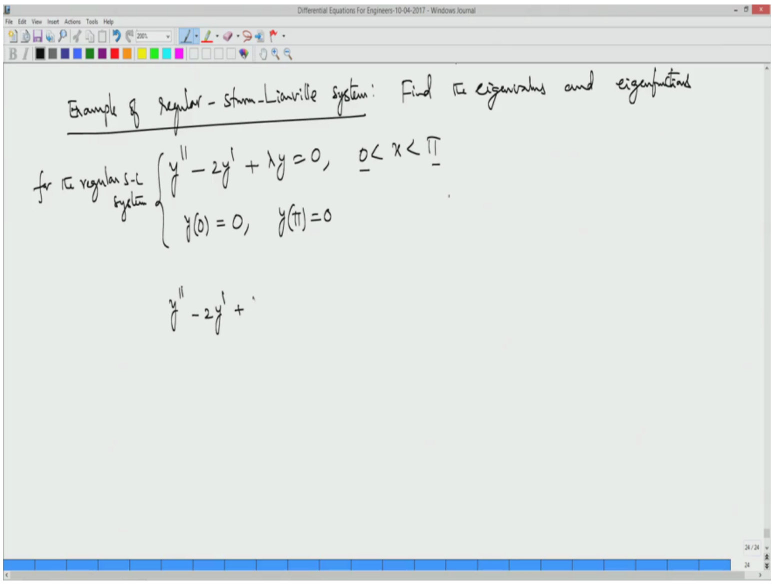
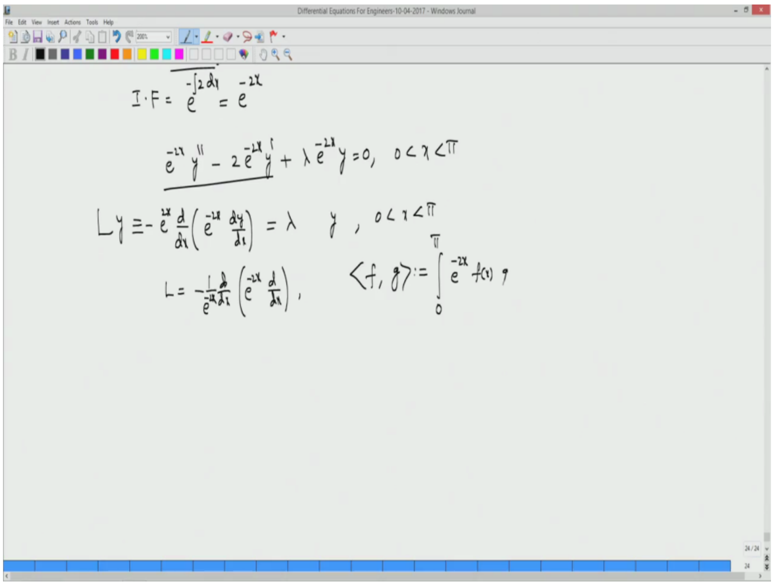
Differenti#al Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
Department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-39.
Examples of Regular Sturm-Louisville Systems.

(Refer Slide Time: 0:54)



Welcome back, in the last few videos we have seen one example of regular sturm louisville system to find the eigenvalues and eigen functions and rest examples for periodic and singular sturm louisville systems we have seen. Today we will just have, we will have 2 more examples of different way of finding eigenvalues and eigen functions for the regular sturm louisville system. So to start with the 1st example which i explained in the last video. So we can see this, this is the example, so for which you can find eigenvalues are eigen functions, so little different from the work we have done for the 1st example of regular sturm louisville system.

(Refer Slide Time: 2:10)

