Differential Equations For Engineers
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Lecture 42
Reduction to canonical form for equations with constant coefficients

Well come back, so in this video will look into the examples of reducing linear partial differential equation of second order will try to reduce them into normal forms uhh, so the simplest examples that we consider first with equations with constant coefficients. So linear second order partial differential equations with constant coefficients we consider and (we have the) with the process, we land in the last few videos we reduce this equation into canonical form or simpler form into new variables.

So in equation in new variables Xi and eta so that if that is simplified equation if it is solvable will try to solve ok, so that is what will see, you start with an example.

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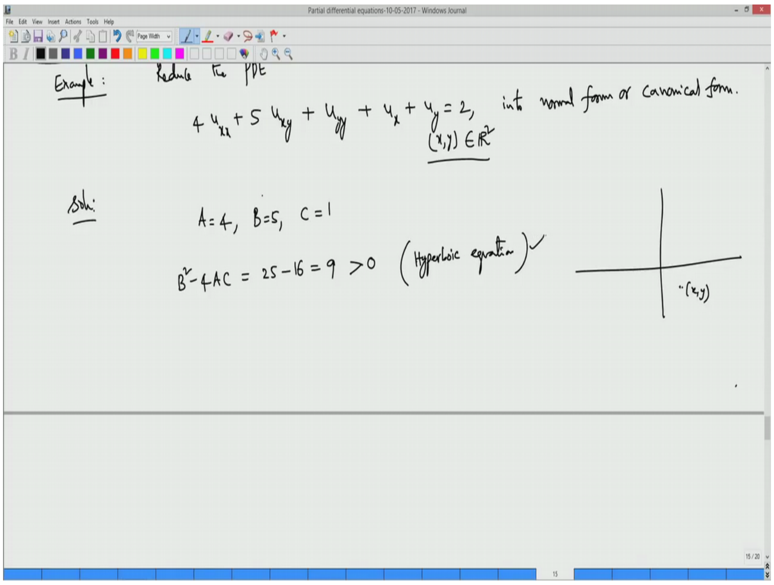

So reduce this if an example of each example we do so for each case of hyperbolic, elliptic and partial parabolic in each of these case will do an example. So will start with an example here so reduce, reduce the equation, reduce the PDE that is partial differential equation four time U XX plus 5 times U XY plus U YY, plus UX plus U Y equal to 2.

So this is the equation into canonical form into normal form or a canonical form. So what is the solution? Solution is we have to reduce this into form, so what is that? So just consider this is your X, so A is 4 just observe that the A is 4, B is 5, C is 1 and D E is 1 1 so that we don’t really need so what you have to see is the discriminate, discriminate is B square minus 4 AC just look at B square minus 4 AC its value is 25 minus 16, so this is going to be 9, B square minus 4 AC is 9 so which is positive.

So that means what we learned is it is hyperbolic equation, so equation is hyperbolic. Where is that domain? So first of all you have to see the domain. So it nothing is given that means X Y belongs to the plane so that is your domain of the differential equation, partial differential equation. So in this you take any point because B square minus 4 AC is, this is the coefficient, coefficients are valid everywhere ok.

So this is a hyperbolic equation in the plane. So this is your domain, so for every XY this domain this equation is hyperbolic equation so that is clear so that is what we can conclude by looking a the discriminate.

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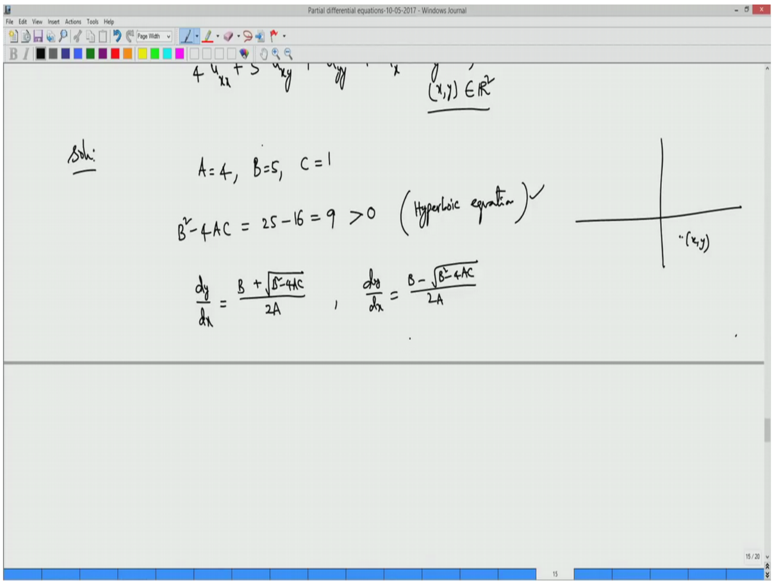

Now how do I get my new variables? Are you consider D Y by D X equal to B plus or minus plus first of all plus you choose one B square minus 4 AC by 2A, this is one and you consider the other one B minus square root of B square minus 4 AC divide by 2A so this is what we have seen.

If you consider that is if you make A star and C star equal to zero, ok this is what you see that the you will see that if you look for those transmission from XY to Xi eta you see that these are the ODE’s you have to solve that fixes your Xi and eta.

