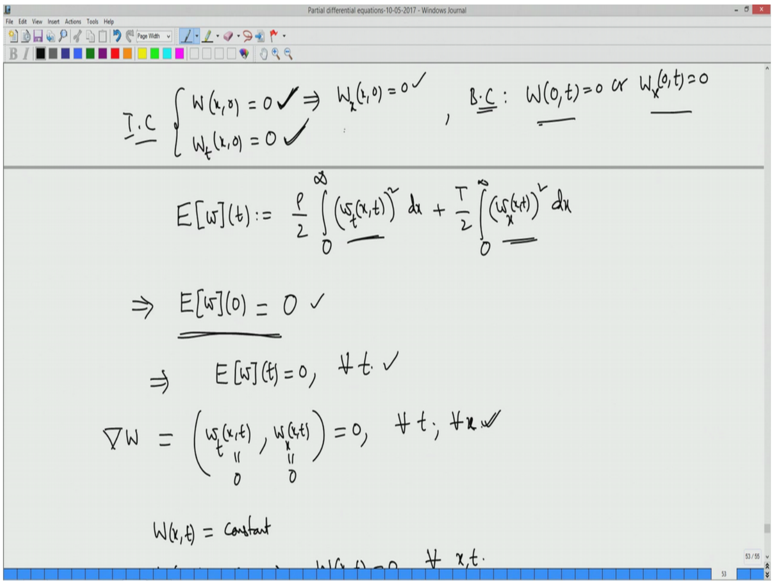

So rest is same so the gradient of W has to be 0 because for all terms if this has to be 0 the integrant has to be 0 each of them that is nothing but your gradient. So implies it is constant because W is 0 at t equal to 0, okay W that is W at W x so from here from the initial data W at x and x is equal to W x dou W by dou x x, 0 is 0 and anyway second initial condition will give you W t at x equal to 0 is 0.

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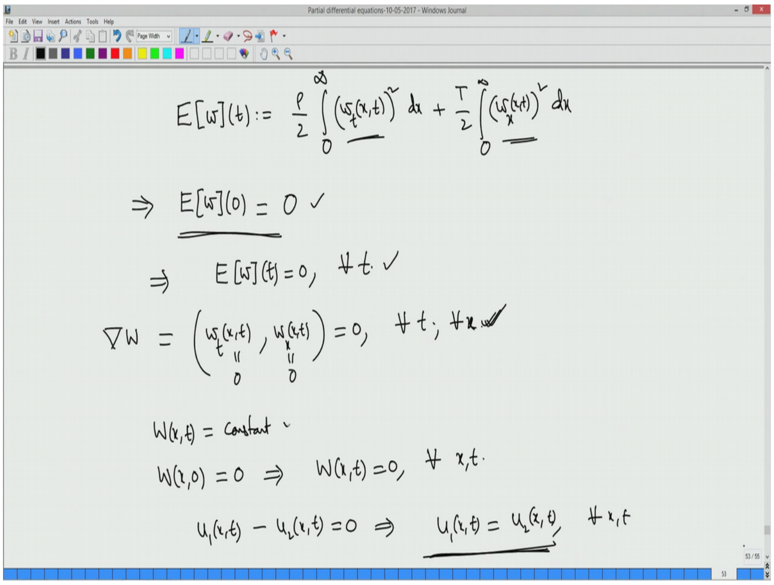

