Differential Equations for Engineers
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Lecture 44
Reduction to Normal form - More examples

So last video we have seen two examples of hyperbolic partial differential equations we reduce them into canonical form and one you could reduce but you could not solve it other example we have consider is a simpler one that is wave equation which is already in the canonical form one canonical form you can get the solution of that equation by reducing into another canonical form, another hyperbolic canonical form by reducing that into another canonical form we could solve the partial differential equation completely, ok.

We can find the general solution of that equation we have given that example those two examples in which one you cannot find the general solution other example you reduce into canonical form and find the general solution ok. So in this video we will see other cases when the partial differential equation is elliptic and parabolic case how do you reduce? Equations, partial differential equations with variable coefficients and each, when this partial differential equation is elliptic or parabolic how to reduce?

We will work out how to get its canonical form in this video ok. So will consider this example.

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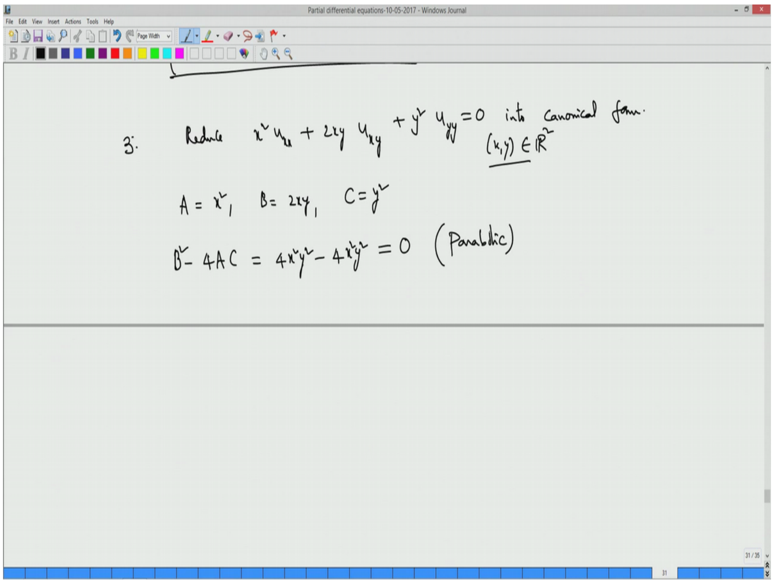

Example number thee so see will start with the parabolic case, so parabolic case is so reduce X square U X X plus 2 X Y U X Y plus Y square U Y Y equal to 0 into canonical form. So again you consider so procedure we know so you start with A is X square, B is 2 X Y, C is Y square discriminate is B square minus 4 A C is 4 X square Y square minus 4 A C is again is 4 X square Y square that is 0. So every X Y in the plane this partial differential equation is defined is 0 ok, discriminate is 0 so the equation is clearly parabolic.

The parabolic case we have seen that new variables are coming only from, get one variable Xi or eta so Xi you can get from the partial (differential) from the ordinarily differential equation.

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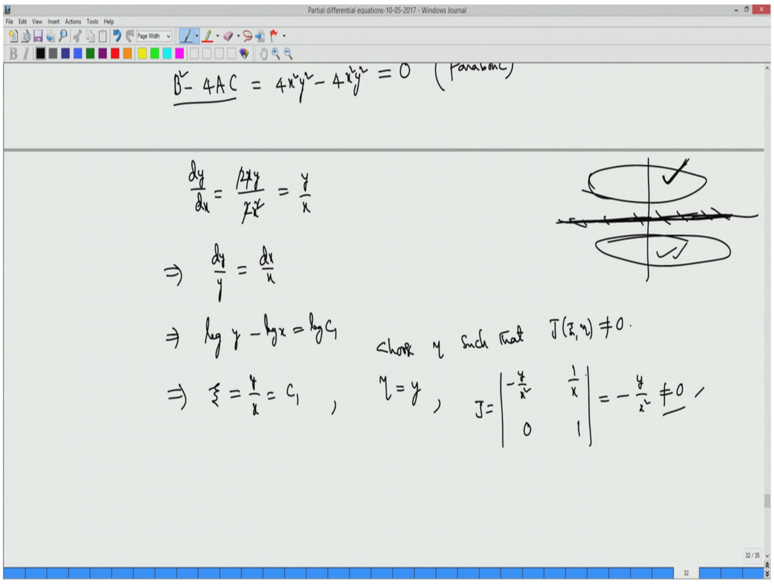

