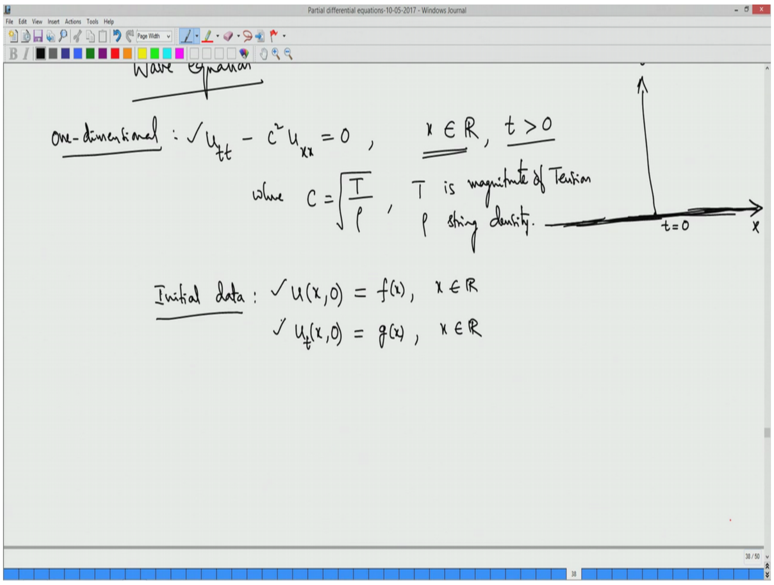
Differential Equations For Engineers
Professor Dr.Srinivasa Rao Manam
Department of Mathematics
Indian Institute of Technology, Madras, Chennai
Lecture 45
D'Alembert solution for wave equation

Welcome back, in the last few videos we have seen how to reduce second order linear partial differential equation into a canonical form that is one of these three typical equations one is wave equation other one heat equation and Laplace equation so these are the three types which is hyperbolic, parabolic and elliptic type of equations.

So today we will just this video will have initial or boundary value problems, initial and boundary value problems for wave equation to start with so will start with the wave equation and we give the (bound) we consider the domains in certain domains will try to solve the wave equation and then if it required we have a boundary, you provide a boundary data and if its and we, and you also provide initial data and will see the methods how to solve this kind of problems ok.

(Refer Slide Time: 01:15)



So to start with a wave equation so we consider wave equation so you can see the wave equation is we have already solved how to solve it you have seen U T T minus C square U X X equal to zero, this is specially you have one dimension, so you have X is in a special variable that is full domain I think it is given so that means is a full domain and T is the time that is T is positive so you have two variables X and T, X is a special variable and T is a time variable. So because T is time T is always you take it as positive.

So you have some kind of domain is this is T and this is X so this is one dimension because the special variable is one dimension so this is the one dimensional evolution equation so wave equation ok, evolution means as the time goes you see the evolution of the displacement of, displacement I mean the dependent variable U of X T ok that depends on the time as time goes you are trying to find what is the value ok, what is the displacement of this infinite string.

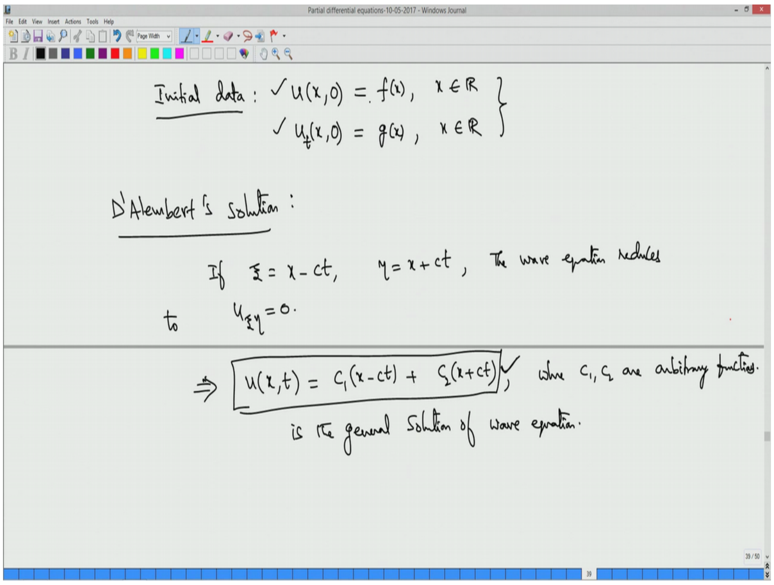
So this is also this is the model for an infinite string having C is, C if you take it as T is tension of the string and rho is the density of the string if you consider this uniform density of the string T by rho is actually C, so T by rho actually you should say root of this is actually your C. So C square is actually where T is tension, string tension a tension magnitude of tension magnitude of string tension ok so you can say simply tension and rho is a density, string density. So uniform is you have infinite string you can model with model that infinite string with this wave equation that is the one dimensional wave equation.

So we can write one dimension, so actually one dimensional wave equation so because special variable is only one dimension. So this is your equation you have your domain. Now what is the boundary? You see that you have infinite in unbounded domain that is X is it infinity but see that the T equal to zero you should provide that is actually kind of boundary if you see as a full domain as the X T domain, but physically this is initial values, initial data and you giving a T equal to zero that is initial data ok.

You don't call it boundary data because this is not corresponding to the special variable X. So you have initial data at T equal to zero so initial data is U X 0 is given as so that means you provide a time T equal to zero what is the displacement of your string that you can take it as some given function F X for every X belongs to R and also because it is a second order you have a T derivative is in the second order you should provide two conditions for the initial data so you provide the velocity of this string at T equal to zero, X0 so if you see this at T equal to zero so it is like D U by D T that is why I am calling velocity at T equal to zero so that is you call this some G of X given function at every point of the (spri) string, infinite string.

So basically you start with this domain X is in full or full X takes in all the real values ok X is full real axis if this is your domain we have to solve this equation with this initial data. So will try to solve this, this is called the D'Alembert solution.