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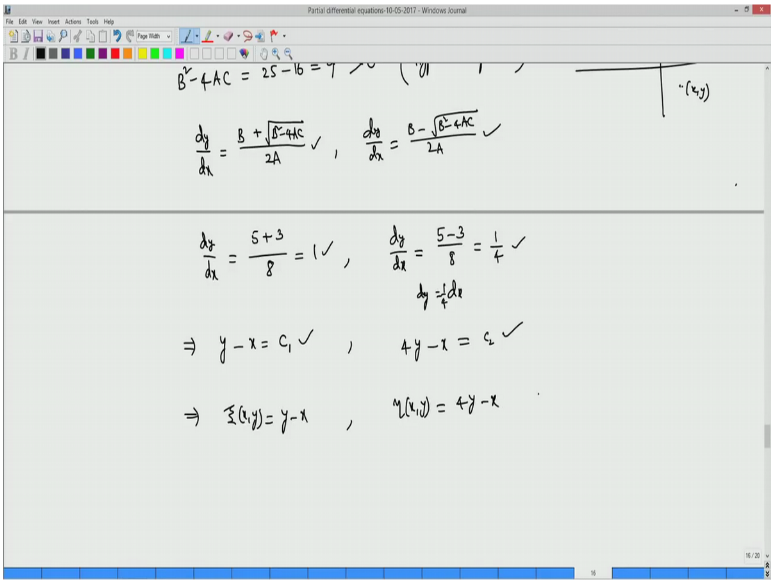

So if you see this D Y by D X, B is 5 plus square root of B square 4 AC is 9, so that is 3 divide by 8 so you have 1 and in this case the other ODE becomes D Y by D X equal to 5 minus 3 divide by 8 so this means 5 minus 3 that is 2, so 1 by 4 ok.

So these are the two ODE’s you have to solve D Y by D X equal to 1 that will give you Y minus X equal to constant C1 and what happens here D Y, 4 D Y minus D X, 4 D Y minus D X equal to 0 so this you can directly integrate and to get uhh writes or you simply D Y equal to 1 by 4 D X so this you integrate both sides to get Y equal to 1 by 4 X plus C2 so if you rewrite this is like 4 Y bring this minus X this side so you have this is equal to 2. So this the, these are the two equations, two solution general solutions of these two ODE's

So this from this we have seen that Xi of X Y equal to Y minus X and eta of X Y is 4 Y minus X is how you fix your Xi and eta.

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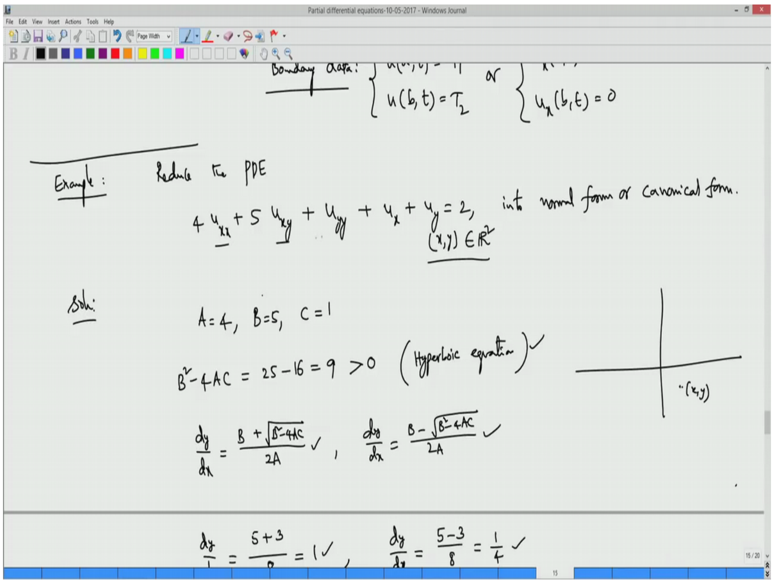

Now, so what you have so your equation is involving U X X, U X Y, U Y Y, U X, U Y, so you need to find U X and U Y.

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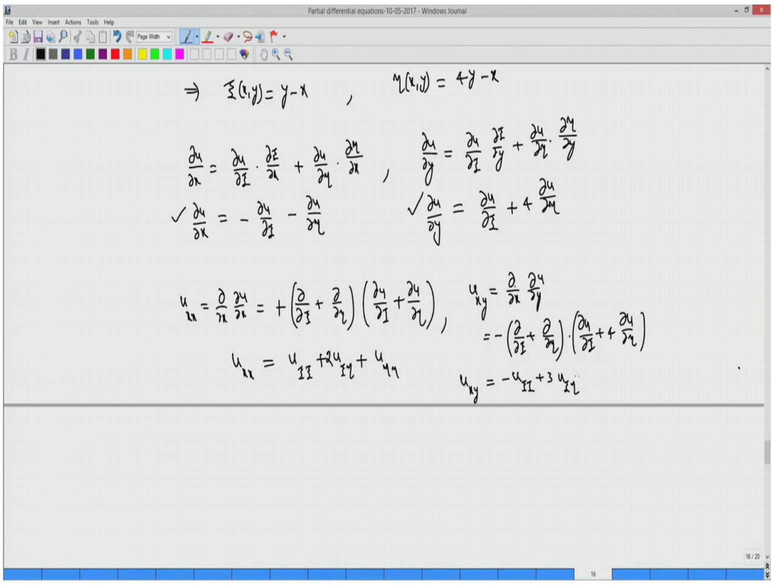

So just like dou dou X dou U by dou X in terms of a new variables so dou U by dou Xi into dou Xi by dou X plus dou U by dou eta into dou eta by dou X. So this will give me dou Xi by dou X that is minus dou U by dou Xi plus dou eta by dou X again you have minus, minus dou U by dou eta. So that is what my dou U by dou X.

And what happens dou U by dou Y replace X by Y so you have dou U by dou Xi. Dou U by dou X you have to replace with dou Xi by dou Y plus dou U by dou eta, dou eta by dou Y so this becomes dou U by dou Y becomes dou Xi by dou Y is1 again so you have dou Xi by dou (wa) dou U by dou Xi and here plus dou eta by dou Y is here up four times dou U by dou eta. So from this so you already got what is U Xi, U Y, U X and U Y but you need U X X so look at this U X X to get U X X what you get is if you want this that is dou square U by dou dou X of U so if you apply this what is dou dou X to get simply you ignore.

