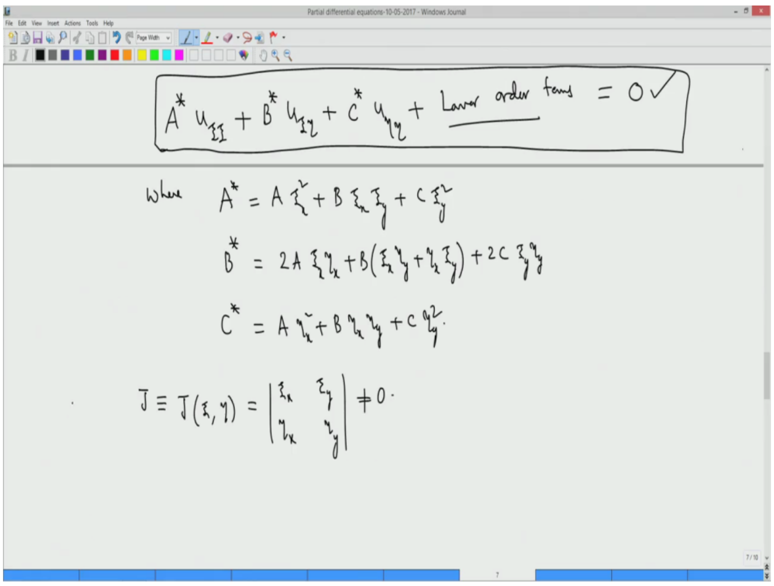
Differential Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
Department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-41.
Classification of second-order linear PDEs.

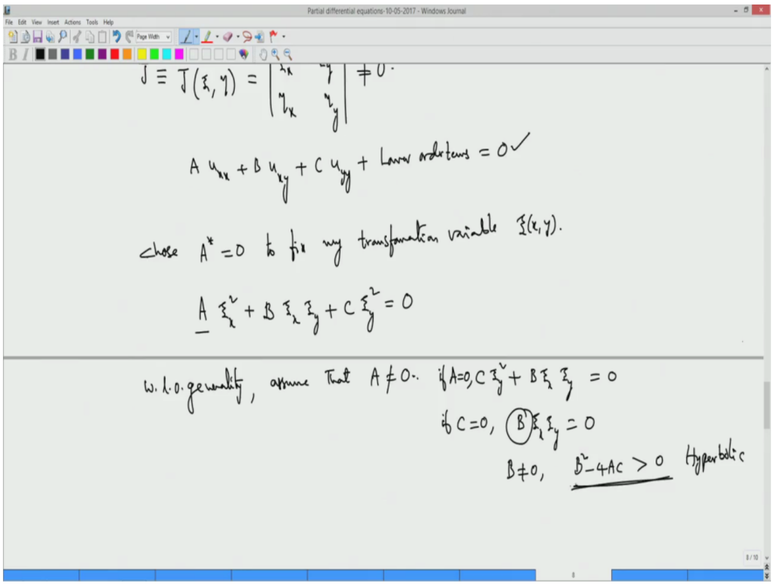
So we were discussing about the canonical forms, that is to reduce second-order linear partial differential equation into another second-order linear partial differential equation with a change of variable from x, y to xi, Eta variables, where xi and Eta are functions of x, y. We have seen in the discriminant is nonzero, we reduce that equation into canonical form of either the special form like mixed derivatives U xi Eta plus where all the terms are zero. So that is either hyperbolic or parabolic, we will see what happens is B square -4 AC, when the discriminant is zero, what happens, so we will get the another canonical form, so we will see that in this video.

(Refer Slide Time: 1:11)



So we will just look at the equation and B square -4 AC is actually equal to 0 so you get, you get this one, so what we feel is this equation, the general second-order equation becomes this equation, this is what we have seen with this A star, B star, C star. And you have the transformation is such that the Jacobian, Jacobian of, J means Jacobian of xi and Eta, so that is what is the function of x, the function of x, y. So this is, we also wrote J, so both are same, so J is actually J of xi, Eta, that is xi x, xi y, Eta x, y, this is nonzero, okay.

(Refer Slide Time: 2:10)



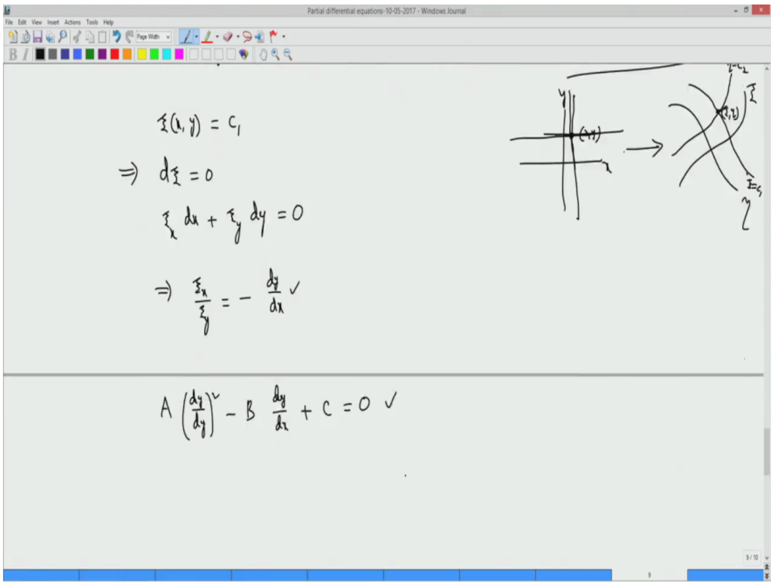
Okay, so this is what is nonzero, and your equation, originally equation is Uxx plus BU xy plus C Uyy plus dux, so you can say lower order terms, okay, lower order terms equal to 0, okay. So this is what we have, this is the original equation that we reduce into this kind of second-order, same second-order equation. So how do I fix my xi and Eta? So the best way to do this to fix this, to find this xi and Eta is to choose A star equal to 0 and C star equal to 0. So let us start with A star equal to 0, so let us make A star equal to 0. So by choosing, choose A star equal to 0. So can I fix, to fix my, to fix my transformation, transformation variable variable xi of x, y. Okay.

So that is why we are choosing. So A Star is, what is A, A is xi x square plus B xi x xi y plus C xi y square equal to 0. So if I choose this, I know a, B, C, they are from the given equation, so if I substitute and make it equal to 0, let a star equal to 0, so what you do is, suppose A is zero, and if A is zero, I can choose, I can, I assume that A is, without loss of generality I assume that A is nonzero, okay. So without loss of generality, assume that A equal to 0. If A equal to 0, okay, so I can write C xi y square plus B xi x xi y equal to 0. Okay.

If K equal to 0, okay, assume that A is nonzero. Without loss of generality you can assume that A is nonzero, if A is zero, this is what it becomes, okay. So the idea, so that I can divide xi y both sides, xi y square both sides, okay. I have xi x, xi x you divide, xi x square U divide both sides, then become, this becomes xi y. So you get like this, if you are equation, if A is zero, your equation becomes this, this is how you write it. If C is also zero, C is also zero, then what happens, your equation becomes xi x, xi y equal to 0, okay. So and we know that B cannot be zero, if B is zero, that means everything, there is no second-order differential equation.

So B, if A and C, both are there, it is already in the form of canonical form, okay, that is one of the canonical form is already there. So B should be nonzero and clearly B square -4 AC in that case is always positive, it is elliptic, sorry hyperbolic, hyperbolic, that is actually already in the normal form, okay. So we will see that, so once, your aim is to make these terms without having Uxx and U yy terms, you want to make it zero after transformation to xi and Eta. That means you choose your A star and C star zero, okay.