That is D Y by D X equal to B is 2 X Y plus or minus, one is plus square root of B square minus 4 A C that is 0, ok divided by 2 A X square. Other one is D Y by D X equal to B minus square root of B square minus(4 A C),here it is zero that is why you get the same ODE so you can both the ordinarily differential equation becomes one.

So this becomes simply Y by X so if you solve this D Y by Y equal to D X by X you solve this you can get log Y minus log X equal to log C 1 so this will become, C1 is arbitrary constant so you have Y by X equal to C1. So this is your Xi variable, Xi is Y by X so what is your eta? You could get only one variable in the parabolic case so eta is you choose eta ok, I choose eta in such a way that the Jacobian of Xi and eta has to be non-zero. How do I choose? I choose the simpler form that is I simply choose as Y ok, if I choose this as Y, what is the Jacobian? Xi X that is minus Y by X square, Xi Y that is 1 by X and eta X 0, eta Y is 1, this is your Jacobian, ok.

So this is simply minus Y by X square which is non-zero clearly where it is non-zero, if your domain is strictly speaking this is non-zero only in the domain when Y is not equal to zero. So except you have to consider the domain minus this X axis if you have to consider either upper of plane or lower of lane in which if you consider is Jacobian is non-zero. So as we have seen earlier this is only one dimensional domain you consider the equation this domain and in this domain separately so that you get your reduced equation canonical form as in this domain separately and in this domain finally just valid you in here, by taking continuity of it.

So with this Jacobian is non-zero so have these new variables Xi is this Y by X eta is Y. So what we need is all X X, U X X and U Y Y and U X Y so just do little calculations here.

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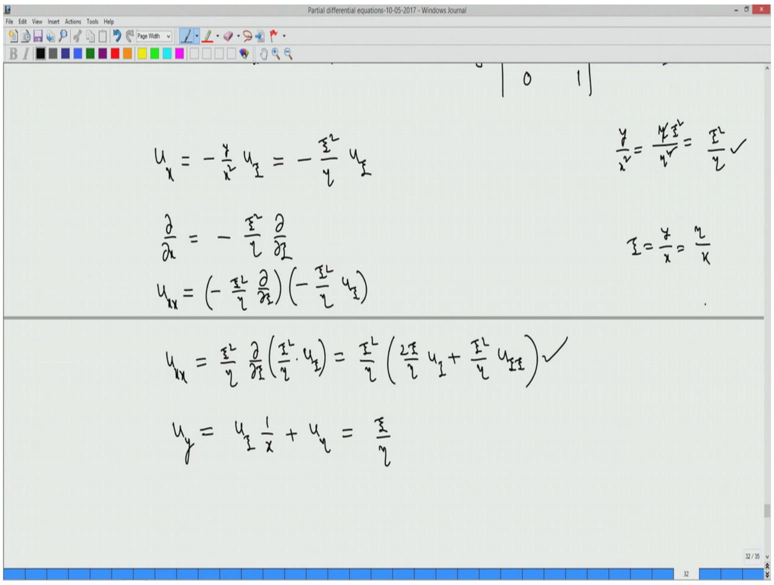

So you have U X start with U X that is U Xi into U X so U X into Xi X, Xi X is Y by X square into U Xi plus U eta into eta X, eta X is 0 so that is 0,so this simply you get this all ok. So what is my Y by X? Y by X square equal to you can write this as Y by X, Y is Y equal to eta so you have eta, Y by X square I have to write in terms of Xi and eta so Y is eta X square is Y square by Xi square. Y is eta so eta square by Xi square.