So that is what is a minus dou dou Xi plus dou dou eta is your dou dou X one more time that so that will give you minus minus plus dou U by dou Xi plus dou U by dou eta. So if you apply this, this you do it so you have U Xi Xi plus U Xi eta plus U Xi eta Xi there are continuous so you have two times so U eta eta so that's all. so this is what use your U X X and from here you can also get U Y Y or first of all you will get U X Y, U X Y is one is minus dou dou X this is like dou dou X, dou dou Y if you or dou dou Y dou dou X of U.

So this both ways you can take. So dou dou X I replace with minus dou dou Xi plus dou dou eta into dou U by dou Y you simply write (dou) whatever you have here dou U by dou Xi plus 4 times dou U by dou eta, so what is this one? So this will give me U X Y as minus, minus U Xi Xi and you have minus U Xi eta and here plus 4U Xi eta so that will become plus 3 U Xi eta, Xi eta, U Xi eta or U eta Xi both are continuous using that the second derivatives or mixed derivative are continuous, ok.

If U Xi eta those mixed derivative are continuous, the it should be same so that is from the calculus ok and then finally get 4 times U eta eta. So because your working here we for the equation that is uhh with constant coefficient that is why look easier. You see this when you are expanding so you have to be careful if these coefficients are functions of X Y ok. So you have to carefully have to expand it otherwise you may go wrong.

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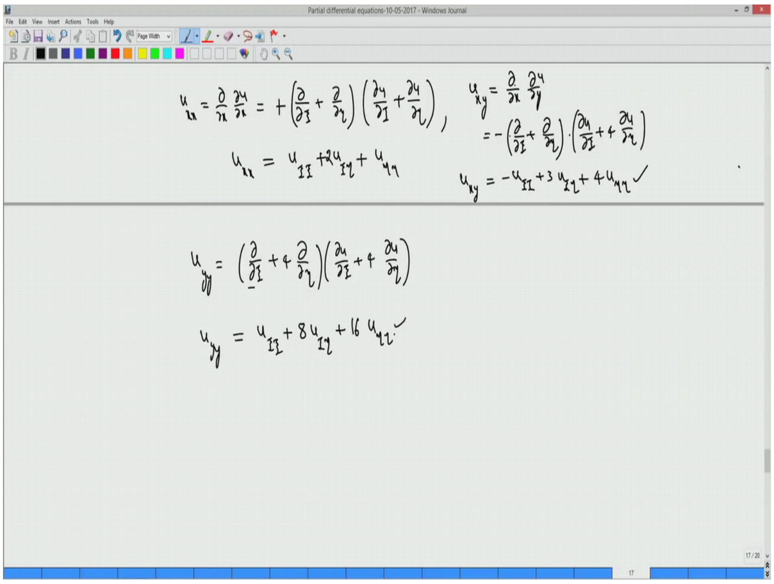

So is what is your U X Y so same way you can get U Y Y, U Y Y is simply dou dou Xi plus 4 dou dou eta that is my dou dou Y from here ok and then dou U by dou Y you write again dou U by dou Xi plus 4 times dou U by dou eta so this becomes U Y Y becomes U Xi Xi this mixed derivative plus this mixed derivative this one and this one so together will give me 8 U Xi eta plus 16 U eta eta is just like A plus B whole square type ok.

So if you do this you can easily see U Xi Xi square so U Xi eta, U X Y so you have two times 4 ok, 2 4 8 and 4 square finally 4 square eta eta so this is how you get ok. So you know all this U X X, U Y Y and U X Y, U X and U Y in terms of Xi and eta only just put it into the equation.

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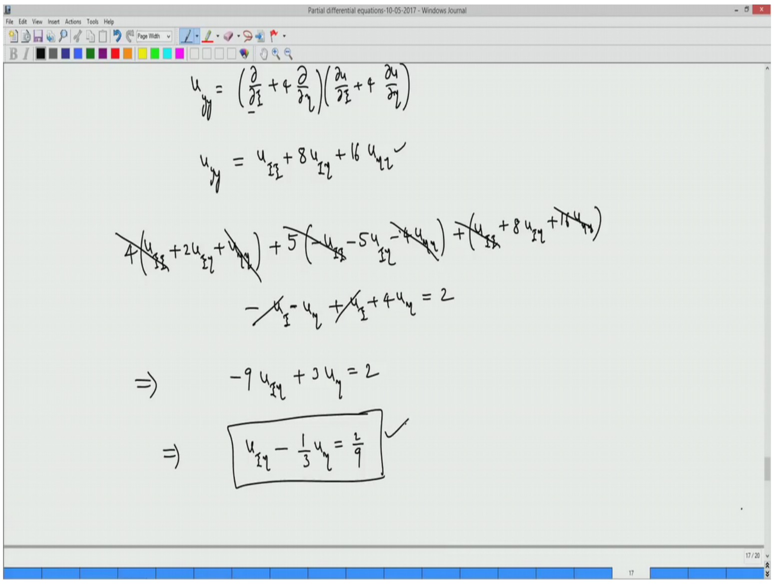

So, what is the equation? 4 times now your equation is 4 times U X, U X is 4 U Xi Xi plus 2 U Xi eta plus 4 U eta eta this is what is my U X X next is 5 times U mixed derivative that is U Xi Xi minus U Xi Xi then I think I made a mistake so it should be minus 4 right so this is plus and this is everywhere minus ok.