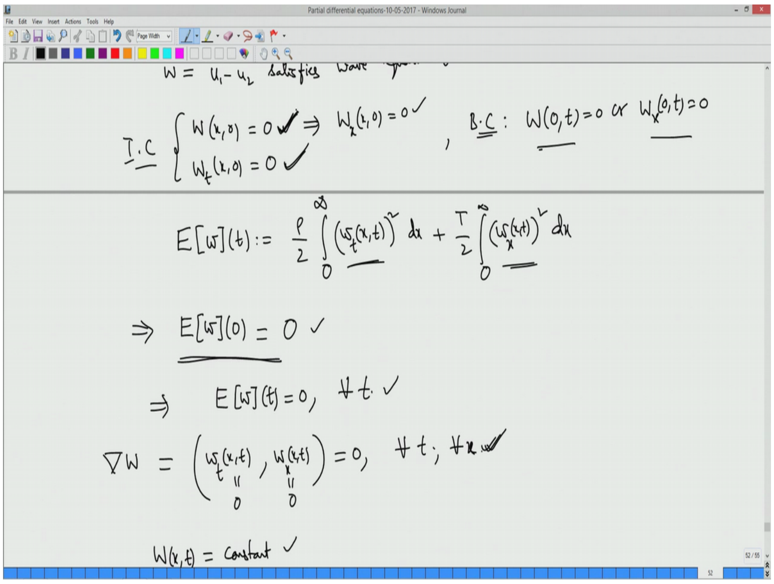

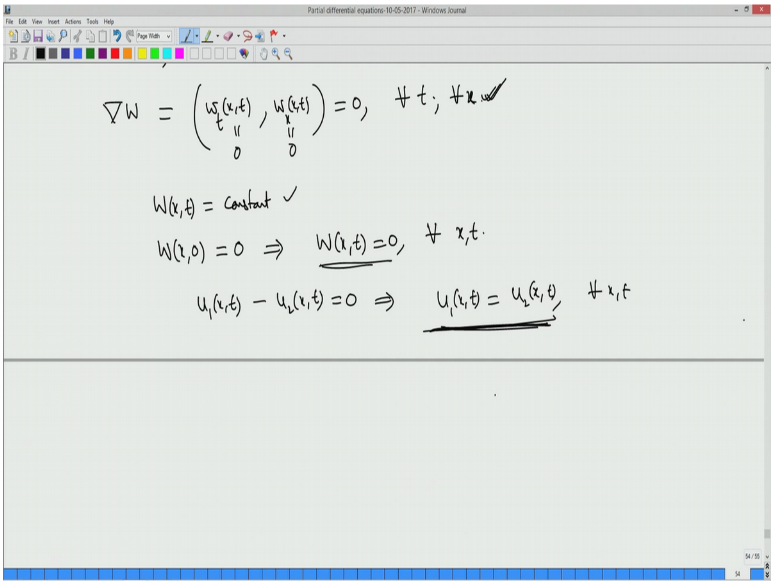

So from that we can say that W at x equal to 0, okay. So this has to be constant and gradient of W is 0 for every x and t implies W x, t is constant this is same, okay. So W at x equal to x, 0 this we know, okay W at x, 0 from this initial condition which is 0 so that implies W at x so that is a constant this has to be same for all t so this is 0 at t equal to 0 that implies (())(10:36) t u have W x equal to 0 that implies u 1 equal to u 2 for every x and t.

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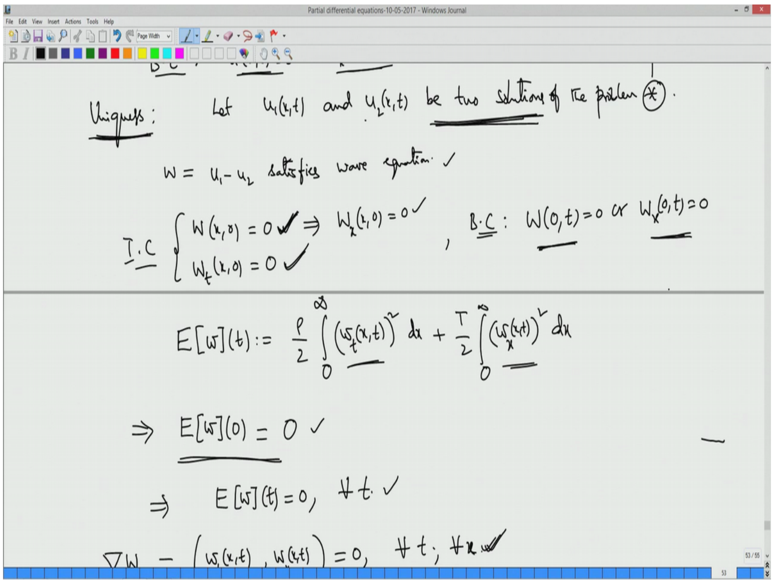

So a same proof is true even if you take the boundary condition either 0 W is the displacement is 0 or that slope of the string at x equal to 0 if you consider as a boundary condition still uniqueness of the solution is guaranteed for this initial boundary value problem for the wave equation in the semi-infinite domain 0 to infinity.

So you have a unique solution so D'Alembert's solution is the only solution for the initial value problem for the wave equation on the full domain minus infinity infinity on a semi-infinite domain if you consider these two boundary conditions u equal to 0 or u x equal to 0 you have shown that you have given a solution and that solution is actually unique by the same energy argument that is what we have just see, okay.

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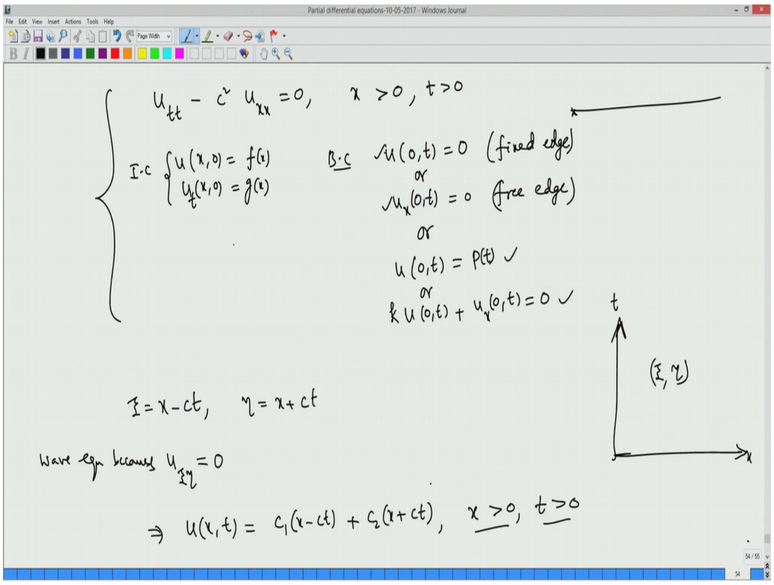