So you simply get Xi square by eta, ok Xi square by eta is Y by X square so into U Xi this is my U X so that will give what is my dou dou X, dou dou X is minus Xi square by eta dou dou Xi so that you can write now U X X that will be minus Xi square by eta dou dou Xi acting on this U X, U X is minus Xi square by eta U Xi this is the function and this the operator. So minus-minus plus Xi square by eta now you differentiate this with respect to this product of functions ok.

So this is Xi square by eta U Xi is 2 Xi by eta into U Xi plus Xi square by eta U Xi Xi. So this will give me that is it ok so this is my U X X. Now you want U Y, U X Y so you calculate your U Y that is U Y if you calculate what is U Y? U Y is U Xi into Xi Y, Xi Y is Xi Y from here Xi Y is 1 by X plus U eta into eta Y, eta is Y so it is one. So you have one by you need one by X. What is 1 by X? 1 by X is Xi by eta, Xi by eta is 1 by X ok. Xi by eta is 1 by X, because Xi equal to Y by X, Y is eta Y X so 1 by X is Xi by eta.

So 1 by X I replaced with Xi by eta into U Xi plus U eta. So this is what is my U Y.

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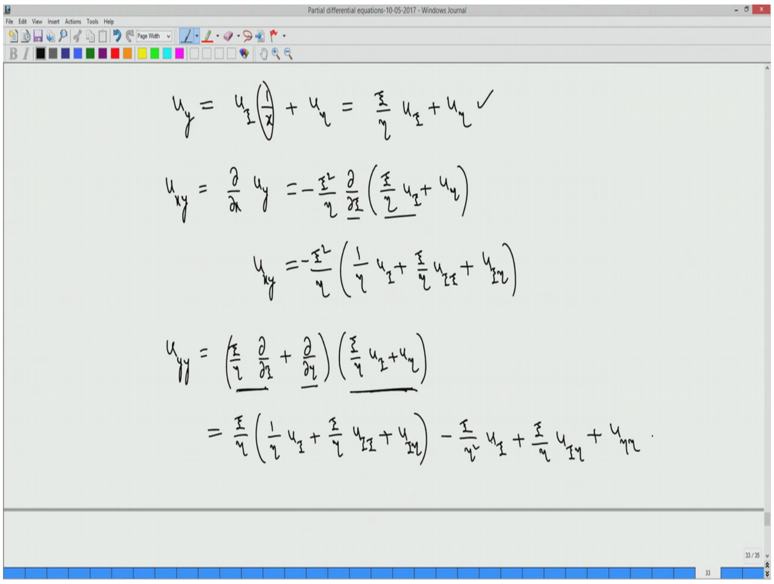

So now you can calculate U X Y that is dou dou X acting on U Y ok. So dou dou X we know that is simply Xi square by eta dou dou Xi is my dou dou X and U Y is this one, this is Xi by eta U Xi plus U eta so have this is equal to U square by eta this becomes 1 by eta U Xi ok. You are simply differentiating this product of functions plus Xi by eta U Xi Xi plus U Xi eta that is my U X Y. Offcourse you have a minus here, minus for dou dou X, dou dou X is actually minus of that so I missed that minus for the dou dou X so you have minus ok.

So this is my U X Y, so you also need U Y Y, U Y Y is simply you from this you can get dou dou Y, dou dou Y is Xi by eta U Xi, dou dou Xi plus dou dou eta this is my together this is my U Y Y, act this on U Y, U Y is Xi by eta U Xi plus U eta so this is a function and this the operator so if you operate this first Xi by eta and now you differentiate this function with respect to Xi you get 1 by eta U Xi plus Xi by eta U Xi Xi plus U Xi eta ok. So this is what you will get if you operate this onto this function.