(Refer Slide Time: 5:49)



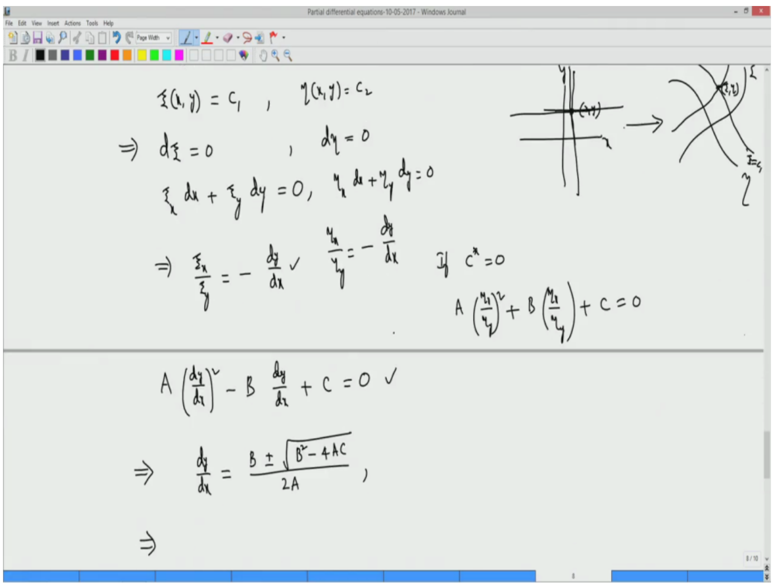
So first I make it at a star equal to 0, so you know A, B, C, if A is nonzero, you can assume, always assume that A is nonzero. If A is zero, C should be nonzero, okay. If C is also zero, that means B is already in the canonical form, already in the standard form, one of the standard forms, hyperbolic form, okay. So, severe, so we can assume that A is nonzero and C is also nonzero you can assume. If C is zero, C can be zero, okay, so you have this one. Now you divide with xi y square both sides, so if you do that, xi x by xi y whole square plus B xi x by xi y, okay plus C equal to 0, this is what you have if A is nonzero.

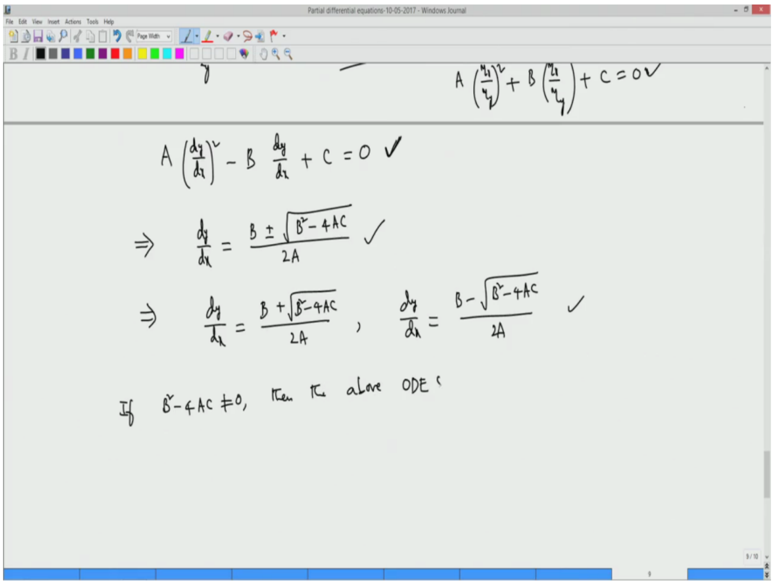
That is what we assume, so you have this. If A is zero and C is nonzero, you work with this, so you divide with xi x square, so you have, so in this case if A is zero, so what you have is C times xi y by xi x whole square plus B xi y by xi x equal to 0. Okay, this is what, this is the equation to work with. So you have this one when A is not equal to 0. Now what is that we are doing actually, you are taking this these xy variables into, transformed to xi, it variables, xi, Eta variables are, this is your xi and this is your Eta, okay. These kinds of variables, so xi, so any point x, y belongs to these 2 lines, this point is your x, y, so this straight line intersection.

So that means here also it belongs to the x, y points belongs to, corresponds to, when you do this, it will become xi, y, so it will be some other point here. So that will be parallel to this, this one, this is actually equal to xi equal to some constant C1, Eta equal to some constant C2. So that is the point, this is your xi and Eta, okay. So we are interested only, because we have chosen A star equal to 0, this involves only xi, so when you transform from here, the instrument to that domain, any point x, y becomes a point in xi, Eta, that is actually passing through this curve xi equal to some constant.

So that means you can have xi of x, y equal to constant C1. Okay. Now what is the total derivatives d xi equal to 0, so what is the total derivatives this means, this means xi x dx plus xi y dy equal to 0. So this implies but what we need is xi x by xi y, so xi x by xi y equal to minus dy by dx. So this you will not replace there, so this if you will put it here, so you can say that, if A star equal to 0, assuming that A is nonzero, so you have dy by dx, dy by dx whole square minus B dy by dx plus C equal to 0, so this is what you get. Okay.

(Refer Slide Time: 9:18)



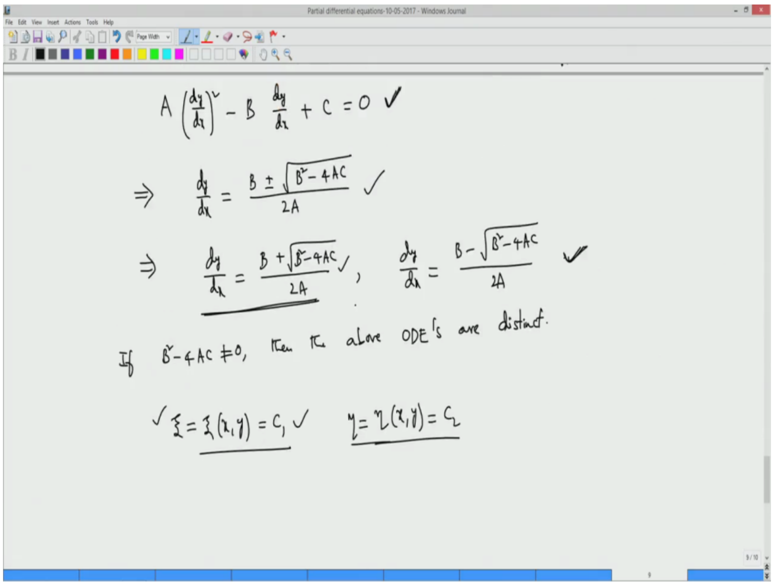


So now I can find what is my what is my dy by dx, okay, from this is the quadratic equation in dy by dx, so you can calculate dy by dx, okay. So dy by dx equal to B the someone square root of B square -4 AC divided by 2A. So this is what is your equation. This is, this is the differential equation, these are the 2 ordinary differential equations. If you solve them, you get your xi and Eta, so you can get, you just try to solve this, you get your xi. Okay, first we see that this is, this is what you get, if you work with same thing, if you choose B star equal to 0, rather C star equal to 0.

You choose the C star equal to 0 and if you choose, if you assume in this case that A is nonzero, so what it becomes, if C star, C star equal to 0, if C star equal to 0, okay, then what you get is A times Eta x. Eta x by Eta y whole square plus B times Eta x by Eta y plus C equal to 0. Okay. So again if we replace, again, C, this is also not true, so if any point x, y to xi and Eta, xi is constant and also Eta belongs to, so Eta of x, y equal to C2. And not only this, you have Eta of x, y equal to C2, again you have the same thing, d Eta equal to 0, total derivative.

So that means Eta x dx plus Eta y dy equal to 0, so what does it mean, so this implies you will get Eta x by Eta y is minus dy by dx. So if you absolute here, you get the same thing. So what you get is, finally you get the same thing, so in both the cases, you try to substitute this here, what you get is this one. So in both, for xi and Eta, if you make A star and C star equal to 0, what you end up is 2 ordinary differential equations, these 2, okay. So one is dy by dx equal to B plus minus square root of B square -4 AC by 2 A and other one is dy by dx equal to B minus square root of B square -4 AC by 2A. These 2 are 2 different odes if B square -4 AC is nonzero, okay.