So mixed derivative so I have a minus and you have U Xi and this is also 4 that is going to be actually 5 minus 5, 4 U Xi eta and 4 U Xi eta if you apply to this here and if you apply eta here so you get 1 so total 5, so you have this one. So that is correct now this is all with minus ok then finally so you see that U Xi Xi minus 5 U Xi eta minus 4 U eta eta plus U Y Y, U Y Y is U Xi Xi plus 8 U Xi eta plus 16 U eta eta plus U X, U X is simply U Xi plus U eta that is what you have. U Xi so you have minus so that is minus ok minus U Xi minus U eta.

That is U X plus U Y, U Y is U Xi plus 4 eta equal to 2. So this is what if you substitute this is what is become now collect the terms U Xi Xi terms 4 plus this is U right, so 4 U Xi Xi, 5 U Xi Xi so I have minus 5 this cancel so you have got gone. So similarly 4 eta eta 16 eta eta's 20 and here minus 20 so you have this is gone, so what you are left with is only 8 U Xi eta and you have a minus 25, 8 minus 25 is minus 17, 17 U Xi eta and you have plus 8 U Xi eta so that is minus 9.

So together minus 9 U Xi eta and here also you get U Xi Xi goes and this becomes plus 3 U Xi U eta equal to 2. So this gives me U Xi eta minus 1 by 3 U eta equal to 2 by 9. So this is what is the equation now, ok. So this is mixed derivative so that means hyperbolic you can easily see so can we solve this equation after reduction that maybe tricky we don't know. So we can solve it actually we can solve it, how do we solve it?

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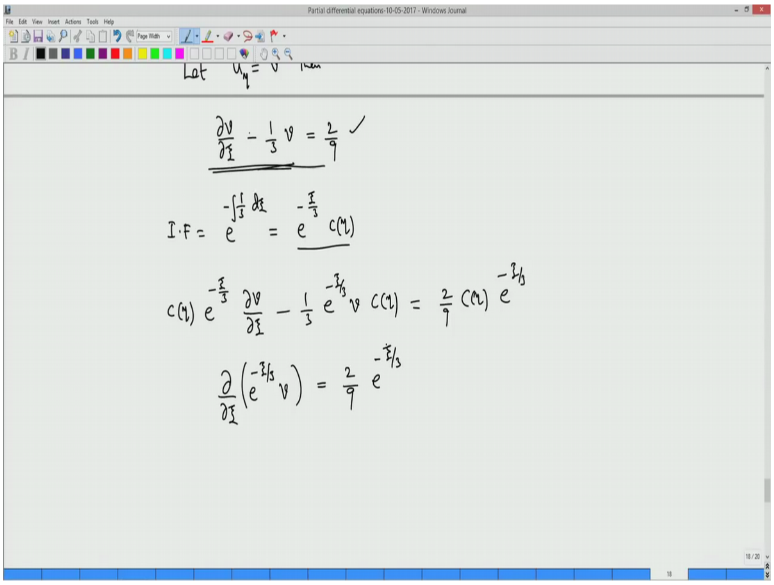

This is a partial differential equation I am just giving you the method to solve this one just using basic methods of ordinary differential equations. So by considering this is what is a canonical form reduce (this) the given partial differential equation of second order into another partial differential equation of second order but in the canonical form nice form simpler form, this is called normal from or canonical form.

So this we can solve now using the techniques that we learned from learned for ordinary differential equations. So if you consider this U eta as some V ok V just take it so then, then what happens to this equation? You have dou V by dou Xi minus 1 by 3 V equal to 2 by 9 ok. This is a ordinary differential equation right, ok. So how do I solve this? So what is the integrating factor? So assumed it looks like a first order ordinary differential equation linear ODE first order linear ODE so E power if I multiply E power integral P is minus 1 by 3, D Xi ok so this is my integrating factor.

Integrating factor is this, so you have E power minus Xi by 3. Xi by 3 plus now this is a PDE so this is a integrating factor right so this is the integrating factor and any cost and into that also integrating factor here we are dealing with the partial differential equation. So this constant if it is a ordinary differential equation this is a constant any constant time is fine.

But here C can be any function of eta so you can choose C times some eta, C of eta into this acts as integrating factor so you substitute you multiply that (C power) E power minus Xi by

3 into C Xi, C eta into dou V by dou Xi minus 1 by 3 E power minus Xi by 3, V into C or eta equal to 2 by 9 C of eta E power minus Xi by 3. That is what I multiplied both sides if you do that so what is this one, this is actually equal to this becomes, this is dou dou Xi of E power minus Xi by 3 into V equal to 2 by 9 E power minus Xi by 3, ok.

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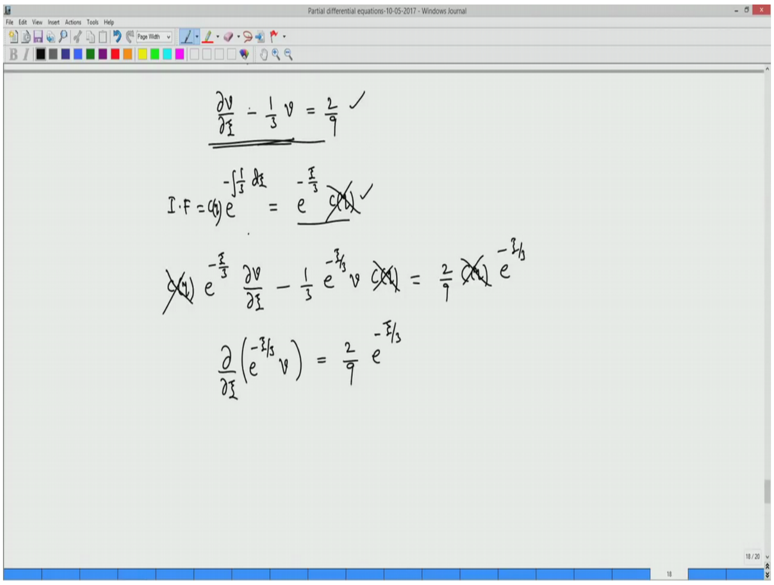

So you need not consider this ok, so finally it is not it is cancelling both sides so is what is also integrating factors can be anything but just for the sake of completeness is not cast at any constant, its constant is actually C of eta, you can choose as a C of function of eta because this is a partial differential equation we are using the methods of ODE's so its possible we can solve.