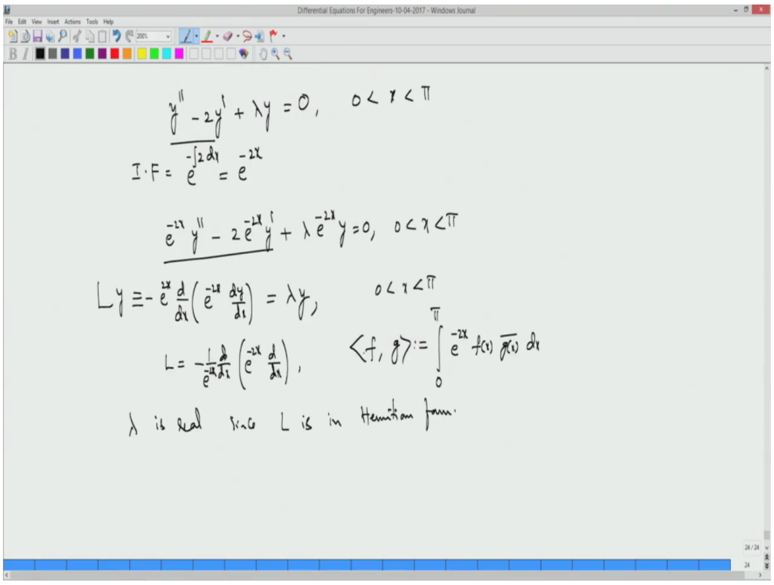


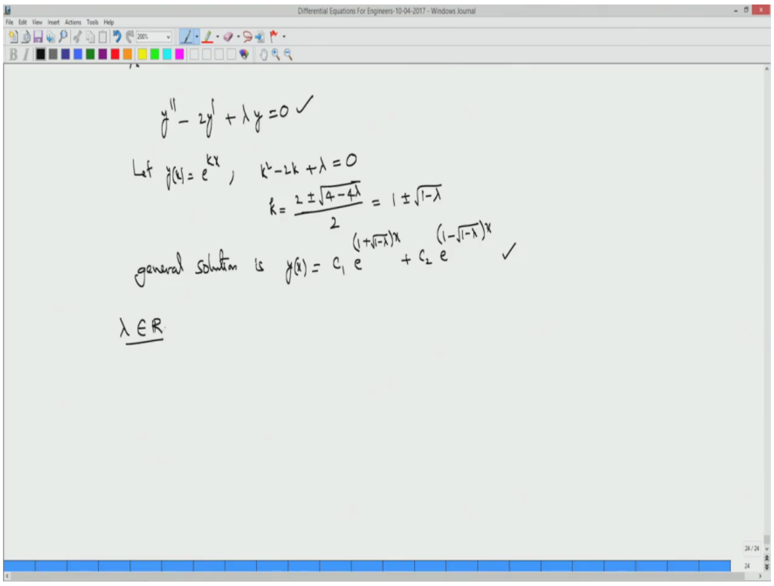
So you need to take the differential equation plus lambda y equal to 0, so this you want to put it in this sturm louisville system, so self a joint form or hermitian form or skew symmetric form, so whatever you say. So x is between 0 to 3, this is what is the domain. You need to put this together as derivative of something, so to do this, this is just a linear equation of time, right. So if you think of y – is a dependent variable, you have dy – by dx -2 dy by dx, okay. So that dy – minus 2y –. So for y – is a linear equation, so you know that integrating factor is e power integral p, p is here -2 dx.

So you have e power minus 2x, that is the integrating factor for these 2 terms. So if you multiply this equation e power minus 2x, so e power minus 2xy double dash - 2 into e power minus 2xy – plus lambda into e power minus 2xy equal to 0, so that x is between 2 pie. Now i can put this with multiplication of the integrating factor, we can combine these 2 as derivative of e power minus 2xy – or dy by dx. Okay 1 you can see that e power minus 2x y double dash, and if you differentiate this, you get e power minus 2 into the e power -2 xy –. So plus q0 and you have, you can put this as outside minus and this will be lambda into e power minus 2xy, okay.

So this is actually your ly, ly has a definition, so x is between 0 to pie and l is minus ddx. Now p is minus e power minus 2x, okay. And q is 0, so 0 into 0, simply q is 0, so we do not write this. So this is what, so once you identify this operator, we can write the dot product of 2 functions. Here the definition is from this domain which is 0 to pie and w, so w is, you have a w here, so e power minus 2x, so, so you can just divide with this okay, so you can write 1 by e power, so you can write, bring this e power minus 2x this side, that is going to be plus, okay. This is going to be plus, so you can remove this, so where l is 1 over e power minus 2x.