(Refer Slide Time: 12:38)



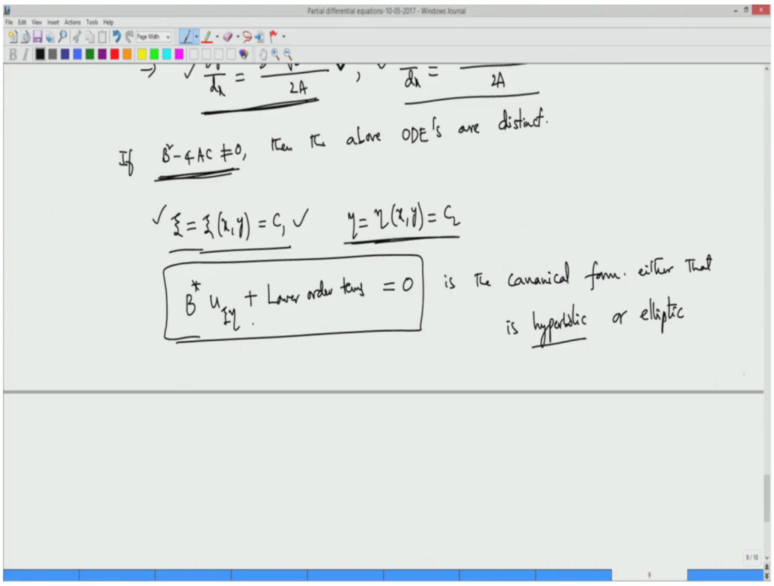
So observe that if B square -4 AC is nonzero, then I have 2 at odes, the above odes, above that odes are distinct, there are different, okay. So you solve them, if you solve them, what you get? So try to assume that this is simply, you try to solve them, what you end up is some function of x, y equal to constant. Okay. This is actually what you get is y equal to integral B, B, A, C are the functions of x, y, so you have dy by dx equal to functions of x, y. So this is, this is the ODE, as such you can solve it. So if you solve it, some, then what you get is let us call this xi of x, y is equal to constant. Okay.

This is your C1 and this one, you try to solve again with this, this if you solve, you end up finely, another arbitrary constant C2, this is what you get. So you can get your, by solving these 2 odes you get a general solution as xi of x, y, you call it xi of x, y, okay. That is your xi and this is your Eta, that is your Eta variable, okay. So if you solve this ODE, you will get, you call some F1 of x, y equal to constant and that is your xi, you take it as as your xi. And you do say for example F2 of x, y equal to constant, after solving this ODE, and once you get that, you call, you take that variable Eta as this is your F2. Okay.

So I myself is calling the xi itself, after solving this you get some function of x, y, we call it xi of x, y, this is my new variable xi. Now I have xi of x, y equal to, whatever xi is equal to some combination of x, y. Similarly what, when you solve this one, which myself is calling, here I am defining as Eta of x, y equal to C2 as a general solution, so that is, so whatever the combination of x, y is, my new variable Eta. Okay. So these are, those the curves actually, those are basic curves. So whatever you solve these odes, they are nothing but the solution, the solution curves of those odes nothing but these comes here, after transformation, because you started with those, from them you derived, okay.

So with this, now you know what exactly those xi and Eta from these 2 odes. So if you are given a second-order partial differential equation, linear partial differential equation with A, B, C, you start with these 2 odes, get, get your 2 curves xi and Eta. Once you know xi and Eta, you simply use the change of variables which are there in the last video and get your form, okay. Get your form, that is here, in which how I derive my xi and Eta, I made, I made that A star and C star is zero. So what you get is B star U xi Eta plus lower order terms will be zero.

(Refer Slide Time: 15:50)

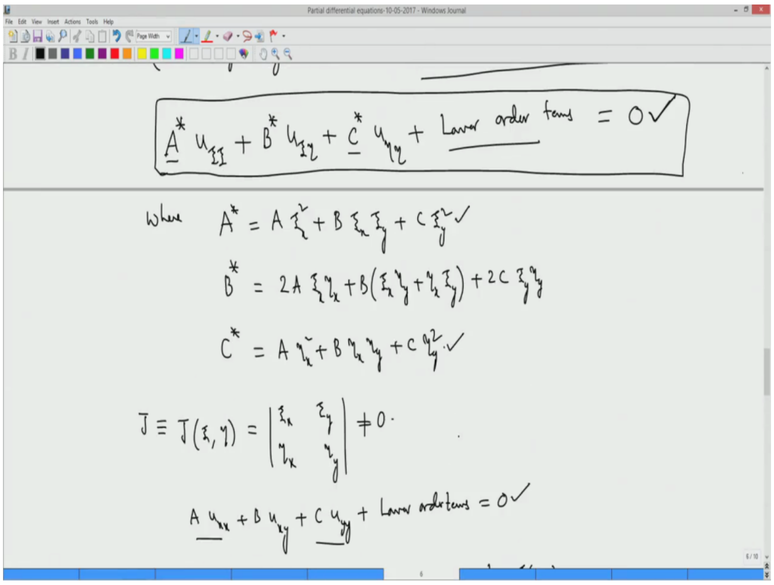


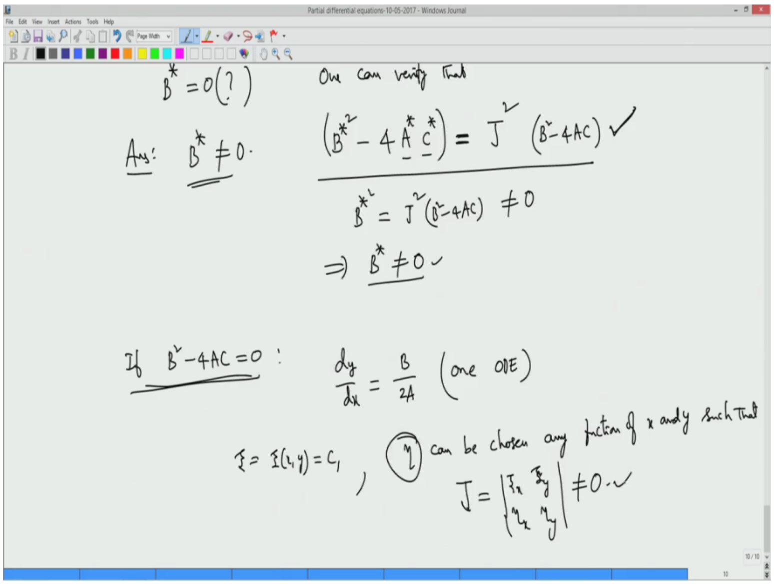
What you end up will be B star U xi Eta, because A star and C star I have chosen to be zero, plus lower order terms equal to 0. This is the canonical form you were looking for, okay. This is a hyperbolic case, this is a canonical form, canonical form, either hyperbolic or elliptic because I have chosen only that it is nonzero, okay. If it is positive, you call this, this canonical form is hyperbolic form. If it is negative, you call this elliptic form, okay, what you get is, in that case if it is negative, what you end up is, your xi, your xi x, y will be complex function. Okay. Similarly Eta x, y will be complex function, and B square minus, this discriminant B square -4 AC is negative.

If it is positive, you have a real root, so there will not be any imagine a number I, F it is positive. So what you end up after integrating this ODE, you have really valued function xi x, y. So you have xi x, y, that is really valued function xi. So xi and Eta variables will be real variables, real functions of x, y is B square - 4 AC is positive. In that case what you get is U xi Eta, xi Eta are real variables, real functions, in that case it is a hyperbolic equation, okay. If they are negative, B square - 4 AC are negative, xi and Eta will be complex functions. In terms of those complex functions if you write U complex functions xi and Eta, okay.

So if it is complex, what you end up with this, these 2 will be same, except that one is taking the bar. So what you end up is Eta will be xi bar, okay, it will be xi bar, so xi xi bar, when xi and Eta are complex functions and B square -4 AC is negative, then that is elliptic form, okay. So these are actually same, so you can derive from this when you consider, by considering B square -4 AC is nonzero. Okay. We will see with examples when it is nonzero, okay. So this is how you get with, so you get, what you got is xi and Eta variables if B square -4 AC is nonzero, okay.

(Refer Slide Time: 18:46)





So canonical form either is hyperbolic or elliptic. This is a canonical, if you have in this form, then xi and Eta are real, that is hyperbolic, I xi and Eta are complex functions, then it is elliptic, if it is in this form. Okay. And there is a question here, you have B star, what happens to B star, B star, can it be zero, question. B so can be zero, if B star is zero, it is nothing, okay. So you have to make sure that B star is nonzero, then only you have in the canonical form. So we have, how do you know that B star is actually nonzero? So the answer is B star is actually nonzero, so this you can see from this one.

