Differential Equations for Engineers.

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Lecture-4

Methods for First-order ODE's- Exact Equations (Continued).

In the last video we were looking at the converse part of the, converse part, that is a sufficient condition to see the given differential equation is exact, okay for the for that is to look at the condition, dow M by dow y equal to dow N by dow x. So if you once see that this is verified, you have seen that it is a necessary condition, if the equation is exact, this is a necessary condition. You want to check whether there is a sufficient condition. So with, if dow M by dow y is equal to dow N by dow x, then you will be able to get some u, so that du is equal to 0, total derivative of u, this function u of x, y, will it be du is equal to 0 is the given differential equation. Okay.

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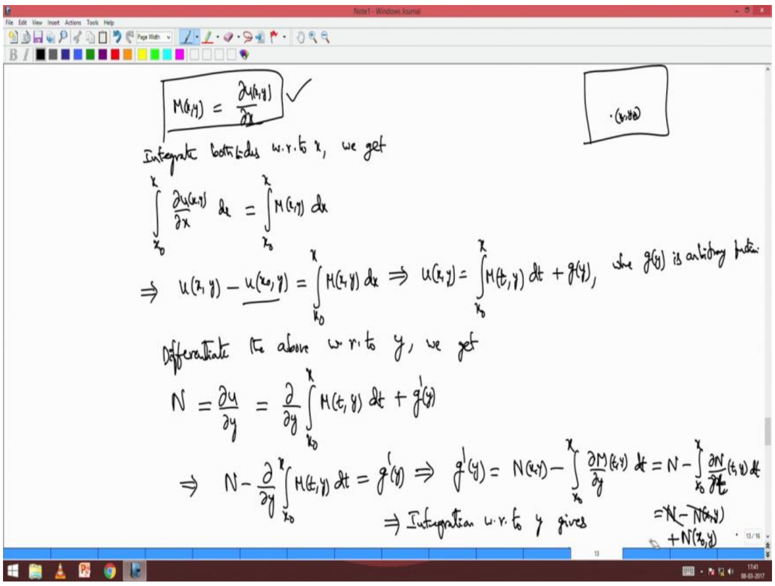

Let us continue where we left in the last video. So if you start with, if you think that there exists a u satisfying this. Okay. So if I come if I make use of this, this condition, will I be able to get my u? Okay, that is the question. So if I use that, what is that you can use? So now to make use of this result, I, I use dow u by this, what I do is I integrate this with respect to y. Differentiate with respect to y, differentiate the above with respect to y, we get, what do we get?

Dow u by dow y is a function of 2 variables and to differentiate with respect to y, that is dow u by dow y, and this is a function of y is involved, you can take it inside, let me write only outside. So dow, dow y of x0 to x M of T, y, dt plus this is only a function of y, this is like derivative dg by dy, so G dash of y I write. Okay. So what is this one? Dow u by dow y is like N okay. So this is actually N. This is actually your N. Fine. Now this implies N minus dow, dow y of x0 to x M of T, y dt is only a function of y.

So this implies G dash of y is equal to N of x, y minus x0 to x dow M by dow y, so because this is function of y, this is a y derivative, this is only a function of x, I can take it inside. Dow M by dow y of T, y, it equals to G dash y we have written. So this you can get it, if you integrate, now this is equal to N minus integral x0 to x, what is dow M by dow y? That is what I use now, dow M by dow y is dow N by dow x, dow N by dow x of T, y dt. Okay.

So now if I integrate both sides with respect to y, okay, integrate, integration with respect to y gives integral of G dash of y dY, now I do from y0, x0 and y0 is fixed, y0 to y, okay, equal to N, so N is function of x, y. So I do y0 to y N of x, y dY minus, so this becomes what? So this now, so what do you have is dow M by dow y of T, y, function of T, y, so this should be T, T, okay. So I made this mistake. So you have dow M by dow y function of T, y, it will become dow M by dow T T, y, variable are x, y. Dow M by dow equal to dow N by dow x, when it is a function of x, y. When it is a function of T, y, I have dow M by dow T.

So this is actually equal to N minus, this already you have, N of x, y, right. So N of x, y minus, plus N of x0, y. Okay plus N of x0, y. If I simply integrate dow M by dow T dt, so this goes, simply total derivative of N, N at x, that is N at T variable, N at x, y, N at x0. So N is a function of x, y this goes, this is simply N of x0, y. So we can integrate directly, so this right-hand side is simply, what you have is N at x0, y.