Now you operate this onto this function you get what you will get if you do this minus Xi by eta square because 1 by eta Xi by eta derivative with respect to eta is minus Xi by eta square into U Xi plus Xi by eta U Xi eta plus U eta eta that is what is the if you operate this onto this one. So this is my U Y Y ok so from this you can see that this if you take this Xi by eta inside this will be Xi by eta square, Xi square by eta square, Xi by eta ok.

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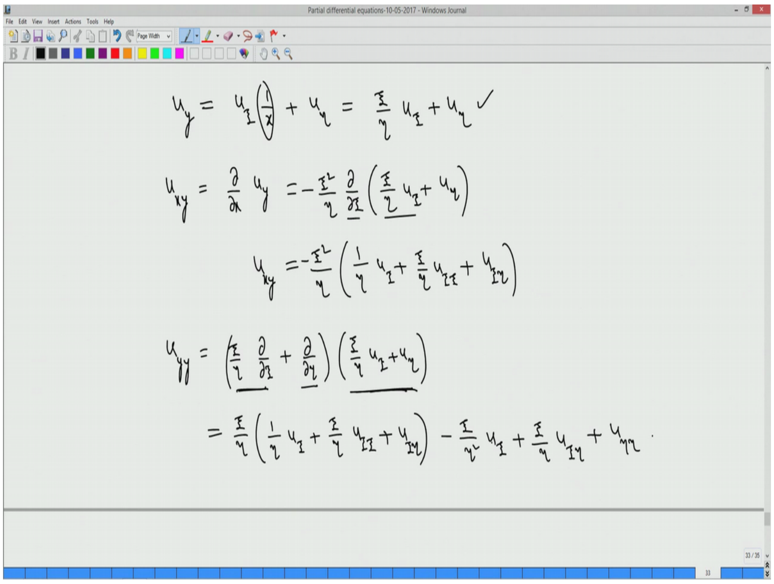

So what you are left with is so you can see that this gets canceled U Xi Xi so finally what is your U Y Y is simply Xi square by eta square U Xi Xi plus Xi by eta into U Xi eta and Xi by eta into U Xi eta so you have two times of that ok plus U eta eta that is it, you got everything as what you require so U X X.

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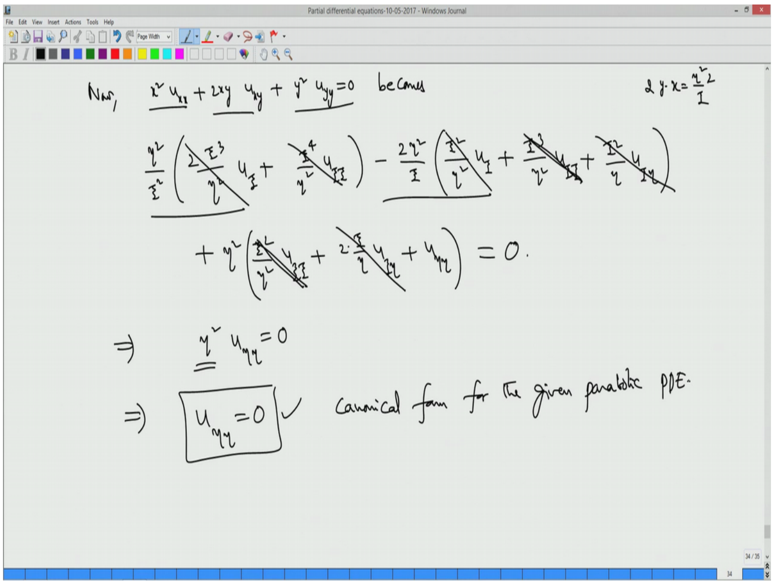

So now your equation that is X square U X X plus 2 X Y U X Y plus Y square U Y Y equal to zero becomes, what happens? Just calculate so you need what is X, X is what is X, from in terms of new variables Xi and eta so 1 by X is Xi by eta so X is eta by Xi, so you have X square is eta square by Xi square U X X.