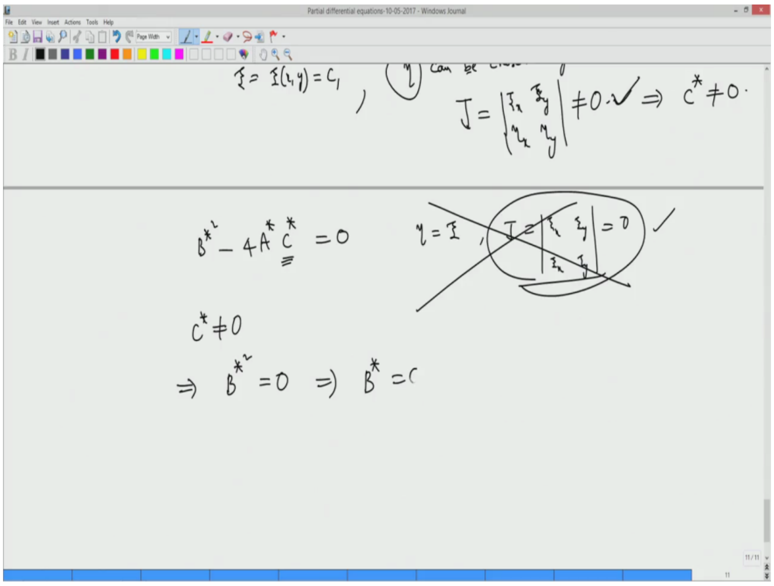
B star square minus 4 A Star C star, this is the discriminant of transformed equation equal to, you can actually observe, you know A star, B star, C star from these 3 expressions, okay, you have these 3 expressions A star, B star, C star, okay. And you know A, B, C, so you here right-hand side you have B square -4 AC, then the relation between this B square, this discriminant and this discriminant, there is a relation. So they are related with this one. So J, that is the determinant of xi x, xi y, it x, y square. So this you can directly by putting A, B, C and J star, B star, C star and J you know, you just substitute and you can see that, you can verify that this equality is valid, is actually, this equality is true.

You can verify that just by substituting A, B, C and A star, B star, C star and J here, okay. So one can see, one can verify that. Now how do I get from this, I can show that B star is nonzero. Now I have chosen A star, C star is zero, so you have this equality becomes B star, B star square is explained J, it is nonzero square and B square -4 AC is nonzero, so this means this is nonzero, okay. Implies B star is actually nonzero, so you have this canonical form actually exist with B star, that is nonzero. Okay. So that can be either hyperbolic or elliptic.

What happens if B square -4 AC is equal to 0 in this ODE? If you take this B square -4 AC, the 2 odes become single ODE, if you consider A star equal to 0 and C star equal to 0, okay, both will give you same, only one, one ODE. So you get only dy by dx equal to B square, B by 2A, only B by 2A you get, one ODE. Okay. Assume, so you integrate, so you get your general solution as xi y equal to C1, that you college one variable xi. Now Eta, so you do not know how to find other variable Eta to get the transformation.

So now in this case when B square -4 AC decade only one variable xi, Eta is you can always choose, okay. Eta can be chosen, any any function of x, y, x and y or x and y or x and y, both, such that, the condition is the Jacobian of, the Jacobian, Jacobian is xi x, xi y, that is from here, Eta x, that is what you have chosen, Eta should not be zero. Make sure that you choose some Eta such a way, such a way that this Jacobian is nonzero. If you do that, that works. So with those, those are the 2 variables xi and Eta. With those 2 variables you can reduce, you can go and reduce your given second-order partial differential equation into canonical form.

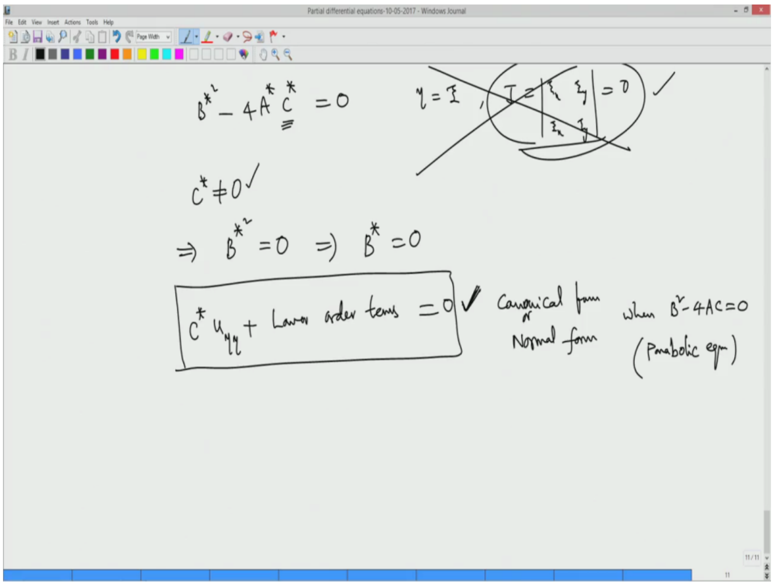
(Refer Slide Time: 23:05)



How do we do this in this case? We have chosen A star is zero, okay. So in both cases , so again, again use this, look at this equality. If B square -4 AC zero, then B star square minus and J is nonzero, so in case, this B star square, B star square minus 4A Star C star equal to 0. In which I have chosen A Star equal to 0, okay. I got this ODE, I have chosen A star equal to 0. Okay. Choosing, choose A star equal to 0, I get this ODE, one ODE. If you choose C star is also zero, you get the same ODE, okay. So it is like assume that you have chosen A star equal to 0, I got this one ODE, now you choose Eta such a way that, if you choose A star in such a way that Jacobian is not zero, so unless this xi is actually, this Theta is actually not same as xi.

You will not see that C star will be zero, okay. If C star is zero means Eta what you have chosen self is xi. That means Jacobian is actually zero, okay. So if you, what are the Jacobian if Eta is actually famous xi? If C star is also zero, what you end up is the same ODE, if you integrate, you get Eta, you can integrate, so this will give you the same integration, okay. So Eta equal to xi of x, y, so Eta is same as xi. So xi x xi y, it is also xi, so xi x xi y, that means it is zero. So this condition is not satisfied, so you cannot just go to new variables so that you can come back. Okay.

(Refer Slide Time: 25:43)



This will be useless, so, so if you choose A star equal to 0, C star is, so the moment you say that it is chosen, the functions such a way that Jacobian is nonzero, immediately tells you that C star is nonzero. Okay. So C star is nonzero and J star is zero, implies B star square equal to 0. Okay. That implies B star equal to 0. So what you end up, so finally equation becomes A Star is zero, B star is zero, so you have C star which is nonzero U Eta Eta plus lower order terms equal to 0. So if the discriminant is zero, we can have, we can see that B star is zero, so in this case C star is nonzero, so that makes it, you have C star U Eta Eta plus lower order terms equal to 0.

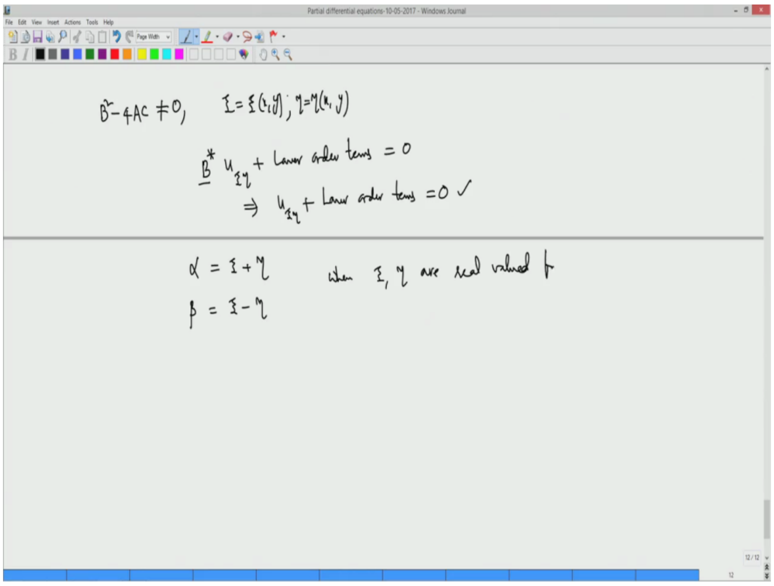
So this is what is second-order linear partial differential equation in the variables x and y, you reduce into some C star which is nonzero U Eta Eta plus lower order terms equal to 0. In the case of discriminant B square -4 AC equal to 0. So this is called canonical form canonical form, so this is a form it becomes after transformation from x, y variables to xi and Eta variables. When the discriminant is zero, B square minus 4H T0, we reduce, we can reduce the second-order equation to this one.