(Refer Slide Time: 4:29)

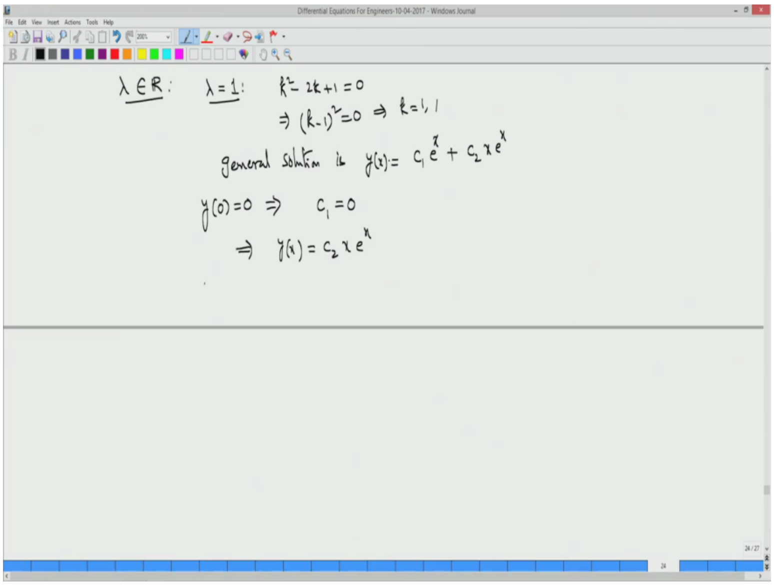




So this is your w, so w is this into fx gx bar dx, this is a dot product you have. So this is what is your sturm louisville equation ly equal to lambda as the eigenvalue problem. Okay. So you need to find the eigenvalues and eigen vectors. Now consider, so consider now l is the self adjoint from, so you have hermitian form implied, lambda is real. Lambda is real since l is in hermitian form or skew symmetric form. So we can now consider, so lambda is real means it can be positive, negative and all such thing. You 1st consider the general solution of the equation, given equation.

So that is y double dash minus 2y – plus lambda equal to, lambda y equal to 0. Okay, so this form, what are the solutions, this is the linear equation with constant coefficients, you can look for solutions in this form e power kx. If you look for this form, you put it, you get k square minus 2k plus lambda equal to 0. So that will give me whatever my k values through plus or minus square root of b square -4 lambda divided by 2. So this will give me 1 + or minus square root of 1 minus lambda. So these are your roots, so implies you have this general solution, general solution is yx which is the second-order equation, so you can get the general solution as some c1 times.

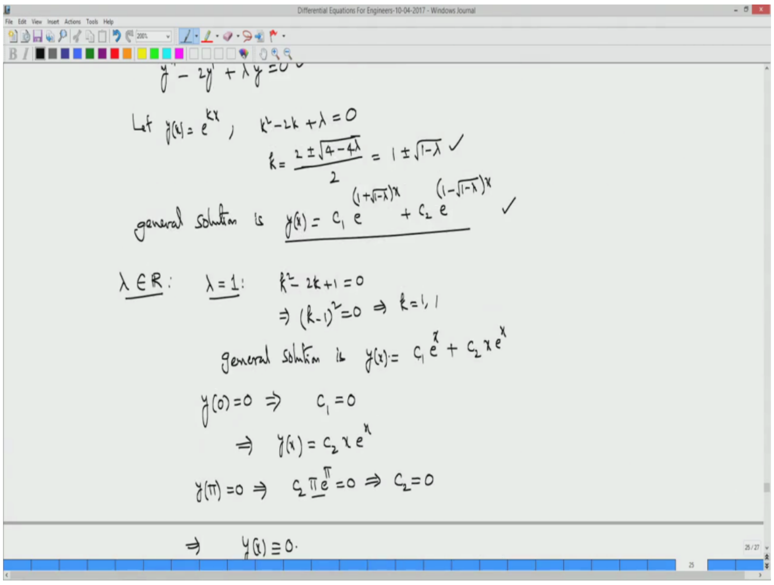
(Refer Slide Time: 6:41)



When you see here, this is c1 times e power 1+ square root over minus lambda into x plus c2 e power one minus square root of 1 minus lambda into x. So this is your general solution. Now, you know that lambda is real, we just start with because, see it involves the general solution involves 1 minus lambda, so let us consider 1st lambda is equal to 1, okay. So you can also think, if it is not there, so you can, you can also look at lambda positive, lambda negative and lambda equal to 0. Because it has at 1 minus in the solution, you have square root of 1 minus lambda, that suggests you work with lambda equal to 1. So if you put lambda equal to1, that is your general solution, so what happens to your roots.