You replace this U X X, U X X is Xi square by eta ok this if you combine this one, this one if you rewrite 2 Xi cube by eta square, 2 Xi cube by eta square into U Xi plus Xi power 4 by eta square U Xi Xi that is what is my U X X so you can see that Xi is 4 by eta square U Xi Xi so that is my this term now plus two times, what is my X? X is eta by Xi Y is eta so you have eta square two is this so you have two, two eta square by Xi is my 2 X Y into U X Y so you have found what is my U X Y, U X Y is this that minus comes here and you have a Xi square by eta ok and offcourse you can write you can put together.

So you have that we can combine take it inside so you have Xi square by eta square U Xi plus Xi cube by eta square U Xi Xi plus Xi square by eta U Xi eta ok that is my second term. Now combine with Y square that is eta square into U Y Y, U Y Y is here so you have Xi square by eta square U Xi Xi plus 2 Xi by eta U Xi eta plus U eta eta of this equal to zero because equal to zero ok, we simplify you can see that this Xi square, Xi square goes Xi Xi goes 2 Xi U Xi here Xi square Xi square eta square so again here 2 Xi U Xi here minus 2 Xi is U Xi so this gets cancel ok and what happens to this part? U Xi Xi part, Xi cube ok so this goes ok so you have eta square eta square goes Xi, Xi goes so you have minus 2 Xi square U Xi Xi, here.

Plus Xi square U Xi Xi this also goes ok, so here Xi square U Xi Xi minus 2 Xi square U Xi Xi so (toge) together is minus Xi square U Xi Xi and here plus Xi, Xi square U Xi Xi so this together will go with this ok. So this one this two combine with this will go so what you are left with is simply and you have one more term so this one, this is eta eta, eta eta goes Xi Xi goes so what you simply having 2 eta by A 2 Xi eta, 2 Xi eta minus 2 Xi eta U Xi eta here plus 2 Xi eta so this this will go.

So finally you ended up both the terms everything is gone except last term, so you have eta square U eta eta equal to 0. Since eta cannot be zero that is how we have chosen ok, if you chose eta in such a way that Jacobian, if you chose eta is zero then Jacobian is zero ok. So you have chosen eta such a way that is non-zero so you (can) so eta square cannot be zero that implies U eta eta equal to zero so this your canonical form, canonical form for the given equation, for the given parabolic PD, parabolic PD, linear PD with variable coefficient.

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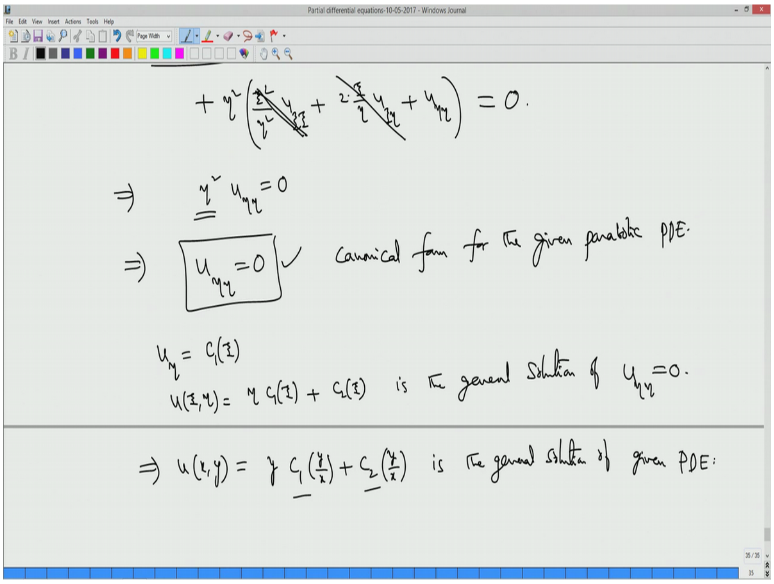

Now I can easily combine this so U eta is function of some C1 of Xi U differentiate with respect to eta you get back this now you again you integrate one more time you have finally U Xi eta will be this will be eta C1 of Xi plus C2 of Xi right. Now you differentiate with respect to eta you get C1 of Xi this will be zero so you have this so this is your general solution in the old variable this is the general solution, solution of U eta eta equal to zero in old variables this you get U X Y which is equal to now eta is Y into C1 of Xi is Y by X plus C2 of Xi by (A) Y by X is the general solution.