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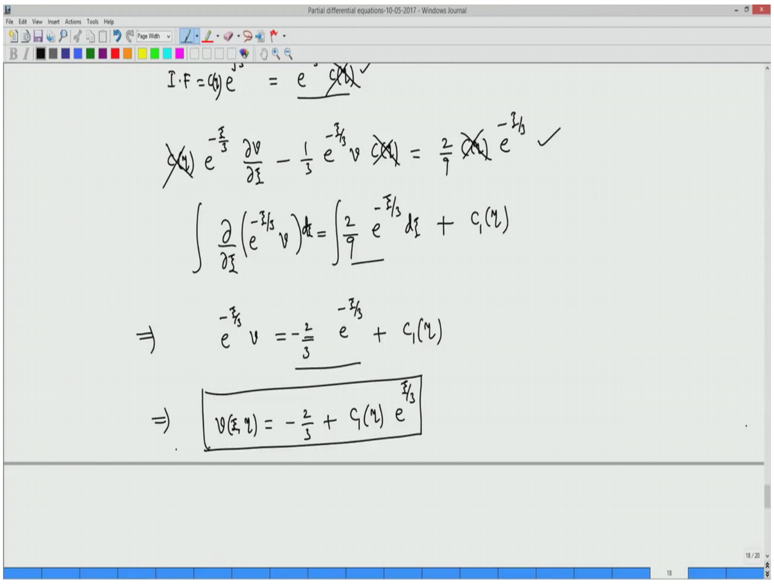

So this is what you get this is the equation now you can integrate both sides so if you integrate with respect to D Xi, so what is that integration constant that can be any function of eta so that I am calling C1 of eta ok if you differentiate now you get back what you want this one ok. So this will give me E power minus Xi by 3 into V of Xi eta ok so this is V in unknown function so this is equal to so this is 2 by 9 comes out E power minus 1 by 3 is minus E power minus Xi by 3 so minus minus 6 by 9 that is 2 by 3.

So this is what if you differentiate this you will get this integral so this is plus C1 of eta. So this implies V of X V of Xi Xi eta is actually now you take this minus 2 by 3 if you bring it plus C1 of eta into E power Xi by 3 so this is what I found as a function of as a V, what is V? V is U eta.

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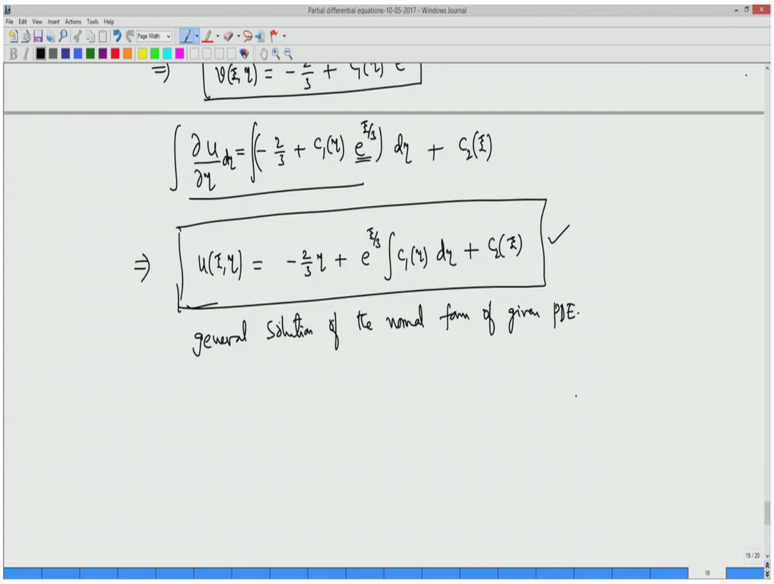

So U eta, U of Xi eta so that is U eta equal to minus 2 by 3 plus C1 eta into E power Xi by 3 but this is a but this may not be, you may not be able to integrate this is like dou U by dou eta so this is what is the equation now.

So if you want to get U Xi eta so you have to integrate this equation, can we integrate this? Actually you can right, so you simply integrate both sides with respect to dou eta D eta so integral, integral of this whole thing with D eta now you are integrating with respect to eta and if some new function that is the integration constant that is Xi, so this will give me U of Xi eta finally equal to minus 2 by 3 into eta plus integral so this is a constant nothing to do with eta so you have E power Xi by 3 comes out integral of C1 of eta D eta that integration of whatever or anything arbitrary constant with integration plus C2 of Xi.

So this is what is your general solution of the partial differential equation ok, this is the general solution of the PDE of the standard form, of the normal form, canonical form or normal form of given PDE ok why I am calling so general solution again so (order) in ordinary differential equations second order equation when you integrate the general solution should involve to arbitrary constants.

So we call in the partial differential equation we call the general solution if it involves two arbitrary functions ok so that is what you have here so U Xi eta is actually C1 of eta that is one arbitrary function and C2 of Xi that is another (arbi) arbitrary function so we have two arbitrary functions here so that is why I am calling it general solution like same way we have done for ordinary differential equations.

So this is the general solution like this you can solve given equation you reduce this into standard form if possible it can be solved if it is after (reduc) after reduction if it is in a simpler form then it can be solved it can be solved for its general solution so this is the solution for the reduction equation.