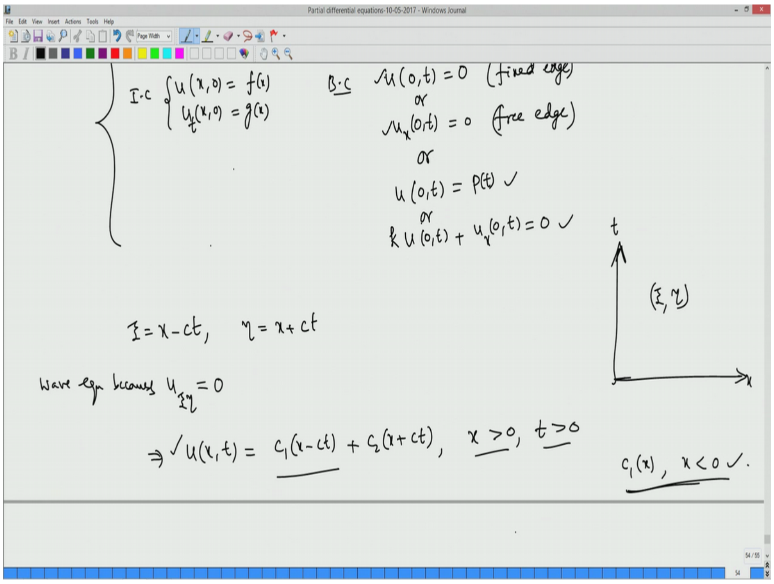

Now I will give you more general approach if you replace this boundary conditions. So if you have a wave equation like this u xx equal to 0 (x belongs) x is positive and t is positive this is your domain and initial value is u at x, 0 is f x, u t at x, 0 is g x as an initial data. Boundary data if you have instead of instead of earlier boundary condition instead of fixing and freeing instead of having this string edge as a fixed one or free one that is u at x, 0 u at x equal to 0 u for all times you are fixing it as 0, okay this is what we have seen earlier or u at u x at 0, t this is the free edge free edge of the string, this is fixed edge, okay what else?

We can also have more generally you can have 0, t equal to some function of t P of t, okay so this this to in order to solve this problem with this boundary condition or u at 0, t plus say some k constant times this plus u x at 0, t equal to 0 for this boundary condition combination of these two this and this combination if u take, you do not know how to solve because this just by extending as a even or odd function will not work so you need more general approach to treat this problem.

So what we do is now this is your domain x is positive and t is also positive so you have this is the quarter plane, this is x and this is t and you bring your Xi and Eta variables as Xi equal to x minus ct, Eta equal to x plus ct you consider this new variables and your equation becomes u Xi Eta this is that is what you have seen equal to 0, okay. So wave equation becomes becomes this this implies u at x, t is the general solution of the wave equation is c 1 x minus ct c 1 is arbitrary function plus c 2 of x plus ct this is what we have seen so right.

Now you try to apply your initial condition now and clearly what u have to observe is x is positive and t is positive earlier x is x can take any value. Now x is only positive and t is positive. So what is required here is x plus ct because x is positive, t is positive, x plus ct is always positive because c is the speed of the wave so that is constant that is positive, okay but x minus ct once you fix your x value which is positive and for bigger times at larger time this x minus ct becomes negative so you need c 1 function further should be define you want c 1 function of x for x less than 0 so you need to find this one this function if you want solution if you want a solution that satisfies the initial condition and the boundary condition.

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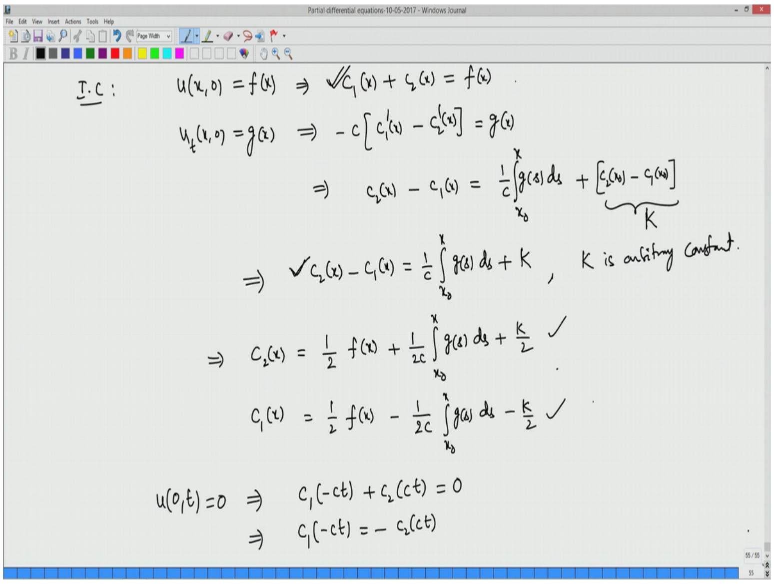

