Differential Equations For Engineers
Professor Dr.Srinivasa Rao Manam
Department of Mathematics
Indian Institute of Technology, Madras, Chennai
Lecture 43
Reduction to canonical form for equations with variable coefficients

Welcome back and the last few videos we have seen how to reduce second order linear partial differential equation how to reduce into linear partial differential equation. We have seen in the last two videos we have seen how to reduce second order a linear partial differential equation. How (to) it is reduced to a canonical form and we have shown with an examples with constant coefficients the examples that we have considered or equations with constant coefficients and so those (work) working with these constant coefficients is really too easy to get the canonical form.

So in these next one or two videos will see how to reduce equations with variable coefficients so those A B C D are there as a functions of X and Y. So if we consider them how to reduce them into those equations with a variable coefficients into canonical form so that you can reduce the equation into simpler form so which some of the equations can be solved ok, and some may not be able to solve atleast is the canonical form so will see how do we do this, we start with an example that is (hyp) hyperbolic equation.

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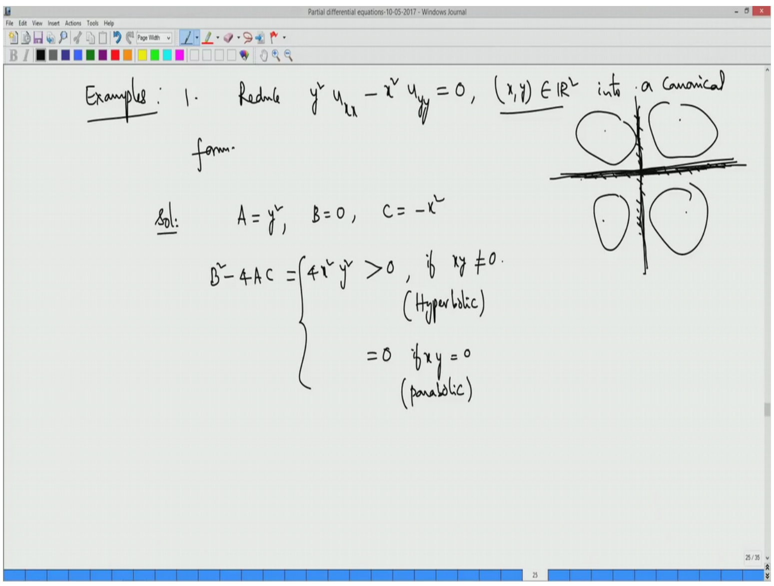

So let's consider the equation examples will consider three examples, examples 1 is reduce with the PDE Y square U X X minus X square U Y Y equal to 0, so domain in nothing is given so that means basically X Y belongs to R2 into canonical form. So this is the question, so will solve it. Will give the solution so will reduce how to reduce, will reduce this equation into a canonical form so the moment you see this one first you like earlier like we have done in earlier for the constant coefficients equation.

You take just what is A, A is Y square B 0 and C is minus X square D E F G are all zero. So when we have to say this discriminate B square minus 4 A C is 4 X square Y square. This is if you consider any domain X square Y square if you consider any of these domains see you see this, this is actually full domain ok, you have full domain and discriminate is 4 X square Y square this is always positive if X Y is non-zero. So that means you remove this from the plane you remove this coordinates Cartesian coordinates this axis ok coordinate axis if you remove that is the (doma) that means this one (domai) this is the domain you are dealing with.

So you have to consider this equation in each of these domains. This is not Q as you have seen in Abel's formula in ordinary differential equations if you know one solution Y1 to find the other solution you found the Abel's found you can find the other solution, other linearly independent solution from the Abel's formula Y2 is given in terms of Y1, Y1 into some integral X knot to X in the integrant involving Y1, there because the 1 by Y1 square is involved in the Y2 so it has to be, Y1 has to be non-zero.

If Y1 is zero at certain points you can remove from the domain and consider the ordinary differential equation in separately in each of the domain and then get from the Abel's formula because Abel's formula make sense only when Y1 is non-zero and then once you get your Y2 in each of the domain and you can take the limits and that those points Y1 is zero it is also you can get because Y2 is continuous so you actually Y1 Y2 both are, both makes sense in the even at the points that are removed.

So in the same way here even though if you want to reduce this equation into canonical form in the on the coordinate axis because X is zero or Y equal to zero we have 4 X square Y square which is zero so when it is a positive that is in each of these domains this is clearly hyperbolic because it is positive but if it zero this can be possible on a coordinates axis if X Y equal to zero this is zero ok. So this is what you have so in this case this is parabolic but see this is a partial differential equation an the domain here is just X equal to zero and Y equal to zero this coordinate axis.

This is one dimensional domain so you have a partial differential equation in one dimensional doesn’t make sense ok so just like ordinary differential equation if you are if it is given at one point so it doesn’t make sense so the same way here partial differential equation should be define on a atleast in a plane domain if you have two linearly (independent) two independent variables ok if you have two independent variables for the partial differential equation domain should be a plain domain, but here this is not a plane domain one less dimension that is simply lines so those you can ignore ok.