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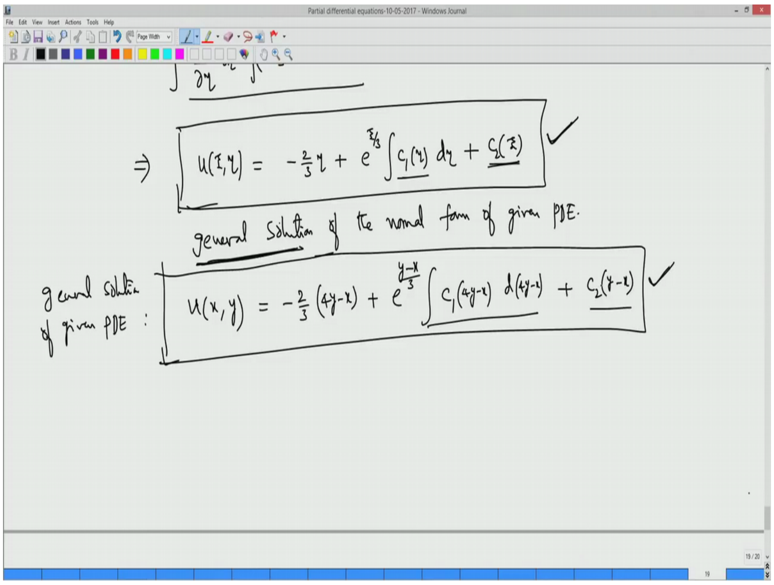

Now to get back my new variables so the actual partial differential equation you can also simply replace X Xi (Y) Xi and eta as your function Xi and eta, Y minus X and Y minus 4 Y minus X ok.

So that you can replace and it becomes function of this is a function of U of Xi eta becomes U of function of X Y this becomes minus 2 by 3 eta is 4 Y minus X plus eta (X) Y minus X by 3 the integral C1 of 4 Y minus X so this is nothing so this is eta D eta so eta eta so whatever the variable so if is replace so you can say that if before we minus X D eta is D of 4 Y minus X ok. If you write again plus C2 of Y minus X Xi is that so this is the function of general solution of given partial differential equation, ok.

This is the general solution of given PDE. Because this also involves this is arbitrary function and this is also arbitrary function so this is your function, everything in terms of old variables this X and Y ok so this is the form. You give any C1 and C2 and this actually solves the given differential equation ok so you chose any function C1 and C2 whatever maybe function it will satisfy the given partial differential equation.

So this is how we solve we can first reduce this given partial differential equation into standard form and then see the normal form reduce normal form if it can be solvable if it is in a simpler form try to integrate directly so only simpler method are simply integration ok you don't know any method how to solve a partial differential equation so if you can only integrate those methods the elementary methods you can apply and get your general solution.

Now will see the other example where you see different other type of equation so will consider one more example.

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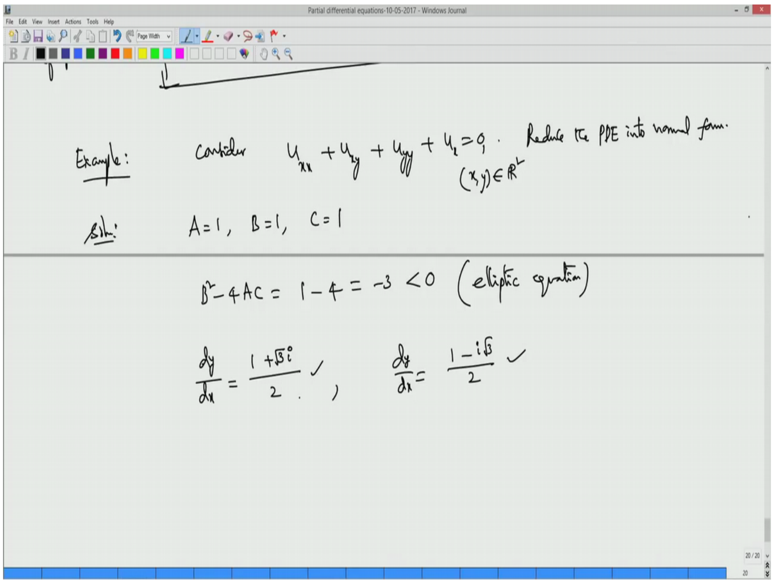

So consider this U X X so the equation is U X X plus U X Y plus U Y Y plus U X equal to zero reduce into canonical form reduce the equation reduce the PDE into normal form so how do we solve again we see look at A B C A is 1 B is 1 and C is 1 so B square minus 4 A C value is 1 minus 4 that is minus 3 so this is negative immediately you can say that the equation that is valid in the full plane is elliptic, elliptic equation.

So equation is elliptic I can directly conclude by just looking at the discriminate B square minus 4 A C. Now you consider your Xi eta from this by solving this ODE's B, B is 1 plus square root of B square minus 4 A C that is 3 I root 3 I ok so divide by 2A so that is 2 and also other equation is D Y by D X this is equal to B minus square root of B square minus 4 A C that is minus I root 3 divide by 2.

So these are the two ODE's so you have to solve so if you do this if you try to solve this partial differential equations.