So this is also called canonical form or normal form, okay, we call this normal form when B square -4 AC equal to 0. So this is how you are reducing. So when you reduce this, the 2^{nd} see that this is actually, this is something similar to parabolic equation, we will see what is rate equation, okay. And this case, this kind of form is called parabolic form, parabolic equation, this is equal, similar to parabolic equation or heat equation. So when the discriminant is nonzero, so this is, this is how you can see the canonical form. So the transformed equation becomes of this type, this is called a canonical form, that is either hyperbolic or elliptic depending on what kind of variables you have.

If you are variables xi and Eta, new variables are real variables, real functions of x and y, then it is called hyperbolic, if it is complex functions, xi and Eta are complex functions of x and y, then it is called, then the canonical form is actually elliptic. If, when the discriminant is zero, B square -4 AC is zero, then you can see equation can be reduced after transforming the variables from x, y to xi and Eta, it becomes this type, this is the canonical form or normal form, this is called a parabolic equation, parabolic form, okay. This discriminant is zero.

So this is how we reduce, so after transforming the variables from x, y to xi and Eta, so you can change, depending on the discriminant value. If it is zero, you can get canonical form like this, okay, you produce into this type. If it is nonzero, you have to reduce in the earlier form, U, xi, Eta. Mixed derivative if you see, that is a canonical form for hyperbolic or elliptic and if it is same variables, U xi xi U Eta Eta, where it is parabolic form. So that is how it is. We will see exactly what happens when you when you consider discriminant is nonzero, what is the meaning of hyperbolic and parabolic there. Okay.

(Refer Slide Time: 28:50)



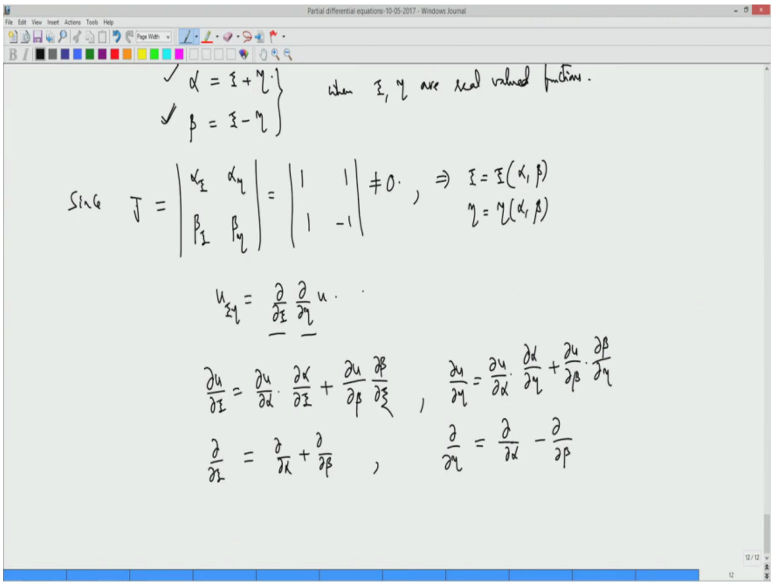
So you just go back, so, let us go back to when discriminant B square -4 AC is nonzero, in this case you have xi of, xi of x, y and Eta of x, y, 2 new things as your xi and Eta, so these are variables. In these variables, what you have got is B star into U xi, Eta plus lower order terms, equal to 0. So this implies, since you have seen that B star is nonzero, so you have U xi Eta plus you can divide it, so you have it is again, so you can divide it, if we divide it, so it will be absorbed into the lower order terms, lower order terms, equal to 0.

So this is what is, if xi and Eta are real valued functions, that is what we have said, it is hyperbolic equation, hyperbolic means it is something similar to waves. So it has different characteristics, so whatever the characteristics if you see, the characteristic of this equation is, if you see, as above solutions, that means the propagation waves, you can see that waves have, waves propagate with a finite speed. So it will have a finite speed of propagation, the solution will have a, you can see that you have a propagating kind of solutions with a finite speed, okay. So we will see that how it is, once you get these xi and Eta.

So what we will do is, you start with new variables, some Alpha Alpha xi Eta, okay, you call this in terms of Alpha xi Eta. Alpha as xi plus Eta, xi and Eta are real valued functions and you call this beta xi minus Eta. You take like this, they are the sum and difference when xi and Eta are real valued functions, real valued function. So this is one canonical form, hyperbolic form when these are real valued function. So if you change the one, new more variables, what happens here when you do this?

Alpha xi, beta xi, Alpha xi, Alpha Eta, these are the new variables, think of x, y as xi and Eta and xi Eta has Alpha and, new transformation, this is the new transformation. So with this new transformation you can see the Jacobian, Jacobian will be Alpha xi Alpha Eta, beta xi, beta Eta. So what is this one, this is actually one, one, one, -1, now clearly this is nonzero. Since this Jacobian is nonzero, we can have the transformation. So the equation becomes, so what you need, what you need is, you need to calculate U xi Eta. You xi Eta is doe doe xi doe doe Eta of U. So this is unique, doe doe xi and doe doe Eta. How do I calculate doe doe, doe doe xi and doe doe Eta?

(Refer Slide Time: 32:05)

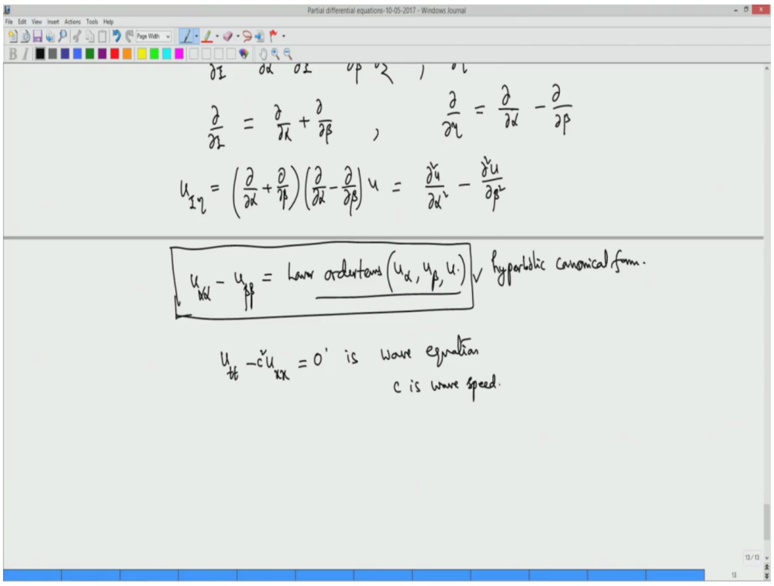


Doe U by doe xi is actually you can write doe U by doe Alpha, then d Alpha by or doe Alpha by doe xi plus doe U by doe beta, because U is a function of xi and Eta now. So here U is a function of xi and Eta. Now xi and Eta, because Jacobian is nonzero, you can actually get xi in terms of, xi in terms of Alpha, beta, xi is, xi in terms of Alpha, beta, okay. This is like x in terms of x of xi and Eta. Implies, so this is this, you can have Etas, in terms of Eta, Alpha beta. So, so you can just solve one equation, so you can go back to the variables say Eta 2 Alpha, beta and then you can come back from Alpha beta to xi Eta. So that is why the transformation is invertible.

So you can see that this derivative, doe U by doe xi, I can rewrite like this and doe beta by doe Alpha, sorry doe beta, doe xi, sorry, doe xi. This is what you have, so I know doe Alpha by doe xi, what is Alpha, Alpha is here, so doe Alpha, beta is one. So that is Alpha xi is 1, so that becomes doe A by doe Alpha, doe U by doe beta, doe beta by doe xi is also one. So we have doe U by beta. So doe doe xi is actually this. Similarly and you can get doe U by doe Eta which is doe U by doe Alpha, doe U by doe Alpha into doe Alpha by doe Eta plus doe U by doe beta into doe beta by doe Eta.