General solution because C1 C2 are arbitrary functions is the general solution of given PD. So this is what this is how you can actually solve some of the equations you can reduce even if it either hyperbolic parabolic or elliptic so we can reduce into canonical form and fortunate here we can solve it, solve the equation. Now we look at the last example that is elliptic case we consider simply you consider reduce.

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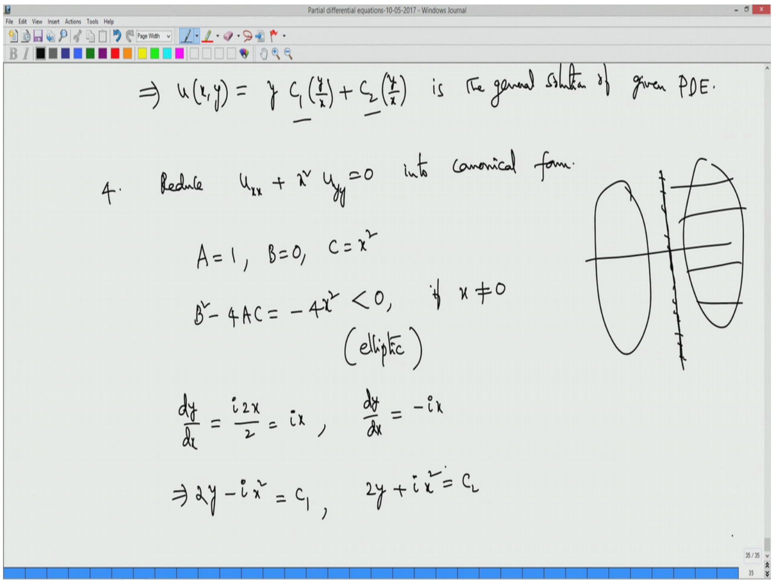

Reduce this equation U X X plus X square U Y Y equal to zero, reduce into canonical form. So here again routinely A is 1, B is 0, C is X square so you see that B square minus 4 A C is B 0 minus 4 X square this is always negative ok for every X positive X non-zero for every X non-zero when X is non-zero so the domain is, when X equal to zero means Y axis so this you remove so that means your domain is either this or this one, so you separate them and you work with this, these each of this domains what you get is your equation in these domains ok canonical form in these domains with the same variables.

So that is how you can do, so what happens because this is negative so the equation is elliptic equation given equation is elliptic clearly then X equal to zero you need not consider because this is the one dimensional line this is that is X equal to zero Y axis ok so ordinarily differential equations for the new variables here is D Y by D X equal to B 0 plus square root of B square minus 4 A C that will give me I times two X by 2 A so that is two. So you have I X other one D X equal to minus square root of B square minus that is going to be minus I X.

So if you do this you have Y minus I X square by 2 equal to C1 ok or you just take this two there so you have 2 Y minus I X square equal to C1. Similarly 2 Y plus I X square equal to C2 this is how you get your so this is my this (how) if you solve this you get these two general solutions for each of these ODE's. Now because these are involve in this imaginary number I so we know how to solve in, we know how to choose this Xi and eta we simply take these are complex conjugates.