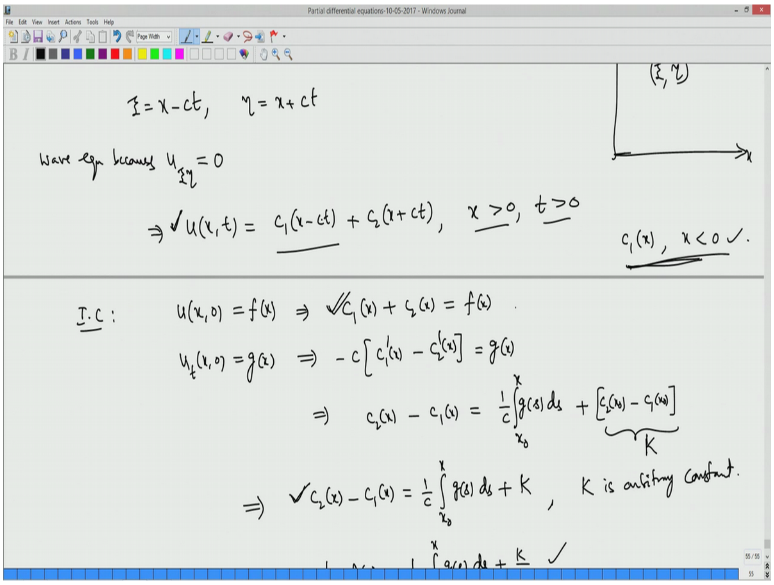

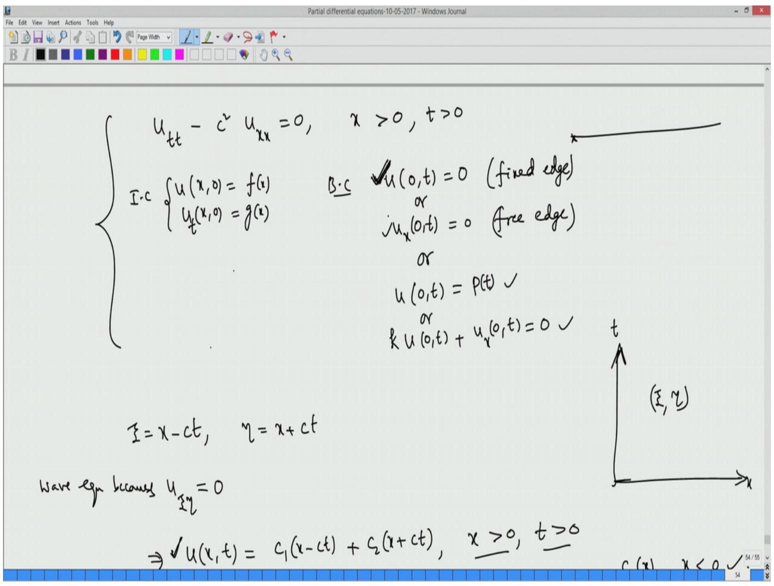

So we have already seen that how to apply the initial conditions, okay let us make this initial condition first initial conditions gives u at x, 0 equal to f x, okay this implies this is c 1 x plus c 2 x equal to f x, okay. Next one is u t at x, 0 equal to g x this will give me minus c times c 1 x minus c 2 of x c 2 dash, okay equal to g x this is what we have seen earlier if you apply this one you can get this, okay.

Now you take the difference now if you simply now this one you simply integrated so if we integrate this one you will see that c 1 dash or you can write c 2 dash c 2 of x minus c 1 of x equal to 1 by c g of x this you are integrating from g of x is as ds this is from x not to x, okay if you do this this difference minus this is going to be plus so you have c 2 at x not minus c 1 at x not this is an arbitrary constant, okay call this K some arbitrary constant K.

So you have this implies c 2 of x minus c 1 of x is nothing but 1 by c this c is this c is actually small c that is well a speed of the wave so this is from x not to x g of s ds plus K, K is an arbitrary constant K is arbitrary constant because you do not know c 1 and c 2. So c 1, c 2 are arbitrary functions fix those function value at some point it should be arbitrary constant. So now considering this one and this one these two equations you can add or subtract to see that c 1 if you add it if you add you will get c 2 of x equal to 1 by 2 f x plus 1 by 2c x not to x g of s ds plus K by 2 so this is what you get if you add.