With lambda equal to1 what you get is k, k minus 2k plus1 equal to 0. So you have k -1 whole square equal to 0, that will give me k is 1 and 1. So you have repeated roots, so the general solution is, general solution is e power, c1 e power kx, k is 1, so you have e power x plus c2 x into e power x, okay. So this is what you have, now you apply boundary conditions. What you have is starting with the 1st boundary condition, y0 is 0, if you apply here, and so y0 is 0, so you have c1, so c1 plus c2 into 0. So that is 0, so this is 0, so you get c2 is 0, c1 will be 0.

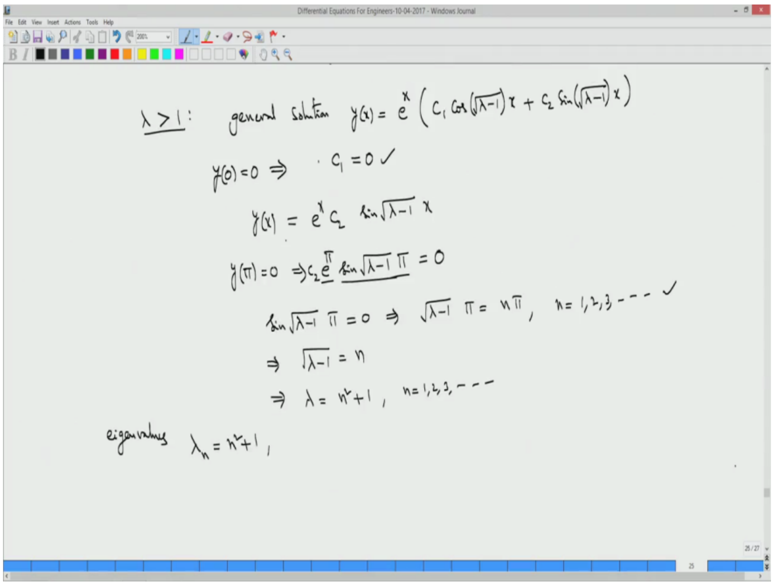
(Refer Slide Time: 8:33)



So this implies the general solution becomes c2 x e power x. Now you can apply other boundary conditions. So apply the boundary, other boundary conditions that is we have y pie equal to 0. So if you apply this, you get c2 by e power by has to be 0, this is possible only if c2, c2 is 0, right. So you have c2 is also 0, this is positive, since this is nonzero, so c2 has to be 0. So general solution finally, that implies general solution or the solution satisfying the boundary condition becomes why x is identically 0. So that means lambda is equal to 1 is not an eigenvalue. Okay. So we can look at lambda greater than 1 and the lambda less than 1.

You have 3 cases that we can look at lambda greater than, we look at lambda greater than 1, then in that case this is your general solution and this will be i, okay. So when lambda is greater than 1, general solution of the equation yx equal to, so e power x is common, so you can look at this e power one x, he power one x is common, so you can take this out and you have c1 and what you get is, when lambda is greater than 1, 1 minus lambda is negative, so you have lambda minus1 i times square root of lambda minus1, lambda is greater than 1. So you have c1 e power minus, so you can rewrite, so you have, that involves e power i square root of lambda minus1 x, here you power minus i square root of lambda minus1 x.

(Refer Slide Time: 11:06)



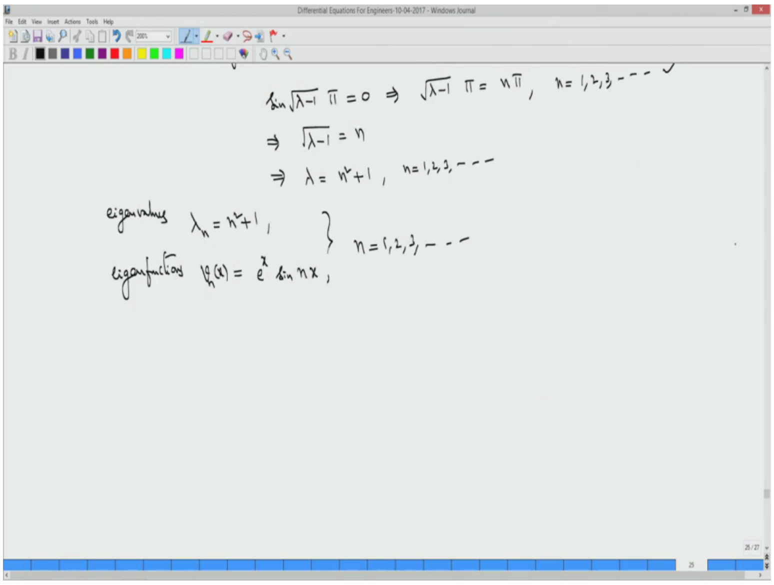
So that you can write also as a linear combination of cosine and sines. So we will write as cos lambda minus1 x under root of whose, square root of lambda minus1 into x, okay and then plus c2 sin square root of lambda minus1 into x, x is outside, okay. So this is your general solution, now you apply the boundary conditions. Y00 will give me, what happens to 0, e power 0 is 1, c1, that is c1 plus c2, when you put x equal to 0, that is 0, so this is what you get c1, this is y0, so this is equal to 0. So you get again c1 is 0, so the yx becomes e power x c2 sin square root of lambda minus1 into x.