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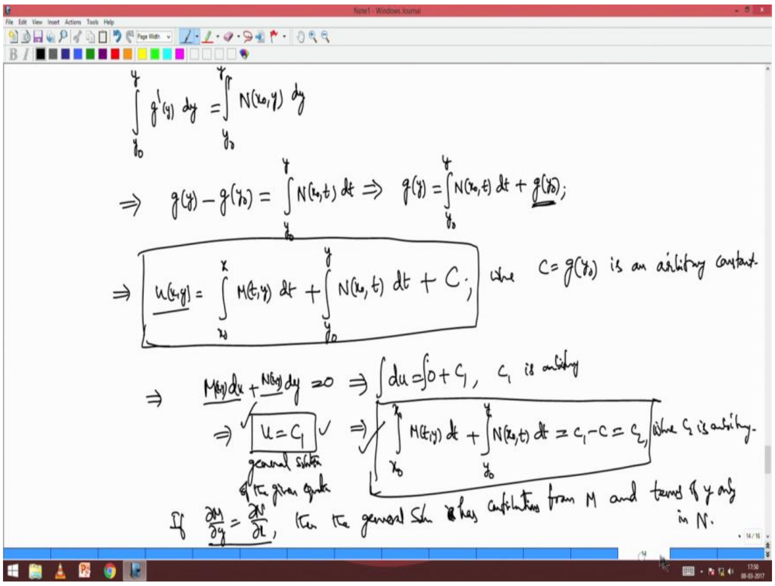

This you are integrating with respect to y0 to y dY. So this will give me my G of y minus G at y0 equal to this y0 to y, N at x0, y dy. So this I can replace as some dummy T dt. T is a dummy variable. So this will give me exactly what is my form of G y. So G y is, G y is integral y0 to y, N of x0 to T dt plus this G of y0. So I am actually solving this equation where G is unknown. Okay. So now G at x0, G at y0 is an arbitrary constant, so you can write it as arbitrary constant or G you can write it as as it is.

G at y0 is an arbitrary constant come actually this, this is an arbitrary constant because G is unknown. So this G, you can go and put it into the equation. You know what exactly the form it takes, u is this, you put it here. u is, actually u form is this. So this u is, now this implies u of x, y which is equal to integral x0 to x, M of x T dt, xT dt is x dy dt. M of T, y dt plus gy, that gy I can replace with this, that will give me y0 to y N of x0 to T dt plus this constant C where C is G of y0.

G is, since G is arbitrary at a some point, it becomes a constant, so it gives an arbitrary constant, arbitrary constant. So what I got finally, if I use that condition dow M by dow y is equal to dow N by dow x, then I can get the equation like I, I can find some u that makes the equation exact. Right. So that is what the converse, right. So this is exactly such a function I constructed, okay. So it is already here, so I will not write here. So this is what is your function.

So what is this function, once you have this function, this du, right, so this implies given differential equation M dx plus N dY equal to 0, I can write M as dow u by dow y, this whole thing where dow u, dow u by dow x, that makes it dow u equal to 0. That means u is equal to constant is by general solution of the equation. What is u, u is here. That implies x0 to x M T, y dt plus y0 to yN at x0 T dt plus C equal to, let us say this is another arbitrary constant.

When you integrate these both sides, what you have with arbitrary constant, okay, C1, where C1 is arbitrary. So what you have is C is equal to C1, so this is equal to, this gives me, this I can bring this C on another side. So if both are arbitrary constants, together is also an arbitrary constant. So finally you can write C2, okay, which is actually C1 minus C which is equal to C2 where C2 is an arbitrary constant. Okay. So this is now function of x and y with an arbitrary constant, this is your general solution.

So, that is how I got, this is the general solution, replace u with this, so that this is your general solution. This is the general solution, general solution of the equation, of the given equation, given equation M dx plus N dy. So this is equivalent to say that this is, this is the general solution of the equation, okay. So if you look at this form, you have this is the equation M dx, M of xy, N of xy, okay, dy is equal to 0.

So what you are doing is finally dow M by dow x equal to, dow M by dow y is equal to dow N by dow x, that condition, whenever it is satisfied, that means if dow M by dow y is equal to dow N by dow x, then the general solution is, general solution is, how we get it, you can see this. M, you are keeping y as it is, you are integrating only with respect to x, you can see this. M, you are integrating with respect to x, y you keeping as a variable, so whatever may be, whether y is there or not, you simply integrate become involved in the coefficient of dx, that is the term M, you simply integrate with respect to x from some x0 to x in definite, definite integration.

So you can fix your x0, y0 as 0 in practice. So you can simply integrate with respect to x, this M and N you can simply, see N what happens, you fix x0 and you integrate with respect to y, the variable. That means you are putting x is 0, if you put x is equal to 0, x 0 is equal to, if you take x0 is equal to 0, that means N, they becomes 0. Okay. So in practice you, you have to integrate this with respect to y, only terms that involving y, not only terms of why only you should integrate, that will continue. Okay.

From this I can conclude the general solution has contributions from M and terms of y, terms of only y, terms of y only in x, in N, okay, you can see this. That is how you get the general solution. Whenever you check this condition, once this check, once this condition is satisfied, you know the reason exact, the shortcut to find the solution is M you integrate with respect x, keeping y as it is and here you put x equal to 0 and you integrate. Once you put x is equal to 0, M may become 0.

When you have only terms of y, even if you put x equal to 0, those stuffs will remain, you simply integrate with respect to y as this x is not there. Okay, by putting by x is equal to x0, that is how you can directly get the solution like this. Okay. In summary what we had is, given a differential equation like this, it is an exact others in the definition. So M and N, I should be able to write in terms of personal derivative of some function of x, y. That is the definition.