(Refer Slide Time: 05:41)



So D'Alembert's solution as given the solution so named after D'Alembert's solution is, how do we do this? How to a first of all you have to solve just like ordinary differential equation this is like a boundary data so this is some given data at the boundary at T equal to zero, this is the only boundary you have.

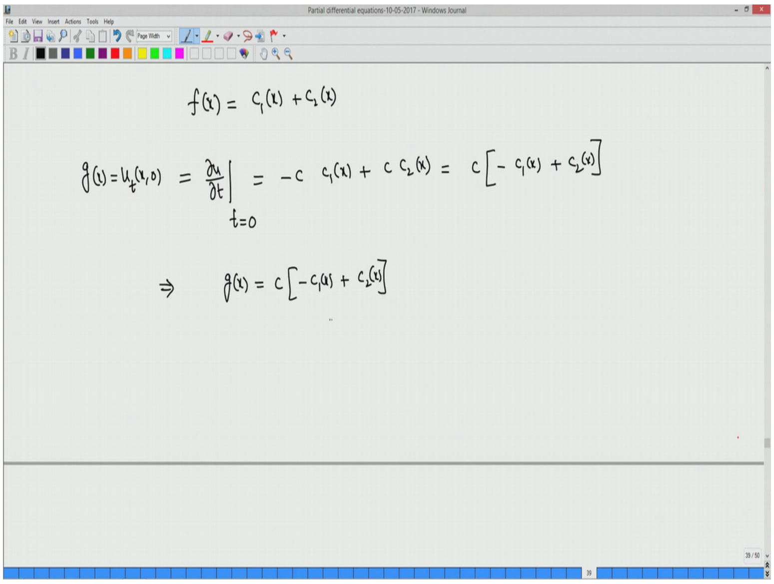
So first you have to find the general solution of this equation and then you apply this boundary conditions get those arbitrary functions just fixed it based on the initial data that is how you get the solution of this problem. So this is the initial value problem so what we do is we have already seen how to reduce this into the canonical form U T T minus C square U X X equal to zero.

This equation if I choose Xi as a X minus C T theta as X plus C T we have seen that what you get is U Xi eta equal to zero ok the equation reduce as the given PD wave equation, wave equation reduces to U Xi eta equal to zero this is what we have seen and immaculately if you try to solve this equation so you have seen that U Xi eta if you simply integrate both sides that we have seen in the last video you can see that C1 of Xi that is X minus C T plus C2 of X plus C T, so C1 C2 are where C1 C2 are arbitrary functions.

So once you have this, this is your general solution of wave equation that is what we have seen ok, this is the general solution of wave equation. Now we simply we try to apply this boundary conditions ok, so what is your boundary conditions? U X 0 is F X so U X 0 so you try to, once you get a general solution in the new variables it is X T this is what it is, if you write U of Xi eta that is C1 Xi plus C2 eta ok and once you bring it to the (new) old variables U of X T this is your general solution of the given wave equation.

Now you apply this initial condition U at X 0 is equal to F X.

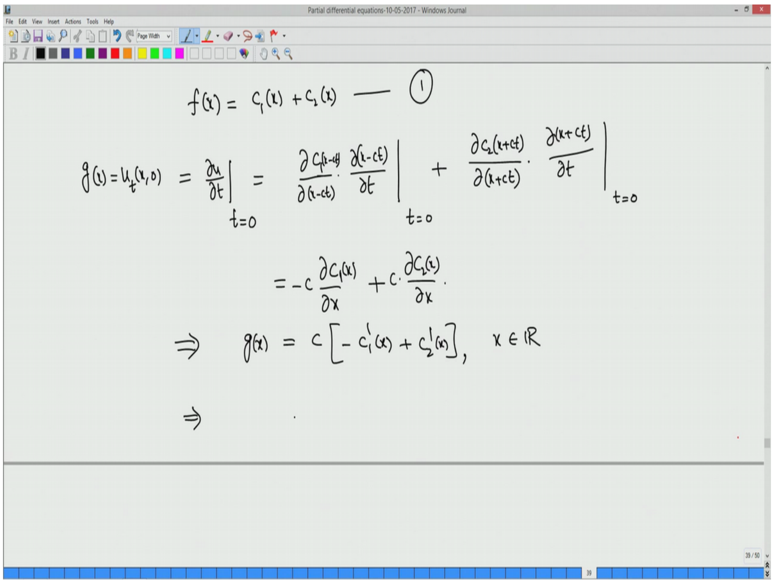
(Refer Slide Time: 08:35)



So F X equal to U at X equal to zero that is C1 X plus C2 X what happens to U T at X equal to zero, so what is this U T at X equal to zero? That is dou U by dou T at X equal to zero at T equal to zero ok, so what is this one? Dou U by dou T at T equal to zero is minus C, C1 of X minus C T at T equal to zero so C1 X plus C, C2 times X ok so this is actually equal to C is common you have minus C1 X plus C2 X, so what is this one? This is actually given as U T at X0 this is my G X that is what is given ok.

This is what is your (())(09:42) your second initial condition this is true for every X so this implies G of X equal to C times minus C1 X plus C2 X from this you can actually, sorry so this is not the way so made a mistake so, how do I get this dou U by dou T so this is C1 dash dou U by dou T is basically (see) how do calculate dou U by dou T?