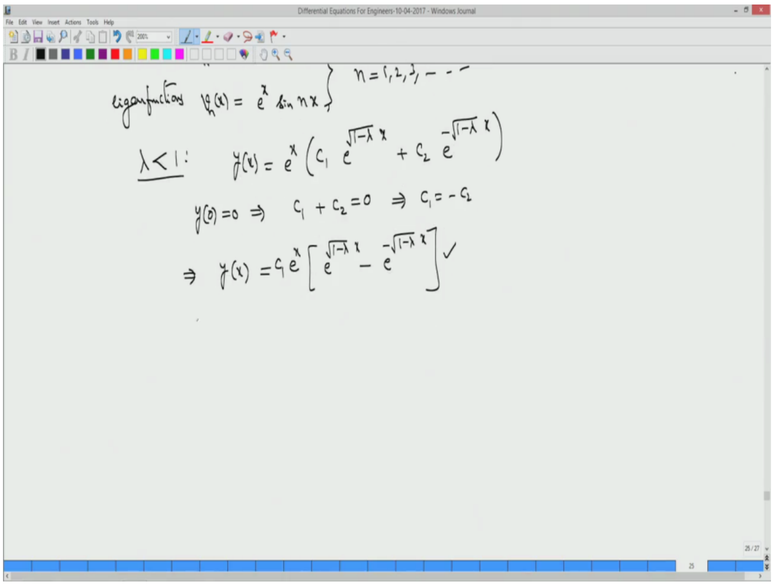
Okay. So now we apply another boundary condition y pie is 0, that will give me e power pie c2, e power pie sin lambda minus1 into pie equal to 0. So this is nonzero, this cannot be 0 and this can be 0 for some lambda values, for lambda greater than 1. So we have to see what are these values, okay. So for such lambda values, c2 can be arbitrary. So you can take this lambda minus1 into pie equal to 0 will give me square root of lambda minus1, pie equal to n pie. And you see that this quantity, lambda is greater than 1, so this, this should be running from running from 1, 2, 3, onwards.

Reason is n equal to 0, if i take an equal to 0, that will become 0, that is lambda equal to1, which already seen that is not an eigenvalue. If i take n negative, then lambda minus1 is actually, it cannot be negative, so it has to be positive. So this is what you are forced so this sin values, positive value, so square root, so square root of this number should be positive. So, right so so we have this. So this implies you have lambda minus1 under root is n, this will give me lambda as n square +1, n is running from 1, 2 onwards. So these are your eigenvalues, lambda n equal to n square +1 eigenvalues for which you have nonzero solution.

So you can, you have c2 can be arbitrary, you can take it as 1, e power x for these lambda values.

(Refer Slide Time: 13:41)