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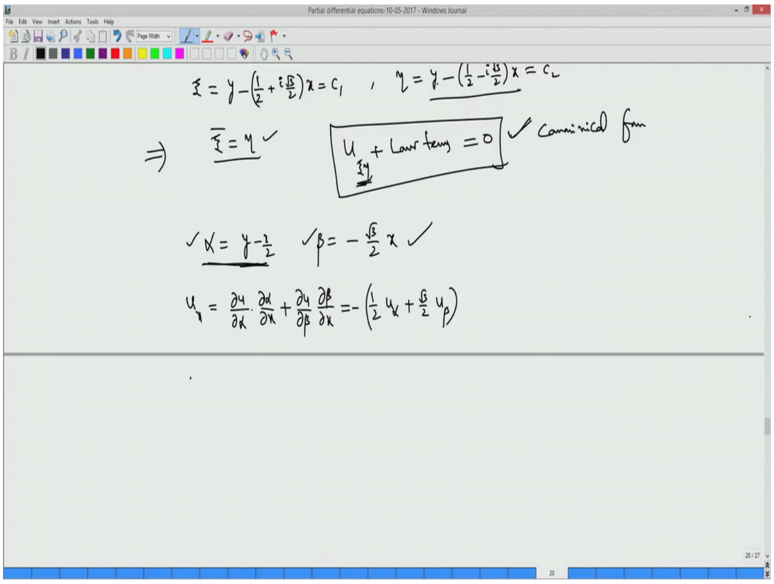

So by solving this ODE's you can say that your Xi, Xi of X Y will be whatever is the solution, so the solution is or you simply solve it first so Y is this is D Y equal to 1 plus or minus 1 by 2 plus I root 3 by 2 times D X so this becomes Y minus 1 by 2 plus I root 3 by 2 into X equal to C1 so this should be your Xi of X Y and similarly here because these are complex things so you can see that if you directly do D Y equal to minus D Y equal to half minus I root 3 by 2 the X so you get Y, Y minus 1 by 2 minus I root 3 by 2 the X equal to C2.

So this is what you get this is your eta. So you can see that Xi equal to Xi bar is actually eta ok so eta is nothing but Xi bar that is difference easily see so this is your Xi bar if this is your Xi, Xi bar is simply eta. So when you have this one so I know that if you actually reduce if you reduce if you now try to look for U X X, U X Y and U Y Y and U X in terms of this new Xi this variable Xi and eta you know that it will become U Xi eta plus lower order terms equal to zero.

That is what it will become ok, this is the canonical form you will get you can expect, canonical form you can expect but then these are complex valued functions so there is no meaning so partial derivati:ves with respect to a complex valued functions variables, variables are complex there is no meaning ok so what we do is we don't do this in between intermediate step of finding the canonical form so immediately once you see that the complex you take the real part so alpha is real part Xi plus eta by 2 that is simply Y.

Beta is these are the variables you look for ok instead of Xi and eta, beta is you may also Xi minus eta divide by 2 Y that is simply minus so here it will be Y minus so real part Y minus

Xi minus eta by 2 Xi plus eta by 2 will be Xi minus X by 2, Y minus X by 2, beta will be simply root 3 to the minus root 3 by 2 X, ok. So this is what you see our place actually so I times yeah minus so doesn't matter so this is what you have so with from Xi if you take alpha beta as real part of Xi this is what you get imaginary part of Xi you take and this is what you get, ok.

Or you can also choose real part of eta that is same this is your alpha real part of eta if you choose here this going to be plus you can choose any one of them so these are the new variables if you look far. Then you try to see what is your U X, U X is dou U by dou alpha dou alpha by dou X plus dou U by dou beta, dou beta by dou X so this is what in terms of Xi not in terms of Xi and eta and their complex functions you choose only real and imaginary parts of the each and any Xi and eta ok, then you can choose these new variables as alpha and beta.

So in terms of these alpha beta you expect this canonical form become U alpha alpha plus U beta beta plus lower order terms that is what we will get it. So U X is this so that is so U alpha by A is a minus half U alpha and dou beta by dou X is root 3 by 2 U beta so you can choose simply for Xi so that you have minus minus that is going to be plus ok. So maybe you can expect the calculation will be simpler if you choose maybe minus root 3 by 2 because you see minus both places so I can take it out.

So you can choose ok I have chosen alpha beta as real and imaginary parts of Xi you can also choose real and imaginary parts of eta, there is no issue ok so you can choose so this is what my U X.

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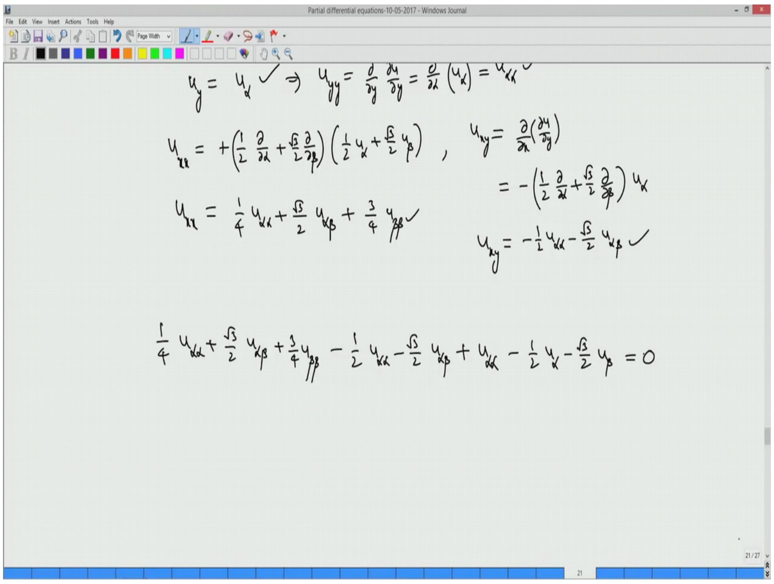

U Y is same thing dou U by dou alpha so that is U alpha into dou alpha by dou Y, instead of dou alpha by dou X you have dou alpha by dou Y that is simply one so you have U alpha and plus dou beta by dou Y that is there is no function of Y for beta, so that is zero.