(Refer Slide Time: 10:23)



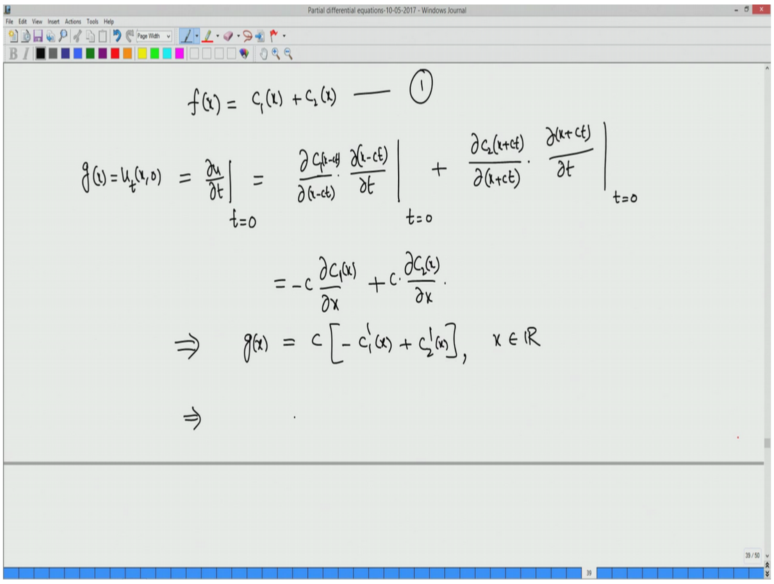
This is like dou U by dou X minus C T into dou X minus C T divide by dou T this whole thing now we put at T equal to zero plus dou uhh this U is C1 first C1, C1 function C1 of X minus C T ok.

And similarly dou C2 of X plus C T divide by dou X minus C T X plus C T into dou X plus C T divide by dou T this whole thing put T equal to zero that is what it is, so this is equal to now you put T equal to zero this is simply C1 is actually what you see is dou C1 by dou X and you put T equal to zero ok that is simply now C1 is a function of X so what is this, this is simply C1 dash of X and what you get here is minus C when you put T equal to zero that is simply minus C times one ok and you put T equal to zero that is what you will get, ok.

So this one and plus dou C2 of X divide by dou X into C and if this one when you put T equal to zero you get C so this together will give me now this you can write now minus C1 dash X plus C2 dash of X. So this is what is my left hand side is G X so this is my second equation, so this is your first equation this is your second equation. So second equation I don't call it second equation you do one more step now you try to differentiate this not differentiation so you try to integrate this you try to integrate this equation ok.

You try to integrate this is the function G of X, X belongs to full R so you choose some so have initial value problem T equal to zero so anyway so you can just integrate from any fix point X knot in your X domain, X domain we is this real line ok you can choose any point X knot to X, X is arbitrary value.

(Refer Slide Time: 13:02)



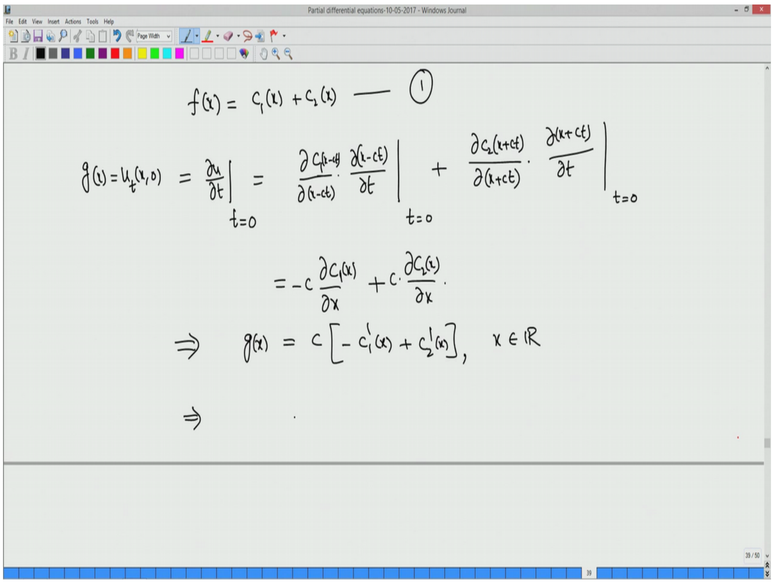
So if you integrate from any X knot to X G of T I am calling it GX DX but I am calling it GS DS ok, is just a dummy variable, equal to C times minus integral X knot to X C1 dash of S DS plus integral X knot to X C2 dash of S.

So this is nothing but C times ok so you can write here so say that C times 1 by C so bring this C 1 by C so you have integral X knot to X, G of S DS equal to so this minus so if you, this is the derivative so it will become C1 of X minus here the upper term will give you C2 of X now here bottom you see that it will be C1 at X knot and here minus C2 at X knot this is what you will get. So this is what is your recreation so right. So this gives me an equation in C1 and C2 that is minus C1 X plus C2 X equal to 1 by C integral X knot to X, G of S DS plus C2 X knot minus C1 of X knot.

So bring this to the other side so that this is what you see, this is your equation number two so from one and two C1 plus C2 equal to F minus C1 plus C2 is this one. So if you add 1 plus 2 gives what you got if you do 2 times C2 X equal to what if you add right side so you have F X plus this part F X plus 1 by C integral X knot to X G of S DS plus 1 by C2 of at X knot minus C1 at X knot. So now you divide both sides 1 by 2, 1 by 2 and 1 by 2 ok, now you take the difference 1 minus 2 gives you get C1 of X 2 times C1 of X actually equal to F X minus 1 by C integral X knot to X G of S DS minus C2 at X knot minus C1 of at X knot this is what you get.

If you simply 1 minus 2, 1 minus 2 is C1 plus C2 minus-minus plus this becomes 2 C1 so F X minus right hand side F X minus this part, so this is what you will get.

(Refer Slide Time: 16:26)



So now again you divide with this side so you have this one, you have this one ok. Now, now we see that C2 X C1 X is this one so what I need what I have the general solution as C1 at X minus C T C2 at X plus C T so try to write what is your U of X T directly.

U of X T is C1 at X minus C T plus C2 at X plus C T, so what is C1 at X (plus) X minus C T, you put it here in the place of X you put X minus C T so what you get is F at X minus C T plus 1 by 2C X knot to X minus C T GS DS plus half this is a constant so that will not change ok this one, a half C2 at X knot minus C1 at X knot. Now what is the next one, so plus half F at X plus C T which I am writing C2 ok, C2 at X plus C T oh C1 is this so C1 should be minus so C1 if you write this is minus and this is going to be minus.