So in that sense the equation you say that is hyperbolic rather than saying a parabolic at these points ok so equation that is (defi) well defined only in these Y and these domains and these quarter planes ok these quarter planes you have equation is well defined which is hyperbolic. So once you have this hyperbolic you know how to find those new variables Xi and eta.

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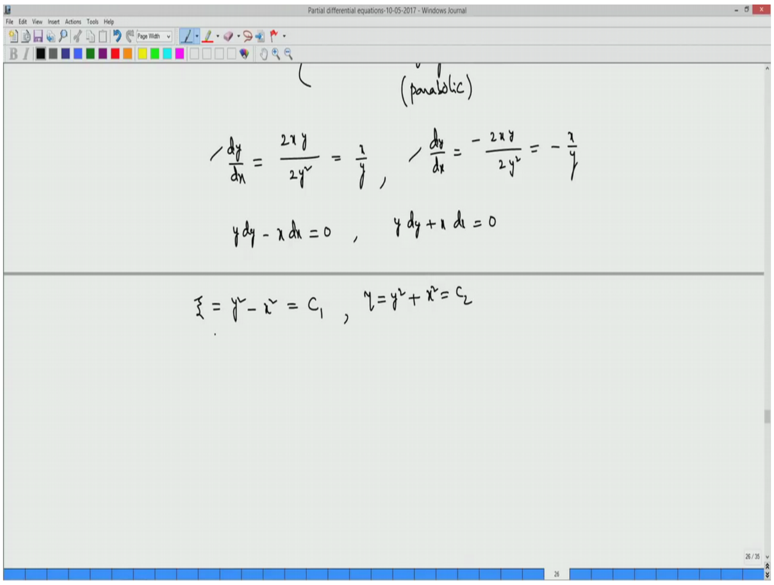

So from this differential equation ordinary differential equations that is B is zero, B plus square root of B square minus 4 A C that is 2 X Y divide by 2A so 2 Y square, so this is simply X by Y.

Other differential equations is D Y by D X is B minus, B (minu) B 0 minus square root of B square minus 4 A C that is 2 X Y divide by 2 A this will give me minus X by Y. So if you solve these two ordinary differential equations what you end up is the new variables so if you do this what you get is Y D Y minus X D X equal to 0 and this becomes Y D Y plus X dou, X equal to 0. So this will give me Y square minus X square equal to constant C1 and similar this will be Y square plus X square equal to C2.

So will get actually Y square by 2 the 2 you can absorb into the arbitrary constant so which I am calling C1 ok similarly here you have Y square by 2 plus X square by 2 the 2 I can absorb into the arbitrary constant. So from this you get your new variables Xi is a function of X Y, eta is a function of X Y that is Y square plus X square. Now what you need, you need to reduce this equation that involves U X X and U Y Y now just calculation part.

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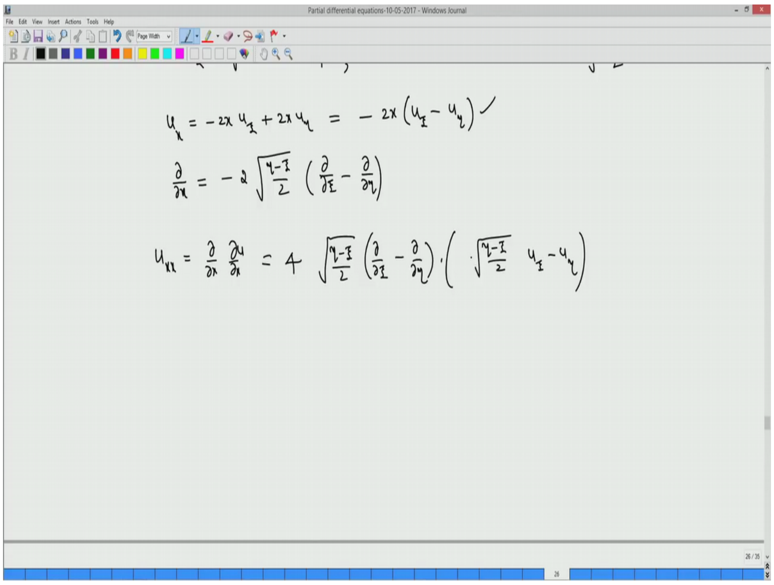

Start with U X, U X is U Xi Xi X that is minus 2X U Xi plus 2X U eta ok U eta eta X , eta X is eta Y is eta X, eta X is 2X, so here 2X here minus 2X from Xi, so this is equal to minus 2X U Xi minus U eta ok so this also gives me what is my dou dou X, dou dou X is actually so what is X, X you have to write in terms of Xi, so what is X ? X is square root of so X is eta minus Xi by 2 that is X square so once you take the square root that is your X.

So finally you get minus 2 square root of eta minus Xi by 2 into dou dou Xi minus dou dou eta. So this is my dou dou X once you know this you can get your U X X, U X X is dou dou X, dou dou X acting on U so this if you write it minus 2 square root of eta minus Xi by 2 ok and then you have U Xi minus U eta ok first you write this dou dou Xi dou dou eta this is my dou dou X. Now dou U by dou X you can pick it from here so that will be again that into this into minus 2 X is square root of eta minus Xi by 2 into U Xi minus U eta ok.