So I have simply dou U by dou Y simpler, simpler form so that's it so you have U X X is minus (minu) half dou dou alpha plus root 3 by 2 dou dou beta into minus minus plus again so half U alpha plus root 3 by 2 U beta. So this is simply 1 by 4 U alpha, alpha plus so what is this one? 2 A B so then this is root 3 by 4, 2 root 3 by 4 that is root 3 by 2 U alpha beta plus 3 by 4 U bet beta. So this was my U X X, U X Y so U X Y also you can calculate that is dou dou X of dou U by dou Y so I know what is my dou U by dou Y that is, dou dou X you can replace from here.

So from this minus half dou dou alpha plus root 3 by 2 dou dou beta is my dou dou X, so this into dou U by dou Y is U alpha so that everything in terms of alpha so you have 1 by 2 U alpha alpha minus root 3 by 2 U alpha beta and what happens from here you can get U Y Y also, U Y Y is simply U alpha alpha. What is this, how do we get this? Dou dou Y of dou U by dou Y. what is dou dou Y is? Dou dou alpha, dou dou alpha of dou U by dou Y is U alpha that is U alpha alpha.

So you have got everything, U X X, U X Y, ok U X X , U X Y, and U Y Y so U X ,U Y are already here so if you substitute into the equation then you get the canonical form that maybe either solvable or not depending on its simplicity. So you have U X X that is 1 by 4 U alpha

alpha plus root 3 by 2, U alpha beta plus 3 by 4 U beta beta plus U X Y that is minus 1 by 2 U alpha alpha minus root 3 by 2 U alpha beta.

Now plus U Y Y that is U alpha alpha ok plus U X, U X is what is U X is half U alpha and you actually have minus, minus minus root 3 by 2 U beta equal to zero, so this is what is your equation so this is your U X (U X)U Y Y and this is your U X Y and this is your X X.

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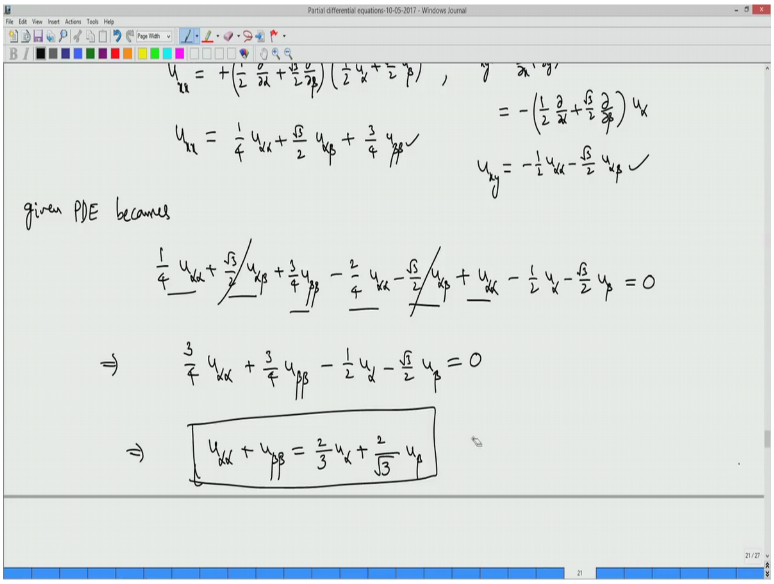

So the equation becomes PDE becomes given PDE, given PDE becomes this one so simplest now you can simplify and you see that 1 by 4 minus half so that is going to be see this is 2 by 4 so you have minus 1 by 4 U alpha alpha.

So that is what U alpha alpha terms and plus 1 so plus 1 if you say so you have plus 1 is 3 by 4 together so 1 minus 1 by 4 is 3 by 4 so 3 by 4 U alpha alpha and what happens to U alpha beta this one and this one cancel so this gets cancelled and U beta beta I have only this one so you have again 3 by 4 U beta beta now this also gone straight so we have considered these and what you are left with is minus 1 by 2 U alpha minus root 3 by 2 U beta equal to zero.

So this is your canonical form so that is U alpha alpha plus U beta beta equal to 4 by 3 that is 2 by, 4 by 3 into half 2 by 3 U alpha plus take it to the other side 4 by 3, 2 root 3 by 3 ok. So that is going to be 2 by root 3 into U beta so this is what the equation becomes this is the canonical form this is the elliptic canonical form ok that is what exactly we have seen so use a eta in canonical from in terms of this new variables alpha beta and that becomes if you actually do two times this is what you will get instead you can directly get into the, you can directly write in terms of X Y ok.

So you choose an alpha beta as real and imaginary parts of Xi and eta so have not done not made two steps like we have done for the general equation. Get the canonical form reduce the equation in term of Xi and eta where Xi and eta are complex valued functions and then choose new variables alpha beta and then reduce that equation into a again one more form like this.

So instead you can avoid that intermediate step and directly chose the variables because you know Xi and eta you take the real part imaginary part as alpha beta that will become this canonical form this is elliptic equation but you cannot expect this to be solve like earlier case because you cannot integrate this is not easy ok this is a partial differential equation which you don't know still you cannot use any ordinary differential equation method to solve this elliptic equation.

So this is how you can reduce second order partial differential equation into canonical form so we have seen two examples one for hyperbolic one for elliptic so in the next video we will try to find canonical form of a linear second order partial differential equation will try to reduce so that equation.

Will first see will try to take partial parabolic type of equation just (())(38:50) looking at its discriminate if the discriminate is zero for a given PDE then it is a parabolic so such an equation we will reduce by choosing we have to choose a properly you will only get one variable either Xi or eta so you have to choose the other variable such a way that jacobian should not be zero, so will see that example in the next video, thank you very much.