So C1 I wrote now I am writing plus C2, C2 is half X plus C T plus 1 by 2C X knot to X plus C T GS DS and this is plus half this is a numeric constant C2 X X knot minus C1 at X knot. So this gets cancel what you see is simply you combine these two terms this one and this one if you combine you see that F at X minus C T plus F a t X plus C T half of the and then this one.

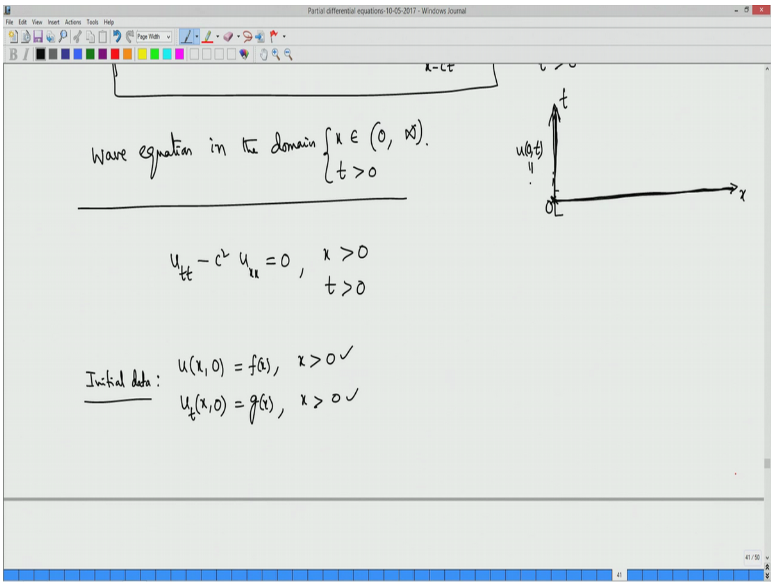
This one you can write X minus C T to X knot ok, X minus C T to X knot if you do so these things these you can write 1 by so 1 by 2C is common what you have here is first term this term you try to write is (goin) is actually minus this going to be plus when you write X minus C T to X knot GS DS and this one you have from X knot to X plus C T GD DS.

So if you combine it now we can just combine it and write this as X plus C T ok so this is what you have so this is what is my U of X T so this is the solution of the initial value problem that I define ok, so this is your solution you have this works for every X in the real line and for every T positive this is D'Alembert solution of the wave equation with the initial data. So you once you have given if you are given initial data that is F X and G X on an infinite string initial displacement and its velocity of the string all along the infinite string if you know it that it initial time that is the T equal to zero you can actually find for all times what is the displacement of this string.

For all times you can get it from this formula so that is what is the D'Alembert solution. This is how you can, what you have is if your domain is infinite one dimension wave equation, wave equation is actually second order linear partial differential equation in two variables so that is T N time and special variable X but special variable is only one dimension so that is why you call one dimensional wave equation.

So string can be zero to infinity semi-infinite also it can be so you may have one side zero other side is infinite if you have (infinite) semi-infinite string ok.

(Refer Slide Time: 20:53)



So wave equation in the domain X belongs to zero to infinity ok T is positive if in this domain if you want to work so again you have the wave equation you have to write so it is U T T minus C square U X X is zero X is in now X is simply positive and T is positive so this is the semi-infinite string ok.

Now what is the, this is the equation this is the wave equation one dimensional wave equation. Domain is again you have a T so you have a time domain so initial data you can provide, what is your initial data? U at X 0 equal to F X this is what is your X axis only from

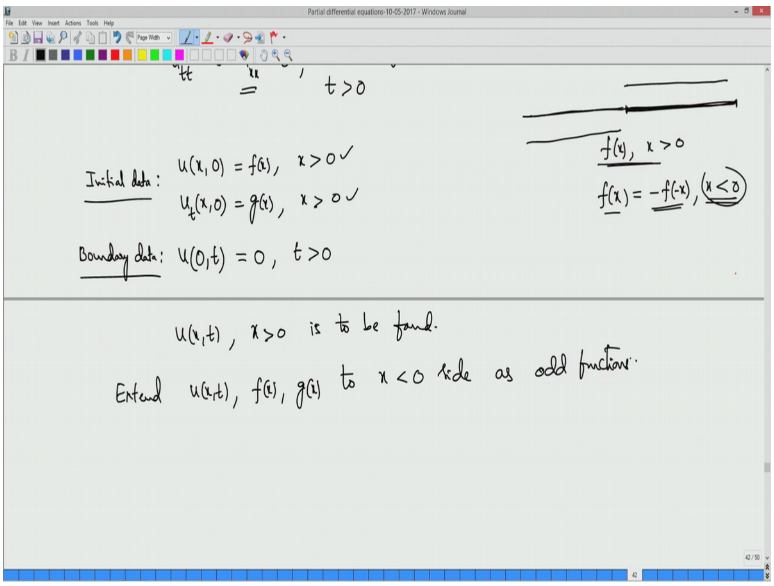
X positive so this is given only for X positive and it is a (())(21:59) U T at X equal to zero is G X , so what is the domain? Domain is zero to X that is the boundary this is the boundary and also at X equal to 0 this is all along the T, this is also boundary ok.

So at T equal to zero this line this with this boundary is initial data you have to provide physically here at X equal to 0 for all times you have to provide a boundary data ok because this is known this is at the boundary this is the boundary point is only zero ok, so the boundary point is zero and there you have other boundary special boundary at infinity so you don't have boundary there so the only boundary is at X equal to zero.

So this is the initial data, initial data is again as usual you have two initial data you have to provide because the second order ok. Now for the because you have the boundary here ok you have only one boundary, boundary data you have to provide special data means special variable you have to provide you don't have to provide just because you have two second order you need not provide two values at X equal to zero ok.

So just for the boundary data that is why for the special variable that is actually physical boundary and that boundary you provide a data you should, the displacement should be known ok. So what is that boundary is at X equal to zero for all times T so on this boundary you provide what is your U at X 0 T so this should be known.