You take this minus and minus that is going to be plus and 2 2 you can write it here as 4 ok and this 2 also you can cancel and write finally 2 ok.

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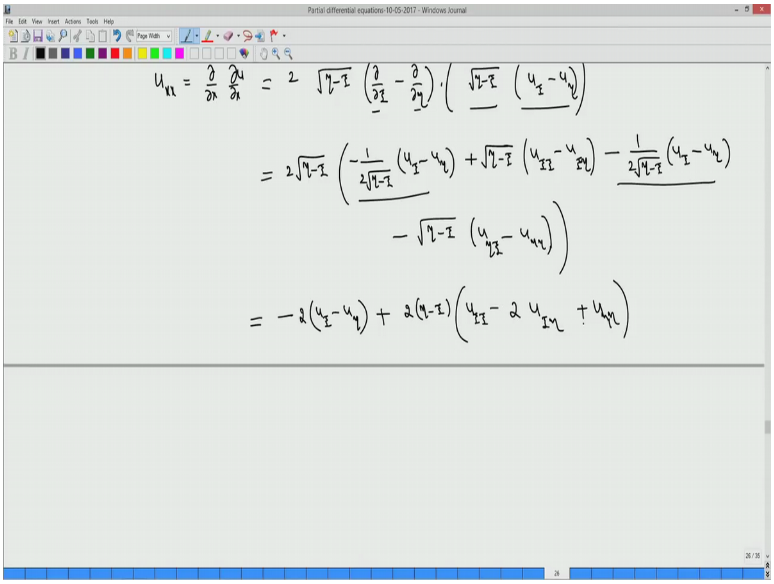

So root 2 into root 2 is divide by that is 2, 1 by 2 into 4 is 2, so this you can write just square root of eta minus Xi into this and similarly here eta minus Xi and the root. So this is what you have to expand nicely this as (assuming) the this is one function this is another function. So you differentiate with respect to Xi and eta to product of these two functions.

So if you do this you get two times square root of eta minus Xi and what you get dou dou Xi of fist function that is minus 1 by 2 root of eta minus Xi into U Xi minus U eta plus now I differentiate this one so this will be as it is eta minus Xi now U Xi Xi minus U Xi eta ok. So this is what is your one part ok this is actually that is what is the first derivative this derivative now you have to take the minus now differentiate with respect to eta this product of two functions.

So you will see that will be 1 by 2 square root of eta minus Xi and you have minus ok (this) that minus comes out and after been the differentiation you don't get any negative sign so you have this into U Xi minus U eta and the last term is minus, this minus and the square root of eta minus Xi this will be U eta Xi minus U eta eta ok. So this is what is the final form so if you now simplify what you end up is I will take this 2 2 goes this this goes and here also so you collect this terms this one what you see is these are same ok so this one and this one are same so what you end up is 1 by square root of eta minus Xi U Xi minus U eta so (mi) with minus so you have minus sign 2 this cancels what you end up is U Xi minus U eta

Plus now you take this one so 2 times square root of eta minus Xi into eta (minu) square root of eta minus Xi is 2 times eta minus Xi into U Xi Xi minus U Xi eta and here again if you multiply here so this 2 times eta minus Xi out and what you end up is minus U Xi Xi U Xi eta, U eta Xi, U Xi eta both are same so you have U eta Xi minus minus that is minus minus plus U eta eta ok this is what you have so finally this you can combine it together so this is because U Xi eta equal to U eta Xi so you can write this as 2 times U Xi eta ok. So this is what my U XX everything in terms of new variables Xi and eta, what is my U Y?

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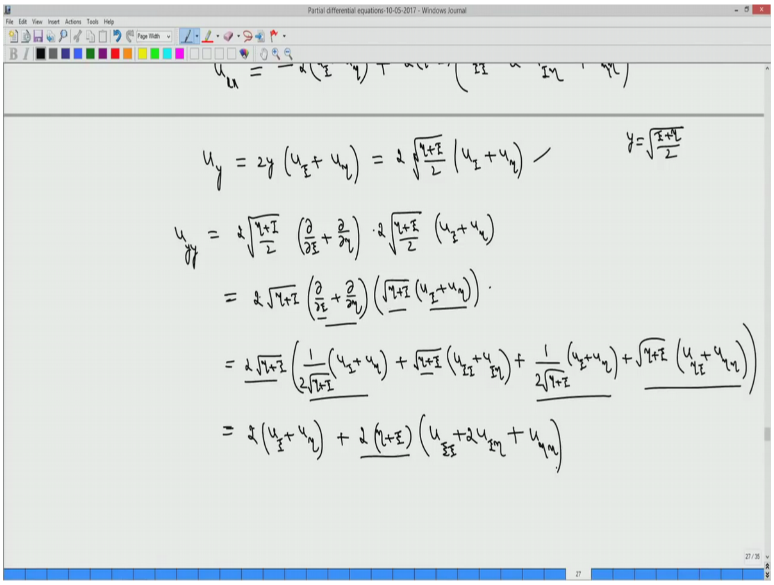