And here you can see that this will be doe doe Eta, if you remove U and doe Alpha by doe Eta is one, so you have doe doe, doe U by doe Alpha and here minus because doe beta by doe Eta is, because beta is this, this is minus1. So you have doe U by doe beta. So you remove this U both sides, you have this transformation, doe doe Eta and doe doe xi. So you come and put it here.

(Refer Slide Time: 34:22)



So if you do that, so if you do, so what you get is U xi Eta becomes, so you can replace doe doe xi by doe doe Alpha plus doe doe beta into, so doe doe Alpha -2 beta on U. So this becomes something but doe square U by doe Alpha square minus doe square U by 2 beta square. Okay. So we go back into the new variables, okay. So this is what we have, so your canonical forms, this is now your equation, your equation is this now, so this in terms of new variables, you xi Eta becomes this one. So you have U Alpha Alpha minus U beta beta equal to lower order terms, lower order terms. Lower order terms here means in terms of U Alpha, U beta, U of Alpha, beta, this is what it is. Okay. The functions of Alpha, beta. So this is another 2nd, another canonical form for hyperbolic, this is a hyperbolic canonical form.

So while xi and Eta are real valued functions, so either this one canonical form for hyperbolic or this one is another hyperbolic canonical form. This looks like a wave equation. If you assume that all the lower order terms are zero but not there, what you get is U Alpha Alpha equal to U beta beta, okay. So this is kind of wave equation, so if you replace with all Alpha

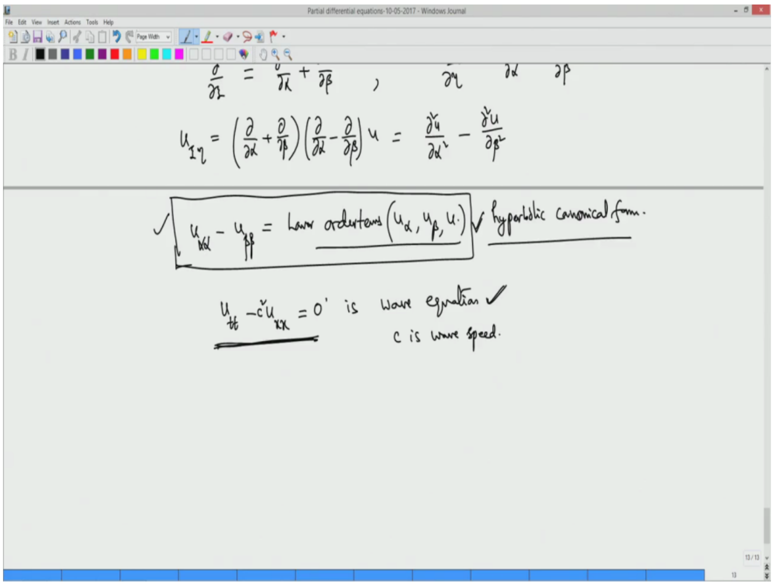
Alpha as TT, the time variable, and this is a special variable beta beta, so what you have is this one is this equal to 0, okay. This is a wave equation, wave equation. So if you actually include the speed C square, okay, you put a C square here, so you have C is the speed of wave, C is the speed of the wave, wave speed, okay, you can write wave speed.

So it is a constant, C square, so this is a wave equation. So similar to, this canonical form is similar to the wave equation, so whatever characteristic of the wave equation. Wave equation if you look at, if you consider string, string if you consider, that satisfies the wave equation. So you can see that waves propagate with a finite speed of propagation and you have certain characteristics. It has it has only 2 kinds of characteristics along with this PDE becomes ODE, okay, that is what is there, it is a constant. So along the curves, along those curves, if you are PDE acts as, it becomes ODE, then we say it is a characteristic curve

Those curves are nothing but, what you get is this. This is the variable, Alpha and beta act as those characteristic curves, okay. The transformation what you get from x, y to xi, Eta, xi are the characteristics along those curves, your equation, your partial differential equation becomes simpler equation. So that is what along those curves you call, because it becomes, simpler means from partial differential equation to U x square B to be simpler form, that this ordinary differential equation which you can solve. So characteristics of this wave equation, solutions of wave equations or it has a finite speed of propagation.

If you look at, now if you look at the wave parabolic equation, for example that we see when discriminant is zero and you have this one. In this case, similar equation will be heat equation where heat, heat propagates at infrared speed. Okay, you cannot have finite speed of propagation. In no time if you give initial values to some disconnects, suppose you give certain temperature at in a rod. Zero temperature at one place and 100 degree temperature at another part of the rod, in no time it will be uniform. So what happens when you give zero zero degree centigrade in parts of the rod and some other degree, nonzero heat at another part of the bar and you see that clearly it is this continuous function, zero at one place and say 100 in one place.

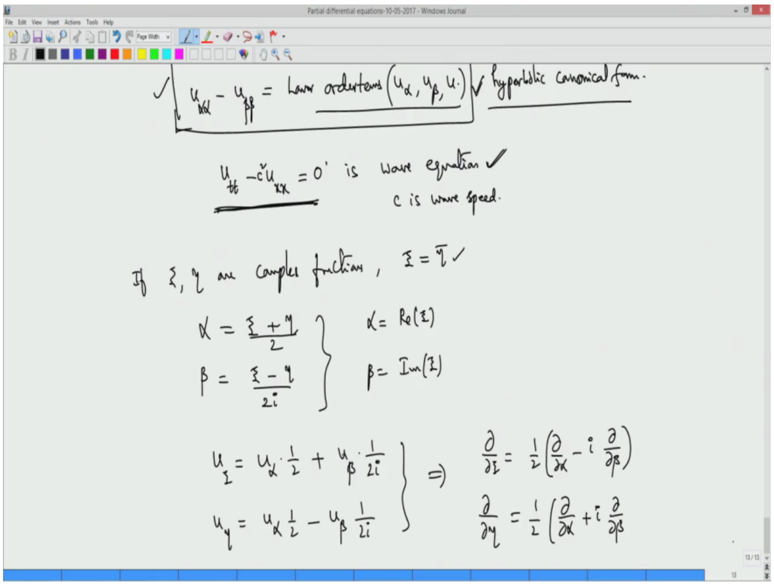
(Refer Slide Time: 37:55)



In no time heat diffuses, okay, the temperature will be distributed in no time, diffuses in no time, heat diffuses in no time and in a few seconds we see that it is a smooth curve. From 0 to 100 it is a discontinuous function, now simply you get a continuous curve. So this is a different form wave equation. So you see that, you have infinite speed of propagation as, if you see as a wave, solution as a wave, the speed will be infinite here. So it has a different characteristics, heat equation has a different characteristics. That is why we categorise these kinds of equations as a parabolic based on, so analogous to the conic sections for the quadratic equation.

So in the same way here we have right now we have this hyperbolic equation and xi and Eta when you transform into U xi, U xi Eta plus where other terms, that canonical form you can convert into new canonical form, this is the 2nd canonical form that is hyperbolic. When you get this, this looks like a whale equation. So this is wave equation, this is hyperbolic. So what happens if, when the discriminant is nonzero, what if you have xi and Eta are, if xi and Eta are complex, are complex functions?