(Refer Slide Time 23:43)



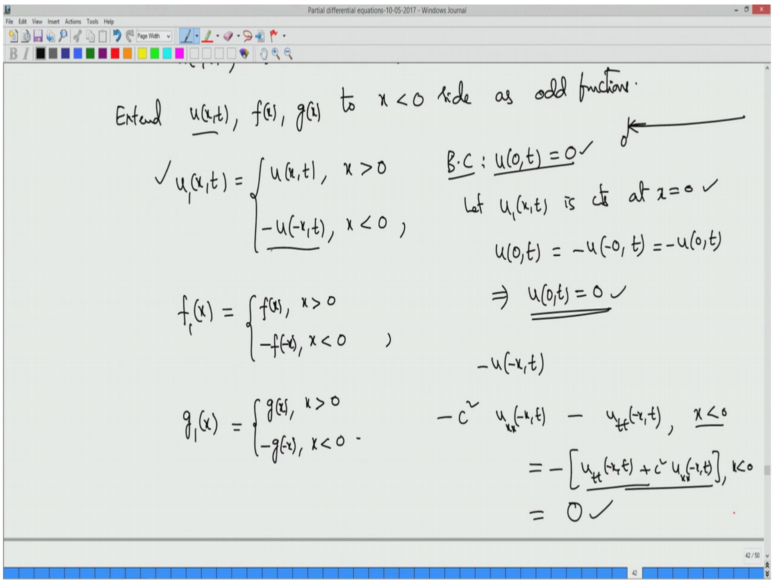
So if you provide this U at X 0 T either this or you maybe you may want the flux that is U what is the derivative of with respect to X at 0 T either this or this, you should provide ok.

The second order or the combination of this you can provide so either U at zero U at X equal to zero or U X dou U by dou X at X equal to zero either these two or combination that is how you can provide so you can see first start with this let say U is given at X equal to zero ok, if U is zero suppose you fix it that means displacement you fix one end of the string you fix is displacement zero then for all times ok. So this is boundary data so are the boundary condition so these are the boundary condition together this equation along with these initial and boundary data forms a boundary initial and boundary value problem for the wave equation in the domain X positive.

So how do we solve this? by looking at this only extra thing is is boundary data ok so if you actually see when do you if you, you have for X positive you have you want to find for X positive you try to extend this , you try to extend this string into you extend as a function U of X T, X positive is to be found ok, what I do is I try to extend this functions ok all the initial data FX GX and this U I try to extend ok I try to extend in such a way that it is an odd function, if it is an odd function for example you take some function FX which is defined X positive side F at negative side I define it FX is minus F of minus X for X negative side.

So that means for negative side I define it as odd function ok I extend it this function as odd function I extend it F is known only here now, now using the data here I extend it to the negative side that means for X negative I simply take minus of F of minus X because X is negative so it is positive that I know here I know, so F at negative side is defined based on the data here if you have a data here like this minus of this ok how do we do this? how do I get so let us say simply so simply constant function let us say FX equal to one minus, minus one is the odd function. So something like this you extend it ok, so extend U S T, FX GX as odd function extend to X negative side as (an) as odd functions ok.

(Refer Slide Time: 27:22)



How do I do this? so that means U at X T equal to U of X T if X is let us call this U ok extension function is U 1 now, U1 is U of X T for X positive ok for all times nothing is T means for all times this is true ok and now I use minus U of minus X T for X negative side, this is how I extend it, this is how I extend my function U of X T ok as an extension so similarly you get U of F1 of X as F X for X positive minus F of minus X for X negative ok.

Similarly this one, this one and G1 F X these are my extended functions that is G X for X positive minus G of minus X as X negative, what is the advantage of doing this? By doing this you can see that now try to use U at 0 T equal to 0 ok now if this this is zero (you) if you want this function to be continuous function U1 of X T, U1 of X T you want to be continuous function so U1 of X at 0 T, zero positive ok positive side you take that is what is this right, this is a U at 0 T means you know only positive side this is zero.

So at this limit U of X T as X goes to zero plus that is zero ok, so this is same as U1 of X T equal to zero ok and if you want this U1 to be continuous this has to be same as U1 at zero minus T ok. So what is this U1 at 0 minus T that is no, not like this so let us say, so you want to make U1 is U1 of X T is continuous at X equal to zero ok if you want U1 is continuous and is differentiable you want you extend it in such a way that is it is smooth ok start like the example like shown. You don't take this continuous function so that see this it is discontinuous here.

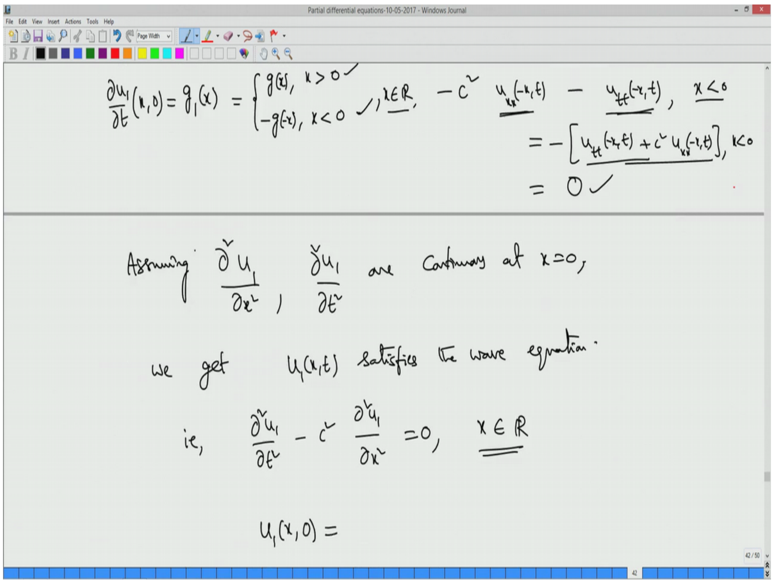
You extend it in such a way that it is continuous like this ok it is also differentiable because it has to be if you want to make use what you do, what you are trying to do is idea is to extend from extend this domain from zero to infinity to minus infinity to infinity so that the extended functions will also satisfy wave equation. So that means it should be differentiable so you have to extend in such a way that they are differentiable. So simple continuity and differentiability. So you want this to be continuous if you want simply to be continuous then what has to be done this U at 0 T should be same as minus U at minus 0 T ok.