U Y is again if we do same way U Y is U Xi, Xi Y that is 2 Y plus U eta, eta Y that is also 2 Y so 2 Y is common U Xi plus U eta this is (equal to) now what you need, you need is Y (()) (14:22) Y is actually from because you know what is Xi and eta Y will be some of these if you take this Xi plus eta by 2 that is your Y square so square root of Xi plus eta by 2 is my Y so if you write this square root of eta plus Xi by 2 ok into U Xi plus U eta so this gives me what is my dou dou Y so once you know that you can write U Y Y as dou dou Y is now 2 times square root of eta plus Xi by 2 dou dou Xi plus dou dou eta acting on U Y, U Y is here so this is into 2 times square root of eta plus Xi by 2 U Xi plus U eta now you combine this together so you have 2 times this 2 this 2 goes ok the denominator will be together to when 2 goes what you end up is this dou dou Xi plus dou dou eta this operating on square root of eta plus Xi and U Xi,U Xi plus U eta ok.

So these are two functions product of two functions and this is the operator so this is all we have to look at it. so this will give me two times eta plus Xi, now you differentiate this on this product of functions you get 1 by 2 square root of eta plus Xi into U Xi plus U eta plus square root of eta plus Xi U Xi Xi plus U Xi eta ok and then now if you differentiate with respect to eta you get again 1 by 2 square root of eta plus Xi into U Xi plus U eta plus square root of eta plus Xi U eta Xi plus U eta eta now close the bracket.

So you have these two are same if you combine it, it is going to be 1 by square root of eta plus Xi into U Xi plus eta so together two times U Xi plus U eta now this one if you combine this, this you see that eta plus Xi this into this will be two times of that and what U is, common is U Xi Xi plus U Xi eta plus here this is also same so you can see that eta plus Xi into eta plus Xi that is two times eta plus Xi into U eta Xi and U Xi eta both are same so you can combine it together with 2 plus U eta U eta.

So this is what is my U Y Y so this are the only things I need because I don't need U X Y because the given differential equation is this one that doesn't involve mixed derivative U X Y.

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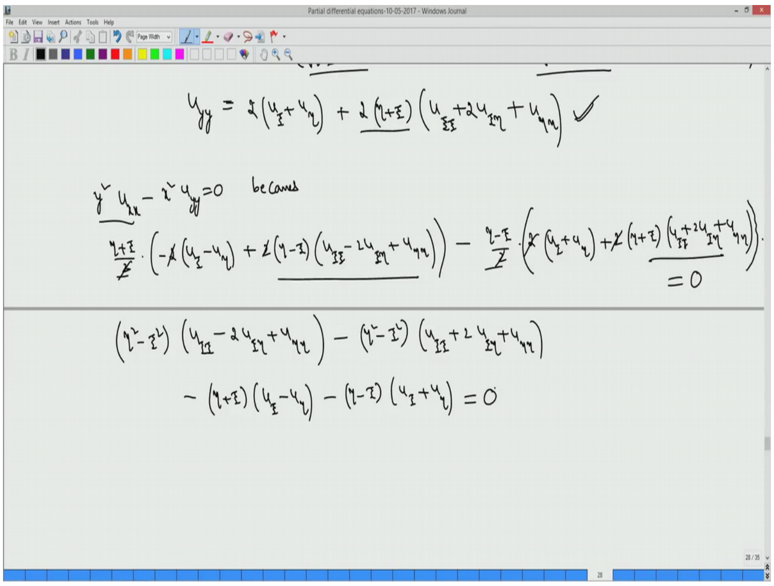

So put together you combine it and you see that U X X Y square into U X X minus X square U Y Y equal to zero becomes what is Y square? Y square is eta plus Xi by 2 into U X X, U X X is you can (use) make use of U X X that is this one so use this U X X minus 2 U Xi minus U eta plus 2 times eta minus Xi into U Xi Xi minus 2 Xi eta plus U eta eta ok.

So this into this one so that is this part minus X square is eta minus Xi by 2 ok into now U Y Y you can make use of with these expression so you have two times U Xi plus U eta plus two times eta plus Xi, U Xi Xi plus U Xi eta plus U eta eta ok this ends that is it. So this has to be equal to zero so that is what is the equation ok now you can combine this two , so two-two cancels here and what you end up eta minus Xi square.

Now we consider this one, this one into this will be eta square minus Xi square U Xi Xi minus 2 Xi eta plus U eta eta ok and here if I combine this one and this one after cancelling this two you see that minus eta square minus Xi square U Xi Xi plus 2 U Xi eta plus U et eta. What is left with these two? So this will be plus eta plus Xi into minus, minus U Xi minus U eta and here eta minus Xi into U Xi plus U eta equal to zero. So you can see that this is common so this terms will be zero this gets canceled.