And if you take the difference you will get c 1 of x two times c 1 of x equal to again later on you divide with half so you have a half f x. Now you are taking the difference, okay this equation first this equation minus this equation so we have two c 1 so we have here minus minus 1 by c so after division 2c integral x not to x g of s ds and you have a minus K by 2 so these are the two equations you will get, okay.

But I need so this is c 1 x and c 2 x, so the general solution is here you want c 1 x minus c t so you need to find c 1 even for the negative values because x minus c t can be negative, okay when x can be x minus c t is negative when x is less than c t, x is positive we know so x is so if you want u of x, t u of x, t for x equal to something you need so for x positive u can clearly see that the solution is you need c 1 the negative values, okay.

So what we do is we apply so this is what if you want this negative values for this function that will give you this that can be got and from this boundary condition, okay. So I start I use this one so I use this boundary condition to get this negative values whatever you required for the solution. So let us see how we do this u at 0, t equal to 0, okay so this implies now the general solution is this from that you can see that c 1 minus c t plus c 2 c t equal to 0. So this implies c 1 of minus ct equal to minus c 2 of ct, okay so you want basically in your solution u of x, t c 1 of x minus c t.

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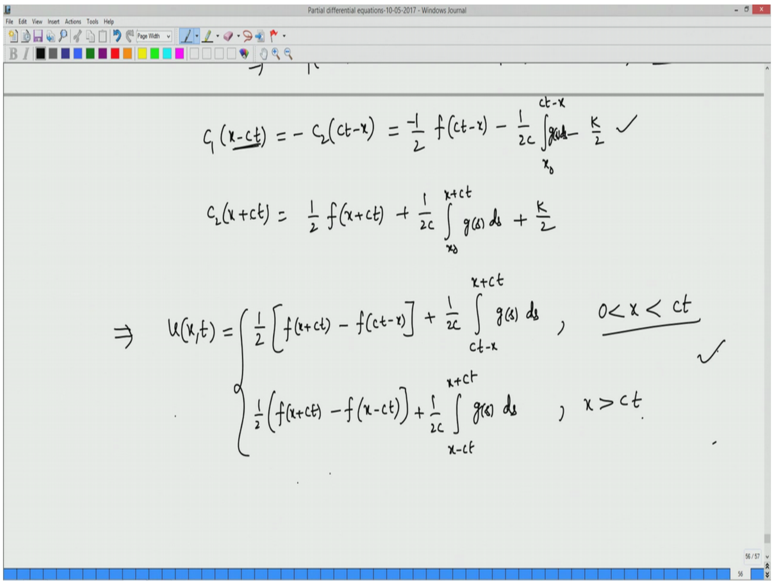