That means this is same as minus U at 0 T that means simply U of 0 T equal to 0 this is naturally satisfied already this is the boundary condition is given ok so this is naturally satisfied if you do this boundary condition is already satisfied ok if you choose this extension this odd extension of your unknown function is satisfied so and you see that U is actually for X positive it satisfies the wave equation if you do even here for minus U, minus U minus X T if you take two derivatives is again U X X of minus X T and minus and then what its derivatives of T is minus U T T minus X T ok.

They already satisfy they satisfy this is different for X positive right X negative. So they together if you put C square minus C square and then plus this and minus this is nothing but minus of U T T minus X T plus C square U X X minus X T ok and this is X negative side so you see that this are all well defined, this is U T T this is a wave equation for X U wave equation for U satisfied by U in a positive domain by, X is negative that means minus X is positive so this is zero.

So U this quantity for X negative is also satisfying wave equation ok. Now you assume that they are smooth enough U X X, U T T they are all continuous at X equal to zero then U1 is also continuous (func) U1 is also satisfy wave equation.

(Refer Slide Time: 32:38)

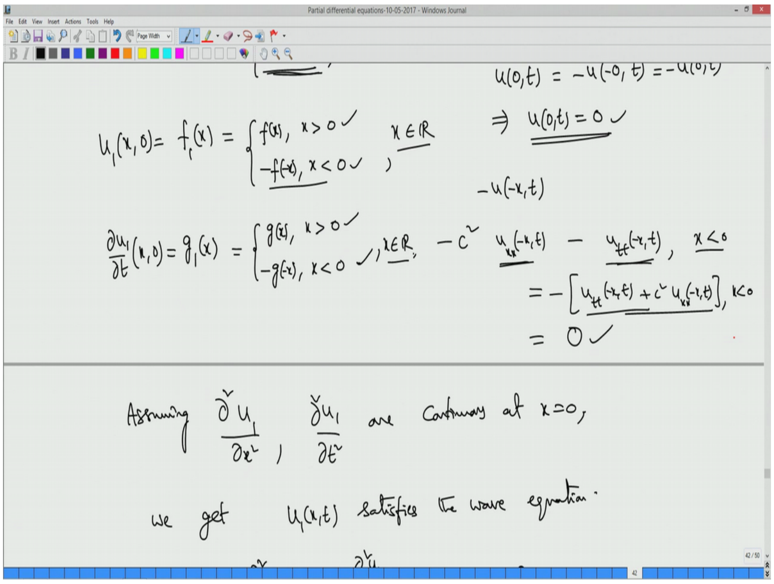


So assuming U X X and U T T are continuous sorry U1 dou square U1 by dou X square and dou square U1 by dou T square are continuous at X equal to 0 we get what we see is by assuming that they are continuous what we are seeing is there U1 of , U1 of X T satisfies the wave equation.

That is dou square U1 by dou T square minus C square dou square U1 by dou X square equal to 0, now U1 is define for every value even for negative side so this is defined for every value of R ok. They are continuous so U1 is now because they are continuous not only they are differentiable (these) derivatives are also continuous so it is U is itself is continuous so that means boundary condition is automatically satisfied. So you have if you extend this your U F and G as an odd function this boundary condition is automatically satisfied, then U1 is automatically satisfy the wave equation.

Now what happens to your U1 at X equal to X T equal to 0? U1 at T equal to 0 is, what happens? U1 at T equal to 0 is, U is nothing but U1 at X 0 that is U at X zero that is F X minus U at minus X 0 ok for X negative. So that is nothing but F of minus X ok.

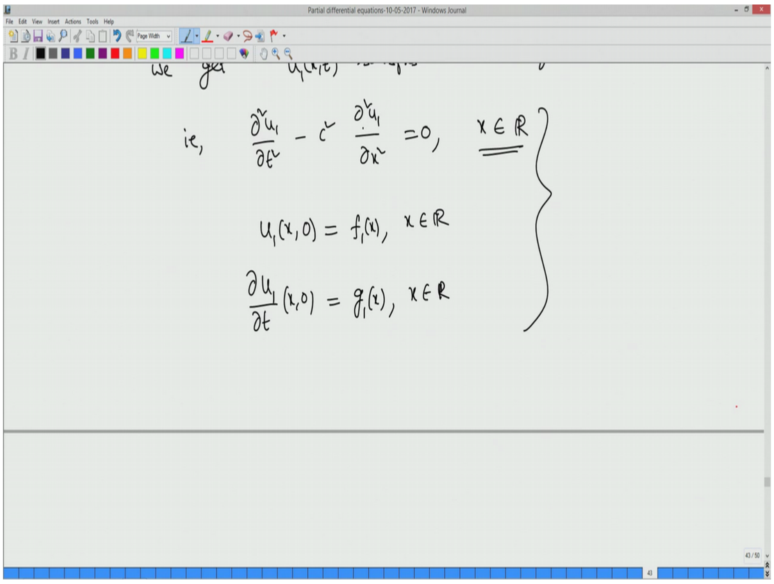
(Refer Slide Time: 34:44)



This is exactly U1 at X 0 if you see this U1 at X 0 is U at X 0 is F X ok minus U at minus X 0 that is X negative side means is actually nothing but minus F minus X, X negative so exactly both are same ok and similarly this is dou U1 by dou T at X 0 is G1 X.

So what is dou U1 by so the time derivative of U1 is U T at X 0 that is G X, X positive minus U T minus X T at T equal to 0 is minus G of minus X so this is how it is what it is. You have a boundary data for every real value now we have the data so you can write here.