(Refer Slide Time: 40:23)



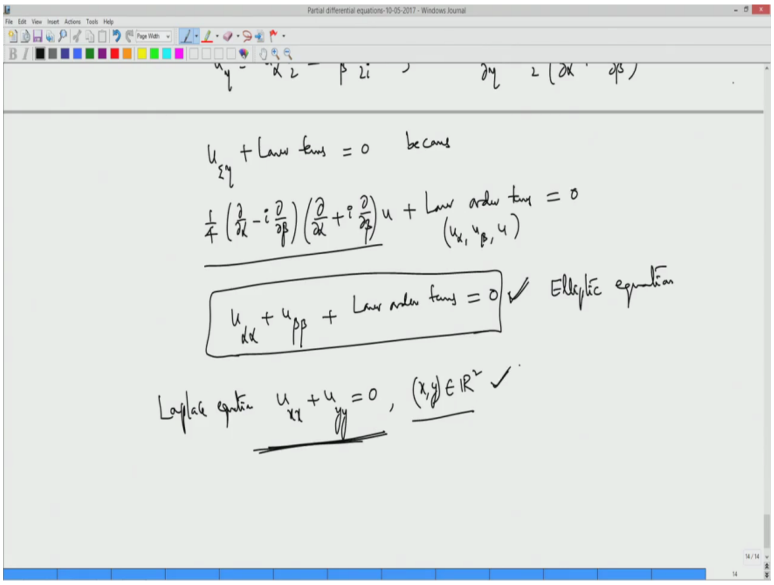
If they are complex functions, do the same thing, Alpha is, so take the sum of the xi plus Eta and beta is xi minus Eta by 2y, you take here 2, then it becomes, this simply real part of what is Alpha then? If you take like this, xi and Eta are complex functions, first of all xi equal to Eta bar, right. So how did you get your xi and Eta, basically when you have the discriminant is nonzero, these are the 2 differential equations where A, B, C are real valued functions, when you take this bar, okay, this gives you xi and this gives your solution y of x is, whatever you integrate, that will be xi of x here, this is Eta of x here.

So when these functions, these are what you get is, if, because B square -4 AC, when it is negative, i comes your, i comes here, okay, you have I comes here, so you have i here, minus i here. So if you, whatever you integrate this equation and take the bar, that is nothing but I will get this point. Or you simply take the bar here, this is nothing but this ODE or you take bar here, that will become this ODE. So what you see is that after integration xi is nothing but Eta bar or Eta is nothing but xi bar.

So when you see that they are complex early functions, there nothing but, they are just conjugate to each other, so there are complex conjugates, the 2 transformations. Now what you do is, when take this and sum and difference as new variables Alpha, beta, Alpha is nothing but real part of xi, beta is nothing but imaginary part of xi. So with these new variables, what happens, you can use the same technique, U xi Eta becomes, so you see that this becomes, doe doe xi and now you have Alpha is different. So Alpha is this one and what you have is U or doe doe xi is, U xi is U Alpha doe Alpha by doe xi, that is half plus U beta, doe beta by doe xi, that is one by 2y. Okay.

Similarly U Eta is, U Alpha, doe Eta by, doe Alpha by doe Eta, so that is half minus U beta, doe beta by doe Eta is -1 by 2 A, minus I already have here, one by 2 y, so is what you get. So this gives me doe doe xi is half doe doe plus or minus i doe doe beta. So when you take this, when you bring this i out, it will become minus i. So similarly you get doe doe Eta is from this 2^{nd} equation you have one by2 is common. So you have doe doe Alpha minus i, when I bring this i out, so it will become plus i doe doe beta. So this is what your doe doe xi and doe doe beta becomes.

(Refer Slide Time: 43:19)



Now if you put it in U alphabet, what happens to your U xi Eta? U xi Eta plus lower order terms becomes equal to 0 becomes in terms of this new variable Alpha, beta as Alpha, doe doe Alpha minus doe doe beta into half of one by 4 and doe doe Alpha plus i doe doe beta U plus lower order terms equal to 0. So here low order terms means U Alpha, U Beta and U, U is a function of Alpha, beta. So what happens to this term, this term is nothing but now U Alpha Alpha plus U Beta beta. Of course 4 you can bring it decide and 4 into lower order terms again, lower order terms equal to 0.

So this is what, this is a canonical form for, when xi and Eta are 2^{nd} canonical form when Alpha, beta, when B square -4 AC is nonzero. And B square -4 AC is nonzero, you get a 2^{nd} canonical form by new transformation, okay. So when xi and Eta are complex functions, you do this transformation to see that this is a elliptic, elliptic equation. Okay, this is a canonical form which is equal to elliptic because if you ignore the lower order terms, this looks like Alpha, U Alpha Alpha, Alpha Alpha you can take it as x, x plus U Beta beta as your yy equal to 0.

This is a Laplace equation, so both x and y are belongs to full R. So we see that, this satisfies the Laplace equation, so it is a different thing, so we do not have tan here, only special variables. This is actually steady-state condition, when you have a plate is having with heat, heated plate and initially had some heat, after diffusion, after sometime it reaches the steady-state, what you see is the heat distribution, heat is at the steady-state. Of course you have some boundary, boundary of the plate is assumed that you fixed it, you say insulated. So what happens is, whatever initially what you have at some point, it may be high-temperature to a temperature just some steady-state.

The solutions of this equations are, steady-state condition for the heat equation. So that is what you see. Okay. So this is what, this is a different, you will not have characteristics here, so this is a different thing. So you do not have time variable here, this is actually a purely boundary data I have to provide for this Laplace equation. So the canonical form of your second-order partial, linear partial differential equation when it becomes this type, elliptic type, that means the characteristics of your equation are basically, so you have to provide the only boundary, boundary data for such an equation so that the problem is well-defined.

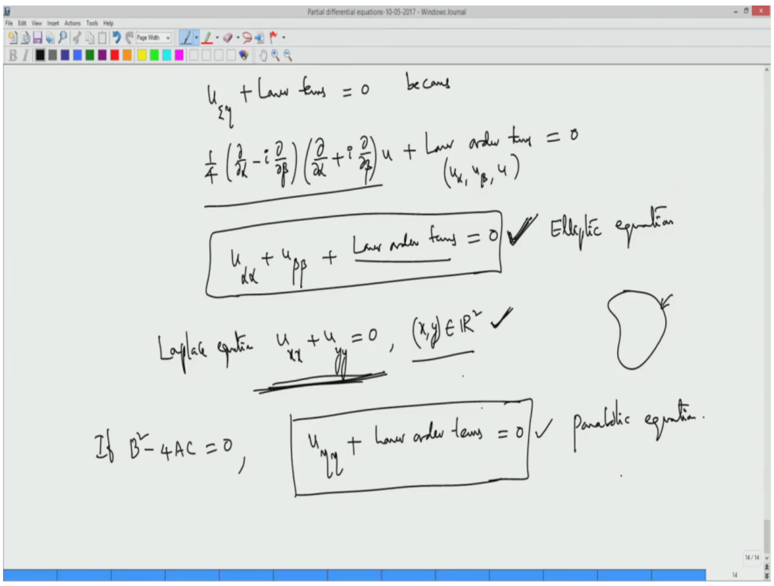
And if you have hyperbolic equation, you have to give initial data because it looks like wave equation. So because it has 2 time variables, okay, T is positive x belongs to R, so because time T, because second-order, time derivative, second-order you have to provide 2 initial data. Initial data at U at T equal to 0 and the derivative, doe U by doe T, velocity, we think of doe U by doe T at T equal to 0. That also you have to provide as initial data and because whatever boundary data you can provide, okay.

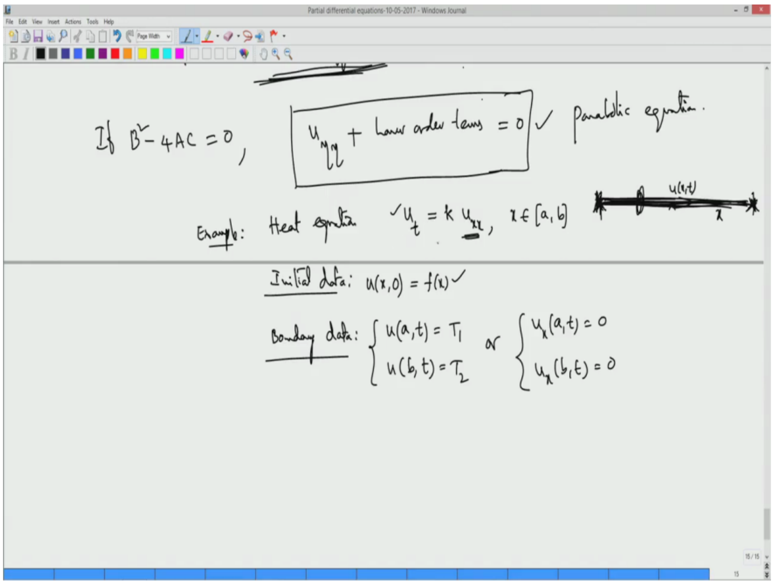
So if these are, if xi and Eta are real and complex valued functions, when you get these from this ODE is, from the odes, what you get is, one of these canonical forms, either hyperbolic equation, it is like wave equation is an example all elliptic equation if these are complex valued functions. Okay. And you have elliptic equation, it looks like, when you do not have lower order terms, it becomes Laplace equation, so it is basically purely, you do not have time involved here. So this, these are purely different equations, so its boundary value problems only you can define because you can only give, when you have a boundary like this for the Laplace equation, you should give the, you should provide the boundary data. There is no T, so only x, y are the special variables, it is a symbol two-dimensional Laplace equation, okay.