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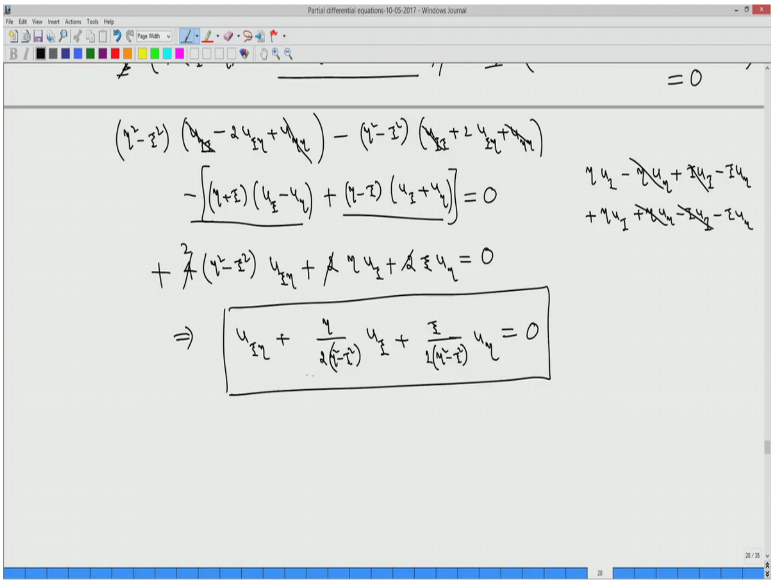

So what you are ending up finally is minus, minus 2 here minus 2 so you have a total 4 eta square minus Xi square U Xi eta minus right so minus this if you write that will become eta, eta times U Xi here eta times U Xi that gets canceled and here if you expand it you expand this and this combine it so will do it separately so you see that eta, eta U Xi minus eta (sorry) eta U, eta plus Xi U Xi minus Xi U eta now similarly you combine this one minus I take it out so what is left with the plus one ok.

So what I am doing is this one whatever in the brackets is this eta U Xi plus eta U eta minus Xi U Xi minus Xi U eta so you can see that these are cancelling so you have to write what is left with here, what is left here is two times eta U Xi minus two times minus-minus plus ok and this becomes plus two times Xi U eta equal to zero. So you can divide it so finally you get U Xi eta minus-minus you can both sides you can make it plus two-two can cancel make it 2 and what you are left with is eta by 2 eta square minus Xi square into U Xi plus Xi by 2 eta square minus Xi square into U eta equal to zero.

So this is your canonical form that is U Xi eta plus lower order terms this are your lower order terms which is equal to zero ok, this is the required canonical from. This we cannot integrate so we know only one technique whether we can integrate directly so which are not possible because it involves second order and first order terms so you cannot solve this partial differential equation even after reducing into canonical form ok.

So this is (hyperbol) (exa) hyperbolic example so will look at other example where so maybe so we have seen how to reduce our linear second order partial differential equation into canonical form a hyperbolic linear partial differential equation we reduce into canonical form in this video so that is we have will do two more examples in the next video so in which we consider the examples equations with variable coefficients for parabolic and elliptic cases ok.

So you have seen this example where the equation hyperbolic equation is reduced to this canonical form which you cannot solve well if you consider now some equation which is our the typical equation is hyperbolic equation that is the wave equation.

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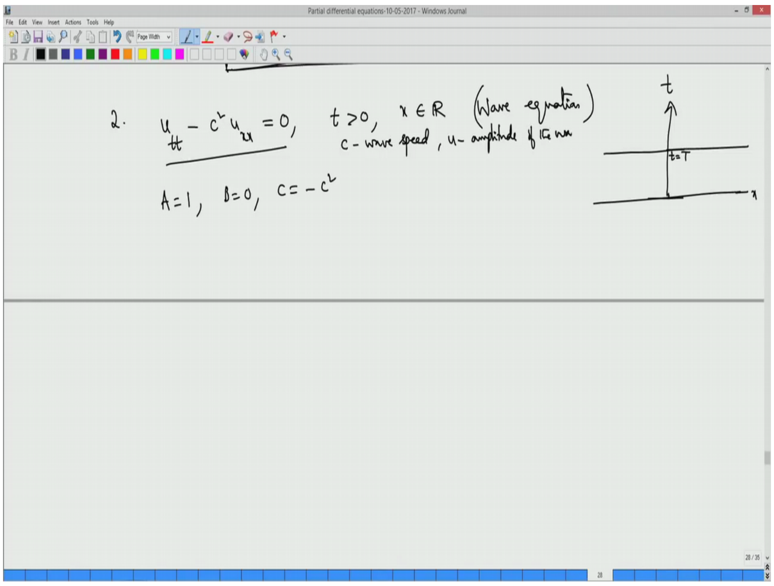

Let us consider one more hyperbolic equation for example U X X minus C square U X X so it actually should be U T T so U T T minus C square U X X is zero and T is positive is the domain X is in full domain let us say this one.

So this is an example two which is also hyperbolic equation this is the wave equation this is called wave equation ok so what happens so how do we don't know how to solve this equation? So far whatever we know ok but because this is a partial differential equation in the domain T is positive X is the full domain and T is positive so if you take this as a T this is your X, X axis at T equal to zero is your X axis at every time T your domain is here ok, T equal to T big T something like this so the domain is upper of plane but we don't know, how to solve this?

But let us see by reducing this into canonical form can we solve it ok. So if you consider here A equal to 1 B is 0 C is minus C square ok C is minus C square what is here physical A, C is the speed of wave, wave speed and U is the amplitude ok U is the amplitude of the wave.