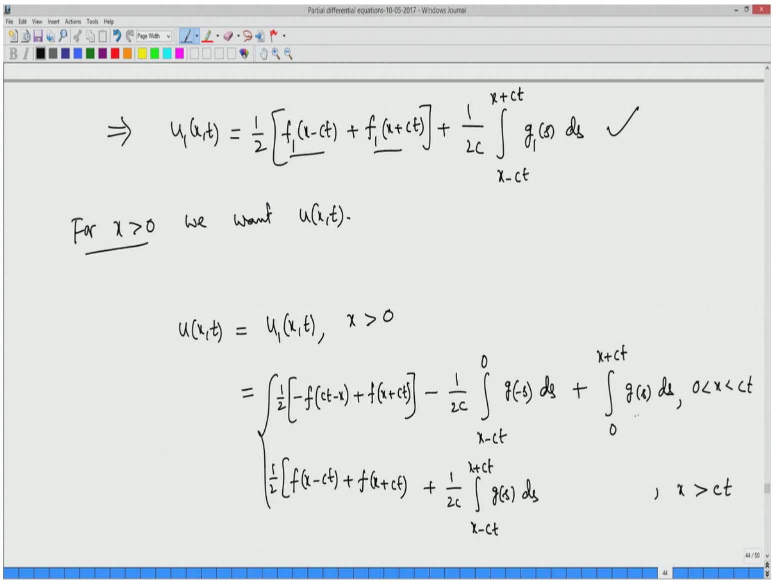
(Refer Slide Time 35:48)



So U1 at X0 is now what is that function? That is F X, F1 and G1, F1 of X now this is for every real line and similarly dou U1 by dou T at X0, this is now G1 of X that is for every real value.

So now this is like earlier problem boundary condition is automatically satisfied ok, this is automatically satisfied making use of this initial data we simply extended this functions as an odd function so that the boundary condition is automatically satisfied. Now what you have is simply wave equation and the full real line that is whose solution we already know ok that is D'Alembert solution.

(Refer Slide Time: 36:34)



So this implies U1 of X T is nothing but half of this data F1 at X minus C T plus F1 X plus C T plus 1 by 2C integral X minus C T to X plus C T G1 of S DS.

This is what is your solution U1 of X T but you want U of X T ok U of X T you want U at X T, X positive ok for X positive we want U of X T so what is my U of X T for X positive this is simply U1 at X positive ok U1 at X positive is U of X T is simply whenever you have X positive so you have two cases when X is positive this has to be you don't know about this one ok, what happens to this function?

So you want we want U1 is actually equal to U1 at X T for X positive ok, what is this one? This is actually equal to you have two cases, one is when X is 0 and less than C T that case X minus C T is negative ok you can see that when zero is less than X is between zero to C T F1 that is my U1 of X T ok F1 is a negative so X minus C T this coordinate of this function F1 is negative so if this negative what is when X , X minus C T is negative what is that minus it is actually minus half rather you write half times minus F of minus X minus C T that is C T minus X ok plus when X is between zero to C T X plus C T is always positive so you have simply write F of X plus C T.

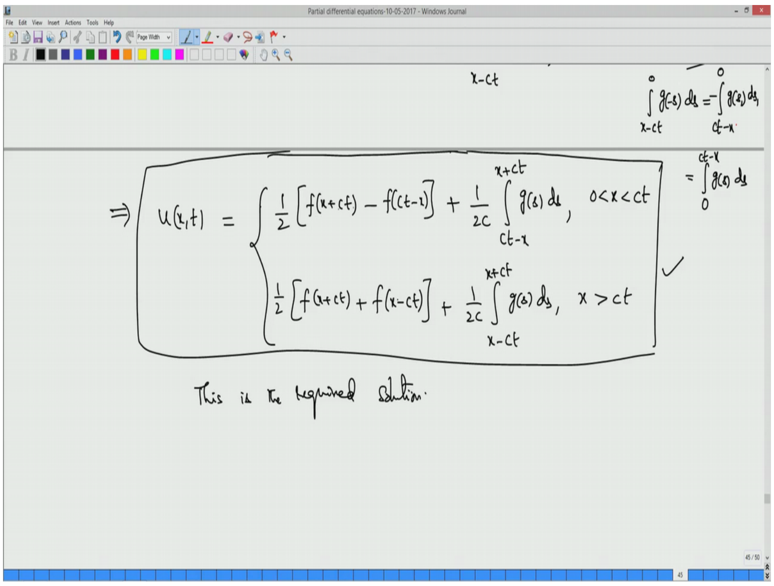
This I am writing from this using this one this function F X is X positive if it is negative you have to write minus F of minus X that is your F1 X so if you do this this is what I wrote for U1 F1 I wrote F1 as this one and now for F1 X plus C T is this term. Now the remaining is plus 1 by 2C this one X minus C T you split this into X minus C T to zero G1 of S DS plus zero to X plus C T G1 of S DS ok this term I split it like this but I have to write in terms of G only not G1.

So when X is between zero to C T X minus C T is negative so clearly G1 is negative side so I have to replace this with minus G of minus S, ok because X minus C T is a negative, negative to zero so that is G is negative G is (define) ok so that means G1 is negative G, S is negative ok S is negative so it has to be G1 of S, S is negative is G of minus S, minus of G of minus S that I put it here. So this is simply G1 is G of S when X is X plus C T is zero to X plus C T, S is always positive.

Now what is the other term is half now what is left is you want X positive side so the X is greater than C T ok and X is greater than C T, F1 is always positive this is also always positive so you can simply write as usual F of X minus C T plus F of x plus C T and this also 1 by 2C integral X minus C T to X plus C T, G of S DS and so because you again like you split it like this G1 of S between zero to, between X minus C T to zero S is always positive that is why I am replacing G of S, G1 of S by G of S only.

Similarly zero to X plus C T, S is still positive so this is the case so combining it together so there is no negative sign so X is greater than C T this is what you get.