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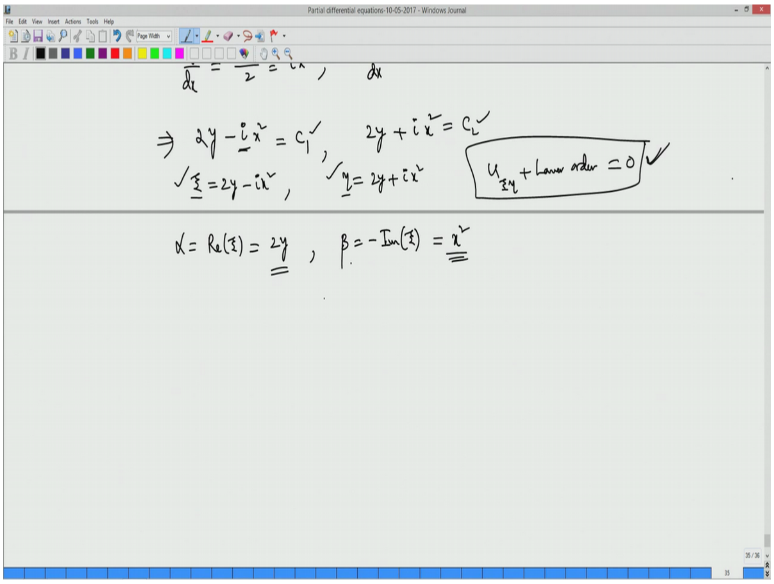

If you take this Xi as 2 Y minus I X square eta is 2 Y plus I X square. If you do like this, with these Xi and eta variables you added up like hyperbolic case U Xi eta equal to plus lower order terms equal to zero but Xi and eta are complex functions of X and Y complex valued functions of X and Y so this is how you get one canonical form so you always choose real valued variables real variables so what we do is alpha as real part of Xi so that is 2 Y ok and then beta equal to real part of Xi that is imaginary part of Xi ok or remaining part of eta anything you can choose ok minus imaginary part of Xi you can consider because I don't want to work with minus, so you have X square.

Anything you can choose doesn't matter ok. So beta is imaginary part of Xi that is minus X square so I don't want minus X square so you can see minus eta so both are so you can work with any of these things. If these two are linearly independent whose Jacobian is non-zero and real part imaginary part these alpha plus beta alpha beta this are simply some of these linearly independent functions Xi plus eta and Xi minus eta with some constants will give you may alpha beta.

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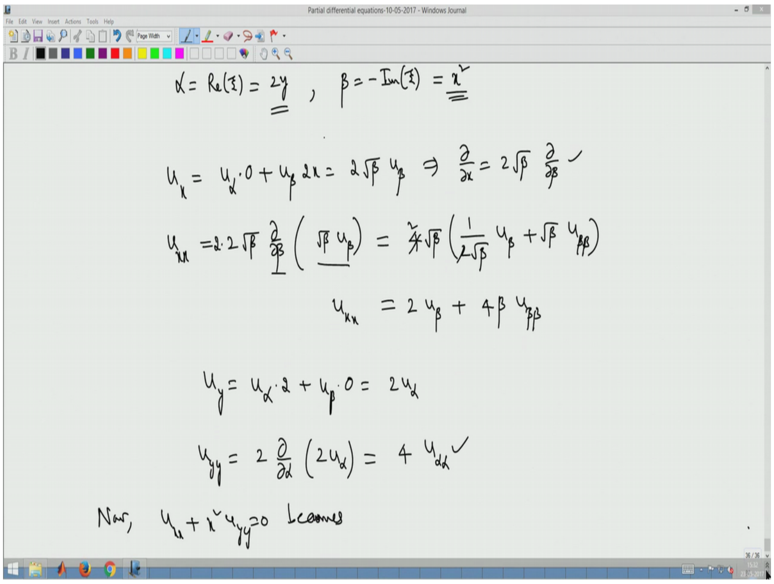

So with these new variables you need to calculate U X, U X X and U Y Y start with U X, U alpha, alpha X is zero plus U beta, beta X is 2 X so we have 2 X, X is root beta into U beta that is my dou dou X so implies dou dou X is 2 root beta into dou dou beta this is how you get the operator this will give me U X X as 2 root beta dou dou beta acting on 2 root beta U beta that is U X. This will give me because it is constant you can take it out as 4 root beta now you operate this onto ok so I operate this onto this product of these two functions I will give A.