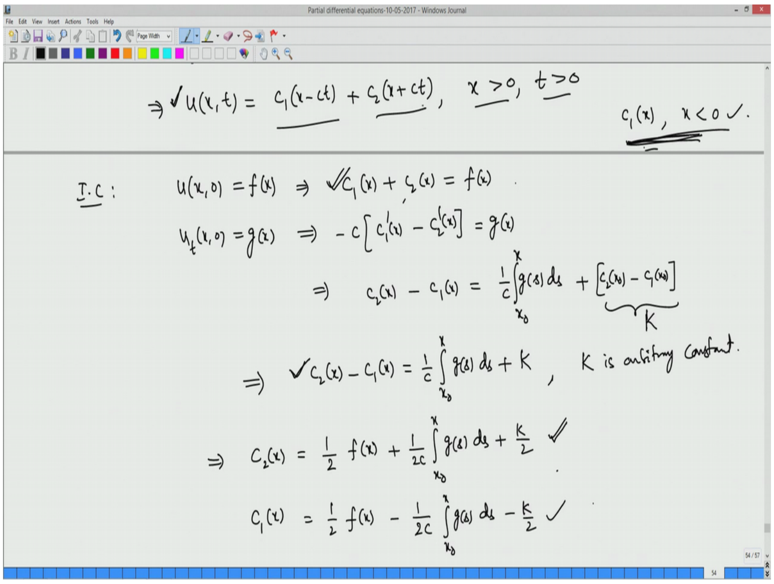

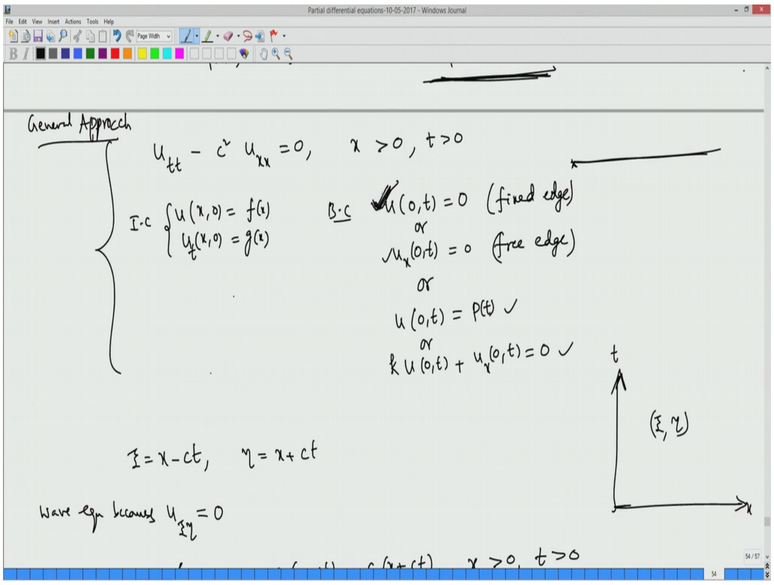

So we just write c 1 of so this implies this is like c 1 of Xi if you call Xi as not Xi let us say some other variable, okay some l c 1 l, l is minus c t so that is nothing but c 2 of minus l. So in the place of l what you actually require is for the general solution c 1 of x minus ct. So the place of l if you put x minus ct is equal to minus c 2 of minus x minus ct so that is ct minus x, okay that we know from here so that we can write it now that is half times minus in fact minus half f of ct minus x minus 1 by 2c x not to ct minus x wherever x is there you replace ct minus x, okay and then minus K by 2 that is my c 1 of x (())(22:05).

And you know what is your c 2 of (x minus) x plus ct that is what is required in your general solution so c 2 of x t you can get it from here directly so that is half f of x plus ct plus 1 by 2c x not to x plus ct g of s ds here g of s ds, okay and then plus K by 2 this is directly from the same c 2. So this implies what is my u x, t u x, t is this one plus this one so that will give you so half f of x plus ct minus f of ct minus x, okay plus 1 by 2c this plus this so this is from ct minus x to ct plus x that is x plus ct g of s ds so this K by 2 that K by 2 cancels.

This is when when (())(23:35) use this this is when I have c 1 is negative c 1 for negative values c 1 this is for l negative values, okay. So when I have so when I will have x minus ct is negative that is 0 less than x less than ct. So in this case x minus ct is negative that is this argument is negative that is my l is negative, so when my l is negative I use minus c 2 of minus l that is what you have, okay.

So this whatever you have written I use c 1 function for the ordinate for the negative values. So that is that is only if x is between this one and for x greater than ct you do not have to worry about anything you can directly write when x is when x is greater than ct both are positive so you know c 1, c 2 both are positive so you can directly get from here, okay c 1, c 2 directly substitute that is same as half f of x plus ct minus f of x minus ct plus 1 by 2c x minus ct to x plus ct g of s ds this is exactly what we derived earlier, okay.

This is the solution for this initial boundary value problem that we are talking about, okay the new general approach this is general approach to solve the wave equation on the semi-infinite domain 0 to infinity, okay and you have this this is the boundary condition we use fixed edge for the fixed edge this is what we found, okay. So far if you want this to be this is what you have this is how you derived but the thing is if you want u to be solution f has to be twice differentiable function and g has to be once it should be differentiable and continuous function because u is u xx this g is then only you get g s, okay if you differentiate u twice with respect to x then you have to differentiate this g with respect to only one differentiate because of this integration, okay.

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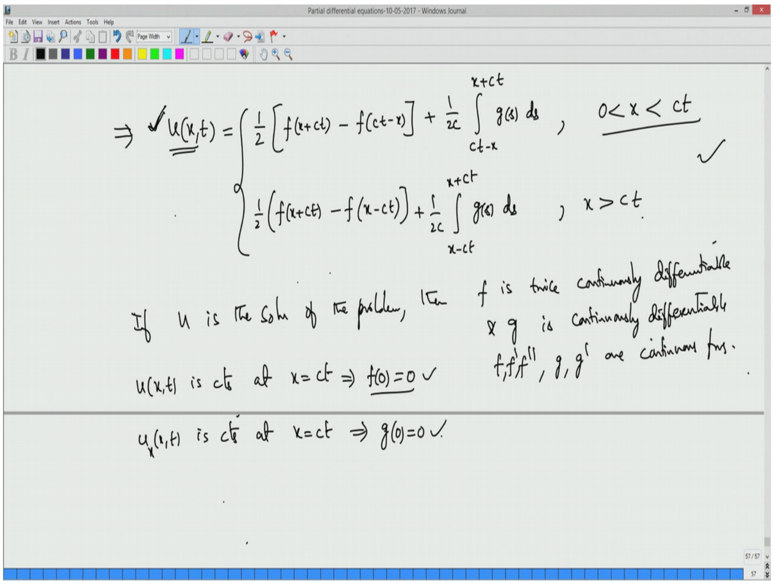

So you have if you want this to be a solution of the wave equation f should be twice differentiable if write you can write it if u is the solution above okay if u is the solution of if u is the solution because we have seen that this is the only solution it has only one solution u is the solution of problem, then f must be f f is a twice continuously differentiable and g is continuously differentiable function that means f, f double dash and g, g dash, f, f dash, g dash or continuous it should exists or continuous functions that is what it means.