(Refer Slide Time: 41:50)



So if you combine it if you rewrite simply nicely so you have nice form here so this is half times F of X plus C T minus F of C T minus X. now this you can combine it (what you) how do you combine it? First you change the order of, first you change the variable minus S as S ok.

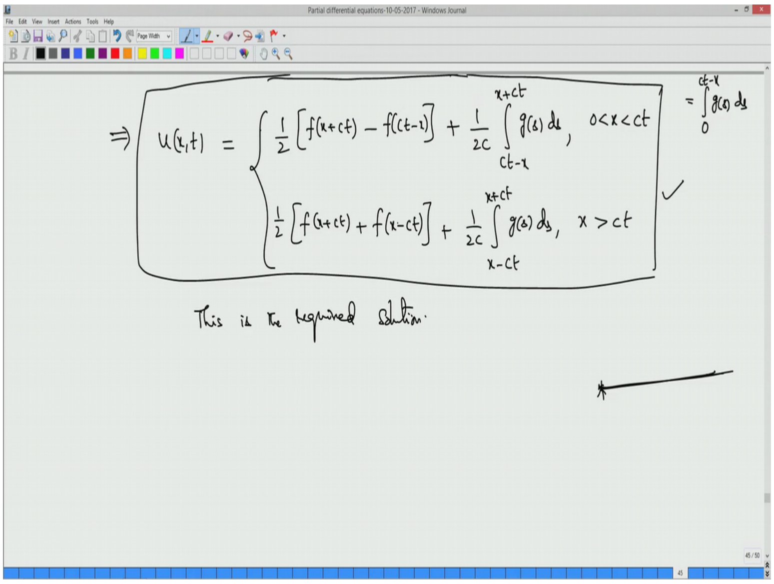
If you do that just replace so what you do is let S this X minus C T to zero G of minus S DS this will be same as C T minus X to zero G of S DS minus ok a simply change S by minus S ok if you do that G of S DS will be minus DS and how to replace this integrents ok so put S equal to T, minus T if you do so you have to write GT DT or if you call the S1 ok so T I already used so you have S1 DS1 so this is what you will get.

So now again you change this dummy variable (S) S1 with S what you have is simply zero to C T minus X ok, zero to C T minus X and zero to X plus C T, C T minus this is exactly equal to zero to C T minus X G of S DS ok. So the whole thing what you see is you have 2 by C, 0 to C2 X and 0 to so minus you have a minus so minus means this is going to be plus from C2 minus X to 0 G of S DS plus 0 to so you have 1 by 2 is C is common here so you can combine it so you have zero to X plus C T G of DS.

So this together you can write you don't have to so you can combine it and say X plus C T ok. So this is what you have for X less than C T otherwise half F of X plus C T plus F of X minus C T plus 1 by 2C X minus C T to X plus C T G of S DS for X greater than C T. So this is your general solution, not the general solution this is the solution of initial boundary value problem in the same infinite domain X positive ok. So this is the solution, this is the required solution.

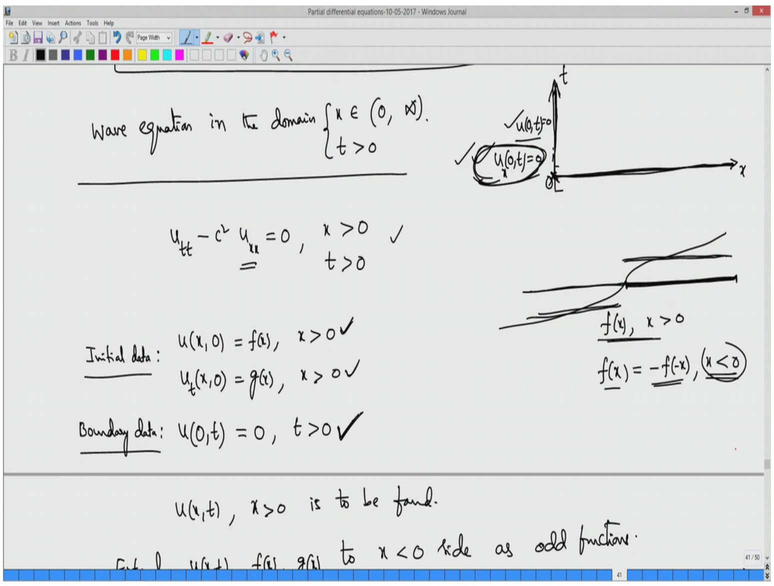
So you see the given data is F at G, F at G is known you know for all times what is your U of X of T ok. You have the displacement for all times even if it is given on semi-infinite domain that is zero to infinite string.

(Refer Slide Time 45:27)



Semi-infinite string but the one end of the string is fixed that means displacement is zero this is how you simply the idea is simply extending this functions as odd functions but in a smoother way ok you extend it in a smooth way so that it is automatically satisfies wave equation and the boundary conditions so that you can (wri) you can make use of your solution D'Alembert solution to write its solution here ok. So will also see we can in the same way we can also see if your (sol) if your boundary data is nuemann data that is U X instead of U 0 of T.

(Refer Slide Time: 46:14)



If you have U X at 0 T is given here at X equal to 0. That means, that means what? U at X, U X at 0 T that means the slope of the string at X equal to 0 is 0 ok, the slope you can you fix in such a way that slope of the string is 0 ok. If you give this zero this condition instead of this if you give this condition, how do we solve this? so only thing is same domain wave equation initial data is same now the boundary data you are changing to this type.

Then only idea is only simply extend the functions as an even function so that this is automatically satisfied ok so this is how you can actually solve this two types either this or this are given if this is given you extend this U and given F at G initial data you extend them as an odd functions if your boundary data is U X at 0 T is 0 then you extend these functions unknown function U of X T and initial data FX GX you extend them as even functions ok in a smooth manner.

So that this at automatically satisfy the wave equation and the boundary condition ok so that you can make use of D'Alembert solution and with that we will see in the next video as I will try to see that one and what are the other boundary conditions we can give at X equal to 0 in the same infinite domain will see that in the next video, thank you very much.