One by 2 root beta into U beta plus root beta into U beta beta so together this becomes U X X equal to two-two goes you have 2 U beta plus 4 beta into U beta beta that is my U X X. what is U Y? U Y is U alpha into alpha Y is 2 plus U beta into beta Y is zero and that is zero ok very simply U beta into beta Y is zero so you have 2 into alpha U alpha so you have U Y Y is two dou dou alpha acting on 2 U alpha. This will become 4 times U alpha alpha ok. So if you combine this now, once you know these are the only things you know so now the given equation that is U X X plus X square U Y Y equal to zero becomes, you can see what happens.

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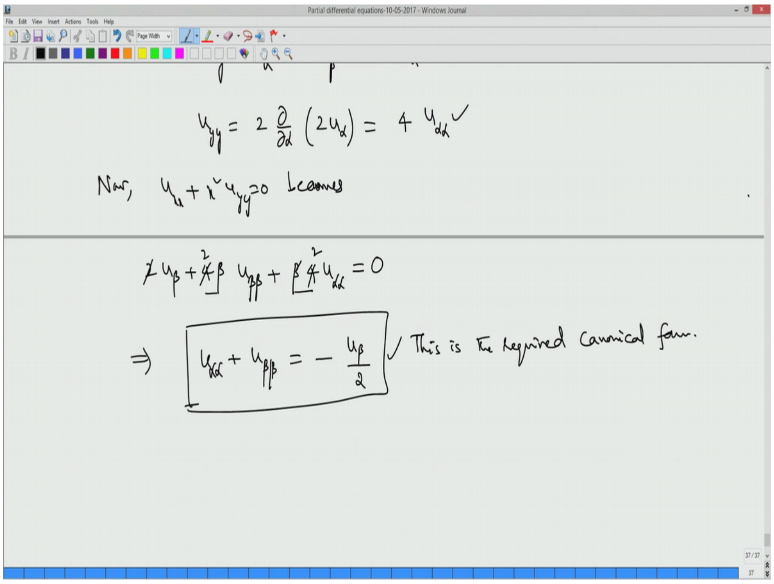

What is U X X ? U X X is 2 U beta plus 4 beta U beta beta, what is X square? X square is simply beta so you have beta into U Y Y is 4 U alpha alpha equal to zero. So this is nothing but 4 beta 4 beta you can divide so first two-two cancels so you have ended up U alpha alpha plus U beta beta equal to minus U beta divided by 2. So this is your canonical form so these is the required canonical form this is the required canonical form. So it looks like Laplace equation but it involves the first order derivatives that is why we cannot solve ok

This you cannot solve even if it is first order terms are not involved still it is not possible because it is a partial differential equation you don't know any method how to solve this one ok, earlier in the elliptic case if whatever you do you may reduce finally only Laplace(equa) equation that is U Xi Xi plus U eta eta equal to zero or U alpha alpha say any new variables Xi and eta or alpha beta can see that the equation becomes U alpha alpha plus U beta beta plus lower order terms zero.

Even if they are no (linear) lower order terms you cannot solve this U alpha alpha plus U beta beta equal to zero because you don't know any method, don't have method to solve it will see the methods later how to solve this Laplace equation ok. So this is how we can reduce second order linear partial differential equation with constant or variable coefficients into a canonical forms some of them after reducing into canonical form can be solved completely to get the general solution. Some are simply in reduced form ok those are called canonical forms this is how you reduce and we have actually seen when we work with our wave equation.

So we are going to solve the wave equation we are going to (sol) we are going to define the problems for wave equations and as a wave equation in the full domain that is T positive and the special domain X belongs to full real lane one dimensional wave equation we have already seen the canonical form and got the general solution ok.

So that we make use when we work with our wave equation and its solutions ok so that we will see in the next video thank you very much