If suppose this is exact, suppose, that means I can find such a u satisfying this, immediately I have this condition is satisfied. So this is the necessary condition, necessary, for an exact equation, this condition is necessary. And is also sufficient, what the converse is, the sufficient condition, this is efficient condition. If this is satisfied, my equation is exact, that method will be able to find my u satisfying this. Such a thing I constructed by assuming that it exists, I use the condition dow M by dow y equal to, dow N by dow x, okay for the finally I derived the solution as this. Okay.

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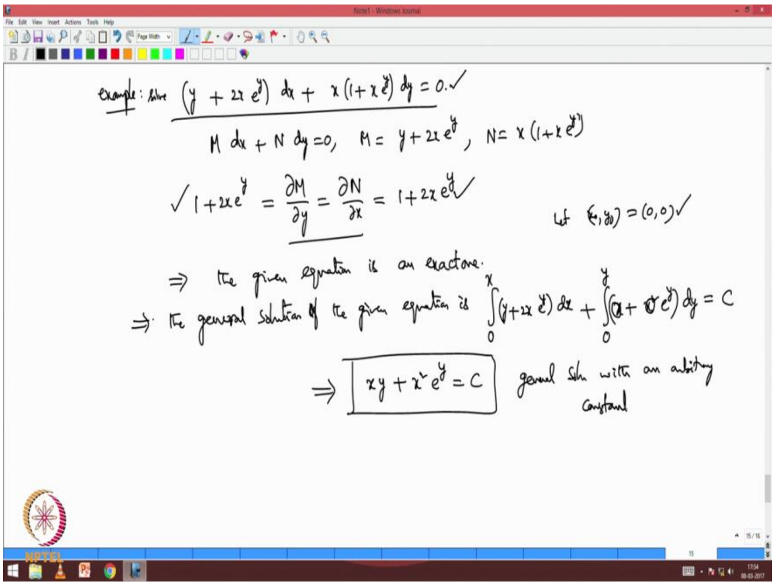

I will do some example so you will understand, so here is an example. You take this y plus 2 x e power y dx plus x into 1+ x e power y into dy is equal to 0. How do we solve this equation? So solve this differential equation is given. So it is in the form of M dx plus N dY equal to 0 where M is y plus 2X e power y, N is x into 1+ x e power y, okay. If this is an exact equation, I know that necessary condition will give me dow M by dow y equal to dow N by dow x.

What is now M by dow y? It is simply 1 plus to x e power y. What is now N by dow x, 1+ 2 x e power y, both are same, necessary conditions. If this is satisfied, that is related satisfied for the functions M, N, I know that the equation is also exact, converse is also true, right. So we do not, when you are given a differential equation, you know, you do not know prior to that the equation is exact, okay. If it is exact, this condition is automatically, should be necessarily, it should be satisfied.

Because the converse is true, that means this condition, if this condition is satisfied, that we have checked here, it is verified, then this condition is satisfied, that makes the converse is also true, that means given equation differential equation is an exact equation, okay. So once you check this, you check this, implies, converse is true, so the equation, the given equation is an exact one. So if it is exact, you know how to get the solution, general solution. We have seen just now sometime back.

So you simply take M, you integrate with respect to x, you simply take N, integrate with respect to y by fixing x0, here you do not fix y0. Okay. You add them, put equal to constant, that is other solution of that equation. If this condition is satisfied, okay. This implies the general solution is, general solution of the equation, the given equation is, what is N, integral, I fix my x0, y0 as 0, 0. Okay. Let x0, y0 equal to 0, 0 because, they are defined, you look at the equation, 0, 0, there is no issue at 0, 0, there is no division with x, y.

So, x equal to 0, 0 is part of the domain, I choose my x0, y0 as 0, 0. Okay. So you integrate from x0 to x, x0 is 0 to x, M is y plus 2x e power y dy, dx, okay, this is the x. y I am keeping is a variable, x I am simply integrating with respect to 0 to x, okay, plus y0, that is 0 to y, N, N is, N is x plus x into, x square e power y dy. I fix y0, inside also N of x0, y, x0 is 0, so if I put 0, this is 0, this is 0, 0 into 0, the whole thing is zero, okay. Okay, equal to constant.

So this implies, if integrate this now with respect to x, xy, okay, plus x square e power y and this is 0, equal to constant. Okay. So this is your general solution, general solution with an arbitrary constant C. In this video we have seen that the given differential equation is, so when, so when it is exact equation? So if it is given, once you are given a differential equation, you can just verify that dow M by dow y equal to dow N by dow x, then that is a necessary and sufficient condition.

That means if you are equation is exact implies it is exact, it is, this condition is satisfied. If this condition is satisfied, then also the equation is exact. So it can only check this condition and you can say that the equation is exact, then we can solve the equation. You have seen what is the form of u, what kind of form u have got, okay. You have got for the u of x, y, so that will give you, so we will do examples, some more examples in the next video.