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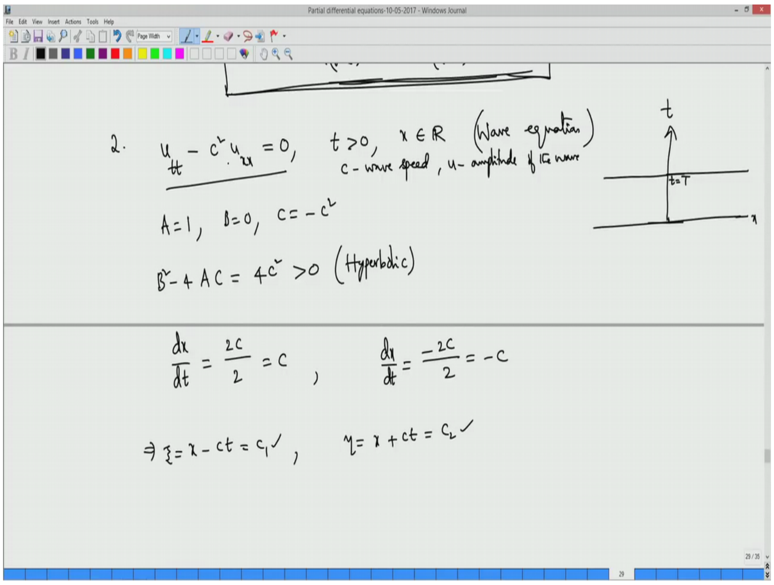

So will see so you consider the discriminant B square minus 4 A C is actually equal to B 0 again minus-minus plus 4 A C, C, C4, C square so implies, implies this is always positive because this C waves speed always positive and 4C square is always positive so it is a hyperbolic equation.

So that is what we know ok so what are the variables, what are the new variables? That ODE's if you solve you can get them so D Y by not D Y by so Y is X here so you have to consider D X by D T, D X by D T equal to B 0 square root of B square minus 4 A C that is 2C divided by 2A, so this is equal to C. What is the other equation? How again? D X by D T is minus 2C minus square root of B square minus 4 A C divided by 2A, so you have minus C.

So if you solve this you can see easily see that X minus (C A) C T ok equal to C1 and X plus C T equal to C2 so this is what if you solve these two ordinarily differential equation. So consider this as Xi this as your eta. Now what you need U T T and U X X you calculate them.

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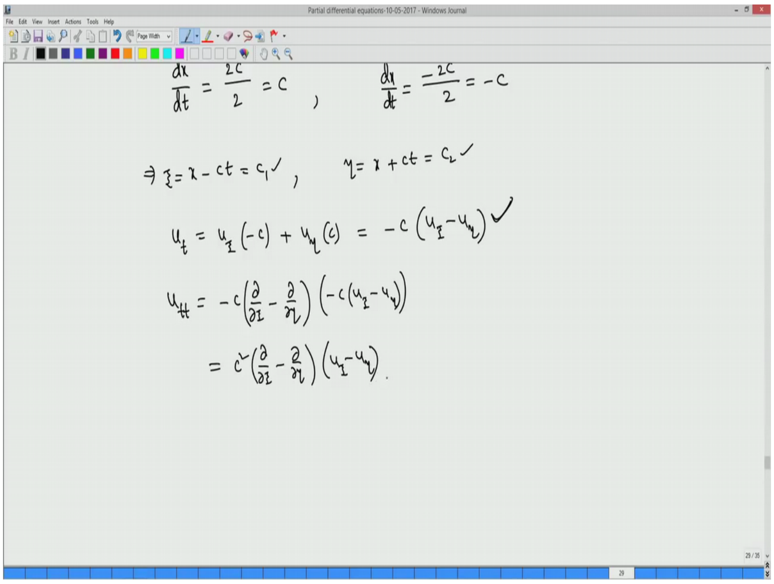

So U T so start with U T that is U Xi into Xi T, what Xi T? Minus C, Xi T is minus C plus U eta, eta T is plus C so together you see that minus C comes out U Xi minus U eta ok.

So this is what is my U T so U T T is from which from these you can get dou dou T that is minus C which is a constant dou dou Xi minus dou dou eta or U T so that is minus C times U Xi minus U eta ok. So C C common so you have C square dou dou Xi minus dou dou eta just like earlier what we have done this is acting on U Xi minus U eta.