And also again they have to be that and if you make use of this continuity if they are continuous you make use of this u is continuous u of x, t is continuous at x equal to ct, okay if you use this you will see that f of 0 has to be 0 you will get and if u x or u t of x, t is continuous at x equal to (t) ct will if you actually use this use this expression and differentiate and make it equal at x equal to ct you will see that g of 0 equal to 0.

So these are necessarily you have to satisfy, the initial data has to satisfy this is exactly what we have solved earlier by considering by taking an odd extension immediately you see that f of 0 is 0, g of 0 is 0, okay. The way you have extended odd extension and your initial data has to be twice differentiable because it has to satisfy the wave equation.

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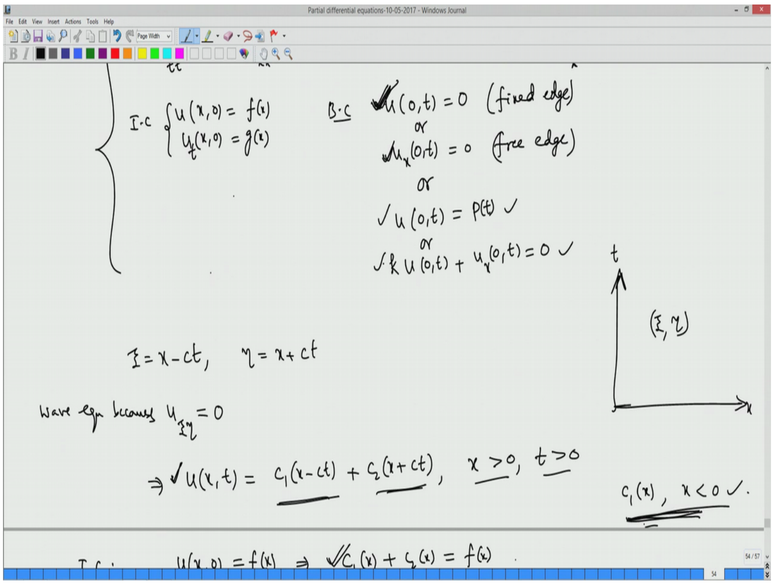

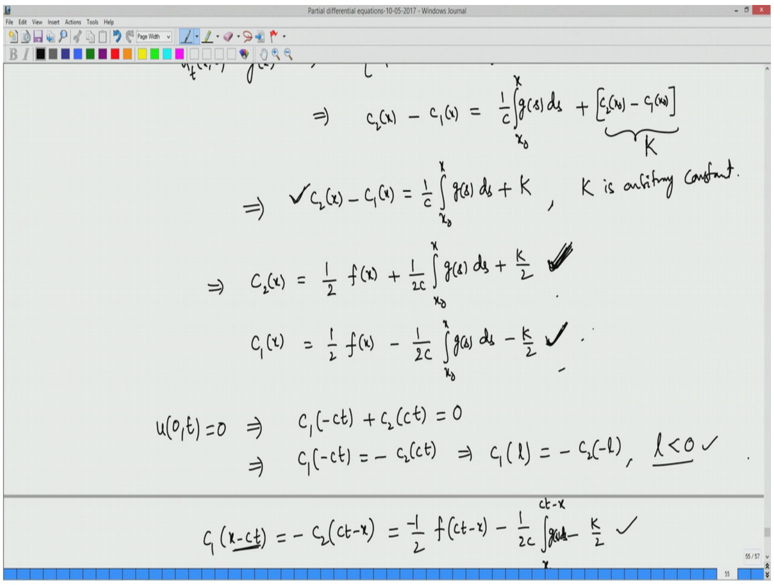

So that is how this is like going in a reverse direction but it works is more general because the same approach if you apply for the other boundary condition so it will work for any any boundary condition so the approach is this, you start with general solution of the wave equation, you take this initial data, you have these two equations, okay only thing is when x is because x is positive you when x is positive x minus ct can be negative, so for those values that is between 0 to x is between x is between 0 to x ct this this can be negative.

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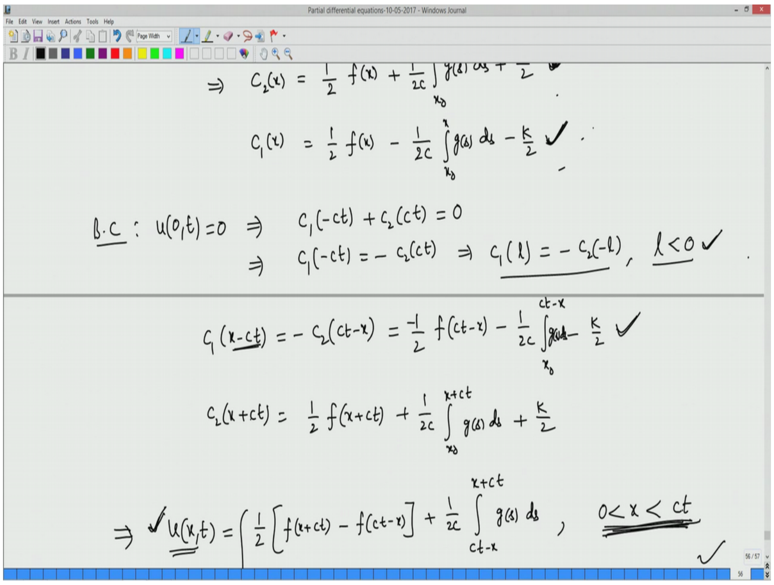