So this is what is the case when discriminant B square -4 AC is nonzero. So either case you can see that either it becomes one of these equations, one of hyperbolic or elliptic equation, okay. By transforming one more time, from xi Eta to some Alpha beta variables which you, Alpha beta you take it as simply some difference between, difference of these functions xi and Eta. So what happens to, when B square 4 AC equal to 0? If B square -4 AC is equal to 0, we immediately, we have shown that can you could get only xi and Eta you have to choose in such a way that Jacobian is nonzero. And B star square -4 AC star is zero, that makes it, because you to choose C star as nonzero, if you choose C star equal to 0, then the Jacobian will be zero.

So that is contradiction to the assumption that Jacobian is chosen in such a way, is nonzero by choosing a suitable Eta function. So that makes it C star nonzero and that but let a star is zero with which you got the xi function. So B star has to be zero. So that means finally you end up C star which is nonzero, that is U Eta Eta plus lower order terms equal to 0.

(Refer Slide Time: 49:15)





So when, we will continue from there, what happens is, if B square -4 AC equal to 0, what you end up is the canonical form is U Eta Eta plus, because C star is nonzero, you can plus lower order terms, lower order terms equal to 0 for the services a canonical form of parabolic, this is a parabolic equation, parabolic equation. So if your equation transforms into this type, only one variable U xi xi plus lower order terms, U Eta Eta plus lower order terms, then you say that it is a parabolic type of equation. Okay.

So we will see, so what is the example here? So in this case, the example, we have not talked about the example. So here typical example is heat equation, heat equation. So what is the heat equation? So heat, heat equation is the temperature distribution, so which is a time derivative, so you have a time involved here, U is a function of x and T, x is a special variable, so basically x is if you consider Rod. Rod of finite length or infinite lines, so this is a one-dimensional, so x is only special variables. So in this what is the temperature distribution so that satisfies this equation UT, the time derivative of temperature distribution x of T, okay.

The temperature at every point at every time T, so this is UT , UT equal to some constant, actually this is a constant kappa, that is specifically constant into Uxx. So x is because I choose this to infinite rod, then this is full R, if it is only finite rod, you have to, suppose you take only finite rod, then you have to take a to b and then you have to have, so this is a boundary, these are the boundary points and initially, so because T is the time derivative and which is only one derivative. So you have hit T equal to 0 you have to provide a boundary condition, initial condition.

So U at x and T is zero, so this you have to provide some function of x, this is the initial condition, okay initial data. Initially what is the temperature of this rod at T equal to 0? Then what happens, if it satisfies this equation, or the solution actually gives you what is the temperature distribution for all times, all along the rod. If it is a boundary, so you should also know exactly, initially what is the our boundary data, for all times you should provide a boundary data. Because it has 2 derivatives, you have to provide 2, you have obviously 2 boundary points, so you should provide the data here. So you have boundary, boundary data, boundary data is U at a if a is the boundary, and T, this is some f1 of T.

And then U at B, T, this is also some function of T. So these are given, if you know this initial and boundary data is given, you can also choose, so physically if you say I insulated these endpoints, I can make it zero. Or temperature, you maintain zero temperature at here in here, this is zero. Okay. And you can also say physically you can, you insulate these ends, initially you have some temperature at the rods and with the rod is insulated and some assumptions, we assume that rod is uniform and the temperature, heat is going only in one direction, all along this lateral side, heat is not dissipated, so that means it is insulated all along the rod. Okay.

So that heat is, heat is distributing only in one direction or all along the x axis only. So under those assumptions if you say that the insulated, in selected boundary condition will be simply the derivative, flux has to be zero. That means nothing is going out, so that means the derivatives, so that is doe Ux Ux at B, both are zero. So these are insulating boundary conditions, if you remove this x and x, only U is zero at both endpoints, that means you have maintained the rod at temperature zero or some constant, okay. You can give some temperature T1 or T2, this is a temperature T1 and T2 at the endpoints. Or you can provide flux boundary condition, zero flux boundary condition, that is inflating rod, that is Ux at A, T is zero and Ux at B, T is also zero, so that means rod is insulated.

So these are also, either this or this you can give boundary boundary data. So this is like, this equation, you see that U x x is plus lower order terms equal to 0. So that is why it is parabolic equation, most of the, so any linear partial differential equation of second-order, you convert it into some new variables xi and Eta, okay. Once it takes either U xi Eta equal to plus lower order terms equal to 0, then it is hyperbolic or elliptic. As a sub case you can also, you can also see that it is depending on what is your xi and Eta. If xi and Eta are real valued functions, again you use a new variable alpha and beta as sum and difference and you can convert U Alpha Alpha minus U beta beta equal to 0.

So that is a kind of hyperbolicity when you see minors and if they are complex valued, then it becomes U Alpha Alpha, xi and it are complex valued functions, again you can use the transformation Alpha, beta, Alpha as the sum of xi and Eta by2 and and beta is xi minus beta divided by 2i. So if you take that, you see that it will become U Alpha Alpha plus U beta Eta is zero, that is Laplace equation type, that plus lower order terms equal to 0, that is elliptic equation like a Laplace equation. And if you see only 2 same derivatives, second-order derivatives plus lower order terms xi or Eta. So that is a parabolic equation, that looks like heat equation, and typical equation is heat equation.

So this is how you reduce linear second-order partial differential equation of second-order, you reduce to another second-order linear equation in new variables xi and Eta, okay. This is not really new what we have done for the PDE, we have already done similar things for the ODE. Say, what we have done for the second-order linear ordinary differential equation, we reduce into normal form, that means we take a some transformation, so from the, we took some transformation, with the transformation we removed the first derivative. So that is, reduction, reducing the equation of second-order equation, second-order ordinary differential equation into a second-order differential equation without first-order derivative in it.

So that is putting in the normal form or standard form, with which if you see that reduced equation is actually, with any transformation if you chose, then it will become again second-order equation with the first-order term also will be there. If we choose specific ways, the transformation, you can choose the transformation in such a way that first-order term you can make sure that is removed, it is zero, coefficient is zero, that fixes the transformation. So once you know that, so you have a reduced equation is second-order without first-order derivative in it, then you can expect to solve if it is in a simpler form.

So we do the same thing here, so that is what we have done here. So here even for partial differential equations of second-order, linear partial differential equations of second-order, we have transformed from x, y to xi and Eta, xi and Eta are functions of x, y, do these new variables xi and Eta, then it becomes second-order linear differential partial differential equation in new variables xi and Eta. So that is there, this does not mean that you can solve this one, okay. So you fix the xi and Eta in such a way that the A star and C star equal to 0. So by choosing that, you reduce that equation to a simpler form, that simpler form as either hyperbolic or that is what you are simply calling, either hyperbolic, elliptic or parabolic type, okay.

Simpler form means U xi Eta plus lower order terms zero or U xi xi or U Eta Eta plus lower order terms are zero. One of them, so the mixed derivative are only should be there or only 2 derivatives of one variable, other variables are not, only lower order terms it will be there. So these are canonical forms. This is like putting in the standard form, so the standard form is in nice form, then you can solve it, you can expect to solve it like ordinary differential equations. Here also if these canonical forms are one of these equations, heat equation, wave equations, Laplace equation, you can solve it or otherwise also if it is in a simpler form, so we can simply integrate by elementary methods you can integrate some of the equations.

So first we will see, so in the next video we will try to see how to reduce, how to actually demonstrate with an example. I will demonstrate with an example in the next video by reducing given second-order linear partial differential equation into a canonical form and then we will try to solve if it is possible, okay. So we will see that in the next video, thank you very much.