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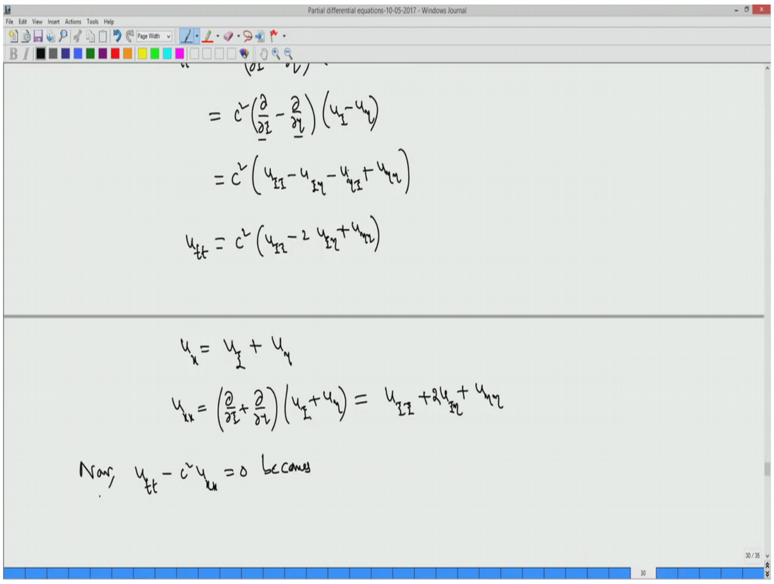

So what happens here so this will become C square now you simply differentiate you see that U Xi Xi minus U Xi eta minus now you differentiate this with respect to eta you will see that U eta Xi minus U eta eta.

Mixed derivatives are continuous so you have C square U Xi Xi minus 2Xi eta minus, minus-minus plus this is actually plus, plus U eta eta is, this is my U T T. What is your U X? U X is U Xi into Xi X, Xi X is dou Xi by dou X that is one plus U eta, eta X again that is one. So you have simply this one so you have U X X is simply dou dou Xi lus dou dou eta acting on U Xi plus U eta. So this is simply U Xi Xi plus U Xi eta similarly U eta Xi so you have two Xi, 2 U Xi eta plus U eta eta.

If you combine this you take U T T that is C square into U Xi Xi or rather U T T minus C square U X X equal to 0 becomes.

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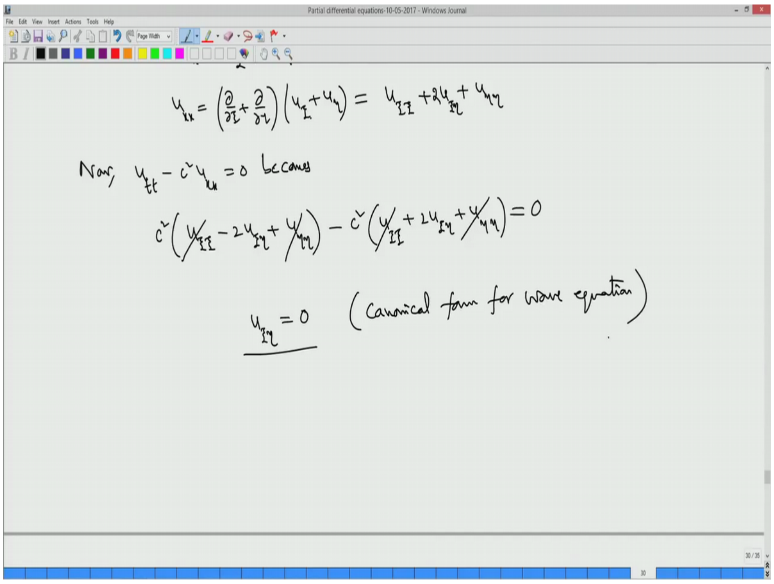

Now this equation the given equation becomes C square (())(30:38) pickup U T T that is this U Xi Xi minus 2 U Xi eta plus U eta eta ok minus C square times this one so that is U Xi Xi plus 2 U Xi eta plus U eta eta equal to 0 ok. So if you see that you can see that these are all cancelling out same derivatives only mixed derivatives that is remain that is going to be minus 4 C square U Xi eta equal to 0 minus C (())(31:13) is 0 so 4 cannot be 0 C square because C is the wave speed which is non-zero C square also cannot be zero.

So that means U Xi eta is 0 so this is your canonical form, canonical form for wave equation in the new variables.

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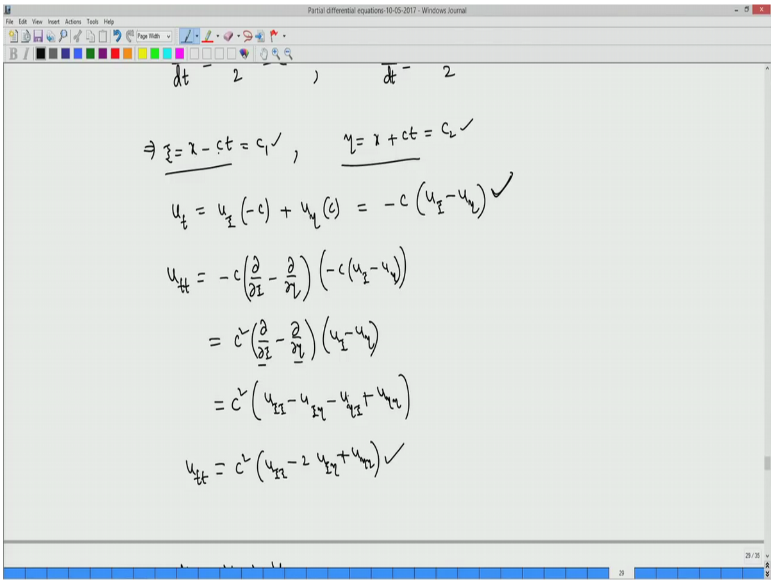

So these are called Xi and eta variables these are called characteristics ok, these are called characteristics variables because along these along with these along these curves X minus C T and X plus C T ok the (par) this partial differential equation actually becomes ODE that you need not worry just let us see so this is the canonical form for which now I can find the solution.