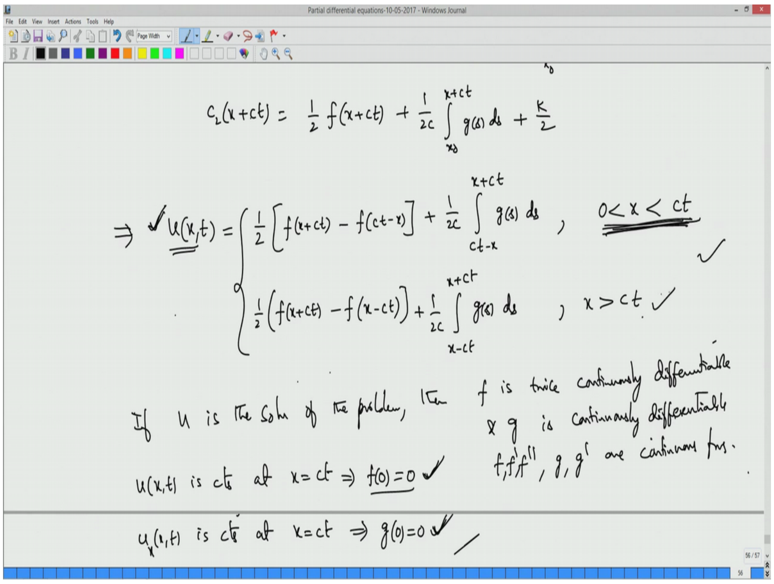

So those negative values that is given by the boundary condition the boundary condition will give you the that c l, l is negative, okay you apply the boundary condition that is actually giving you this negative values of c 1. So that I make use and then finally add it up when x is between 0 to ct and when x is greater than ct you do not have any problem when x is greater than ct both are positive and you already has you do not has you do not need boundary condition, okay so you simply use the initial data this is what you get and you simply add it you will get it that is the solution you get, okay for x greater than ct.

Now if we want this to be this is how you constructed if we want this to be a solution if you want this to be a solution of your wave equation and f has to be twice differentiable and g has to be one time differentiable function and it has to be continuous. So by using continuity of u at x equal to ct we will see that f of 0 is 0, u x or u t is continuous at x equal to 0, x equal to ct will give you g of 0 is 0. So the initial data has to be just like odd extension when you do the extension you will see that f of 0 is 0 and g of 0 is also 0, okay.

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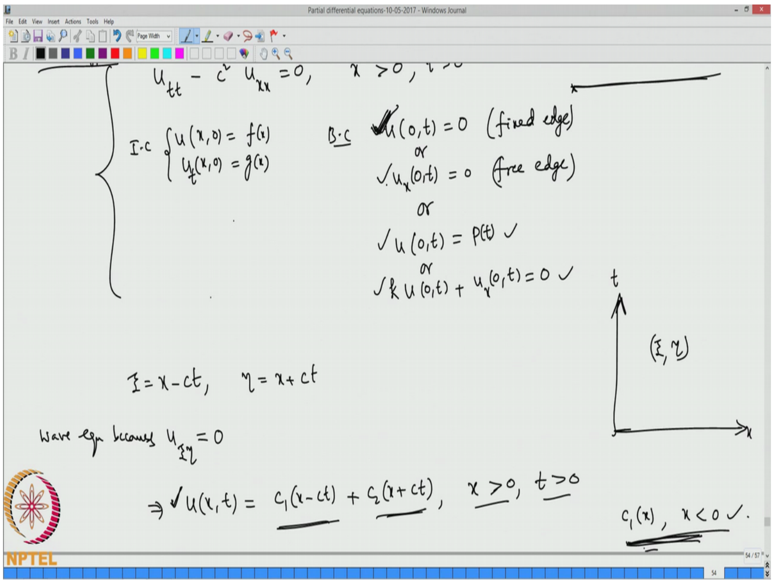

So what we do is in the next video we will try to do the same technique this more general approach for the other boundary condition that is free edge that is u x equal to 0, if we use this u x equal to 0 how we can do the same thing same technique and that we will see in the next video, for the free edge u x equal to 0 as when you consider this as the boundary condition for the wave equation on a semi-infinite domain 0 to infinity we will try to give you the same approach that we have seen just now we have seen just now this is the more general approach. So what we do is we will try to see this solution in this approach in the next video, thank you very much.