So how do I find the solution?

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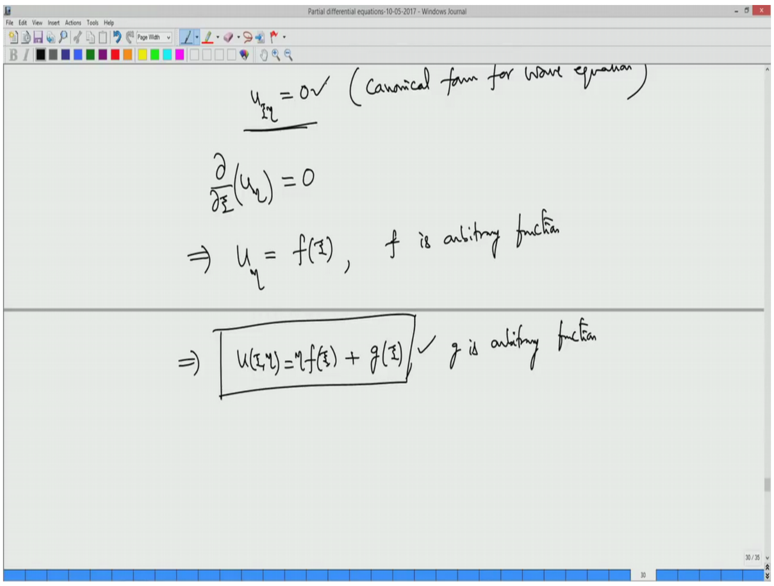

You see that U Xi first which is if you differentiate dou dou eta so what is this one, this you can rewrite dou dou Xi of U eta equal to 0 now if you integrate you see that U eta equal to some function of Xi where F is arbitrary, F is arbitrary function. Now one more differentiation (U) then you get U Xi eta equal to F Xi into eta ok plus some G of Xi, G is also arbitrary where G is arbitrary function.

So this is the general solution of this canonical form.

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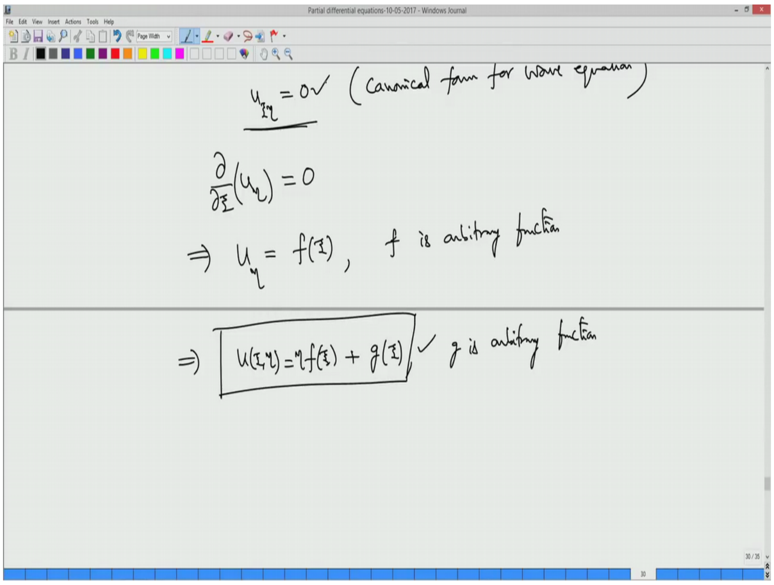

So in the new variables U of X T and old variables U of X T is this eta A into F of Xi right, this eta into F of Xi is it, no U eta is F of Xi yeah its fine so this should be U eta you are differentiating with respect to eta so it should be eta ok, U eta is (())(33:39) if you differentiate with respect to (eta) Xi then it will be U Xi eta this will be 0.

So your original equation will get ok so this should be function of eta so you have function of eta here so this eta you can combine in this eta into F of eta X is another (arbi) F is arbitrary function into another function eta that will be another arbitrary function you call this C1 of eta plus C2 of Xi call these G is same as C2 so this is your uhh this is your ok if you write this as the same Xi and eta you can say that this is your general solution where C1 and C2 are arbitrary functions ok.

So general solution because you have two arbitrary functions involved in your partial differential solution of the partial differential equation. Now in the old variables U X T what happens if you rewrite C1, C1 of eta is X plus C T and C2 is, C2 of Xi is X minus C T so this is your general solution of the wave equation in the old variables the partial differential equation is this one, so this is your wave equation who's general solution is simply two arbitrary functions each function takes, one function takes X plus C T as a ordinate other function takes X minus C T ok.

So this is your general (solu) so you can actually find general solution of the partial differential equation which is a wave equation once you reduce into canonical form. So with this we will close the video and we will stop here and we will look at other two examples of the partial differential equations those are elliptic and parabolic case we will reduce them, we will reduce those equations with variable coefficients into canonical form in the next video, thank you very much.