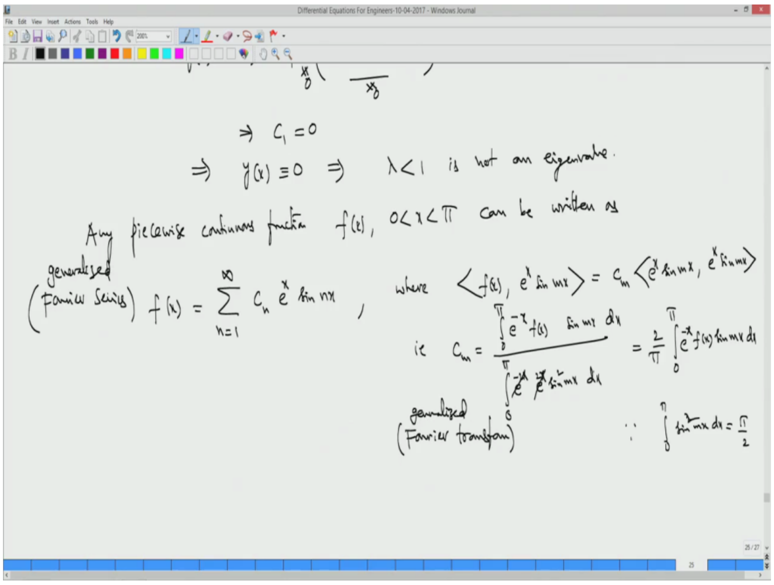




If you put it, so eigen functions will be, you call them vn of x which is e power x, sin root lambda -1 is n, so you have sin nx. So for both n is running from 1, 2, 3 and so on. Even if we put n equal to 0, actually becomes, you see that vn is 0. So it is not an eigenvalue, corresponds to lambda 0 equal to1. So these are your eigenvalues and eigen functions when you consider lambda greater than 1. So you have another case, that is lambda less than 1, if you consider this case, the general solution we can rewrite. So yx is eta were x is common, so you have c1 e power and lambda is negative, when lambda is less than 1, this will be positive, okay, so this will be positive. So you can write as it is.

So you have, simply write 1 minus lambda x plus c2 e power - square root 1 minus lambda x, okay. So now you apply the boundary conditions, y0 is 0 implies c1 plus c2 equal to 0. Okay. So that will give me c1 equal to minus c2, so implies y x equal to e power x, c1 you can take it out, e power x is common, what you have is e power square root of 1 minus lambda x and you have c2, i am replacing with minus c1, so you have minus e power minus square root of 1 minus lambda x. Okay. So this is what you have, so this is your general solution have to plan the boundary condition.

(Refer Slide Time: 15:36)



Now you apply ghee at the boundary condition y pie is 0, you get c1 e power pie the power square root of 1 minus lambda pie minus e power minus square root of 1 minus lambda pie. So this quantity has to be 0. And you see that this is e power something, fix lambda less than one, this is infinite number. So e power, let us call this c, c pie minus e power minus c pie, this will ever be 0. So this cannot be 0 and this cannot be 0, that means c1 has to be 0. So that implies yx is identically 0. That implies lambda less than one is not an eigenvalue. Because you do not have nonzero solutions for the system.

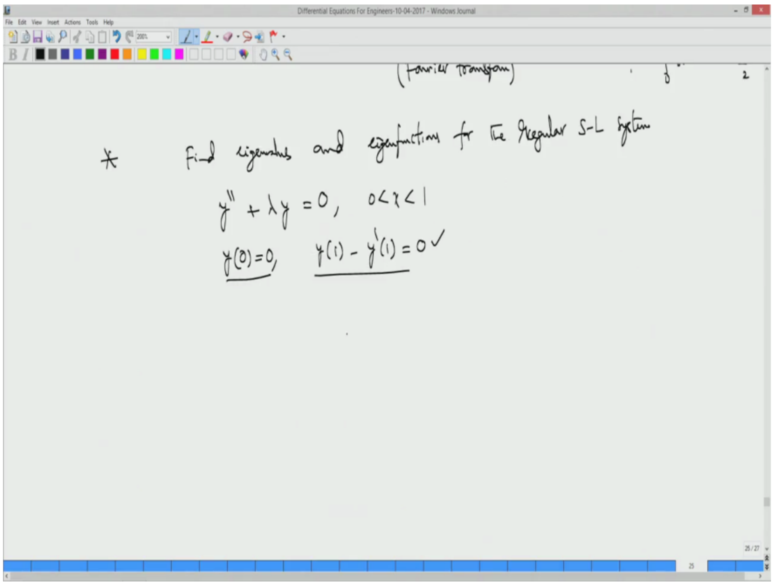
So what you have is, what you have found this, only these are your eigenvalues and these are all eigen functions. So any piecewise function, any piecewise continuous function, piecewise continuous function f x which is the domain between 0 to pie, i can write, can be written as fourier series, fourier series is fx is from, n is from 1 to infinity, you have some cn and what are your eigen functions, e power x sin nx, e power x sin nx, okay. Where cns is simply, we integrate, so take the dot product, where cns are, how do we find my cns, you mult, you just take the dot product with vns, e power x sin mx equal to, so when you multiply this e power x sin mx, only cm will be contributing, m equal to n only will contribute.

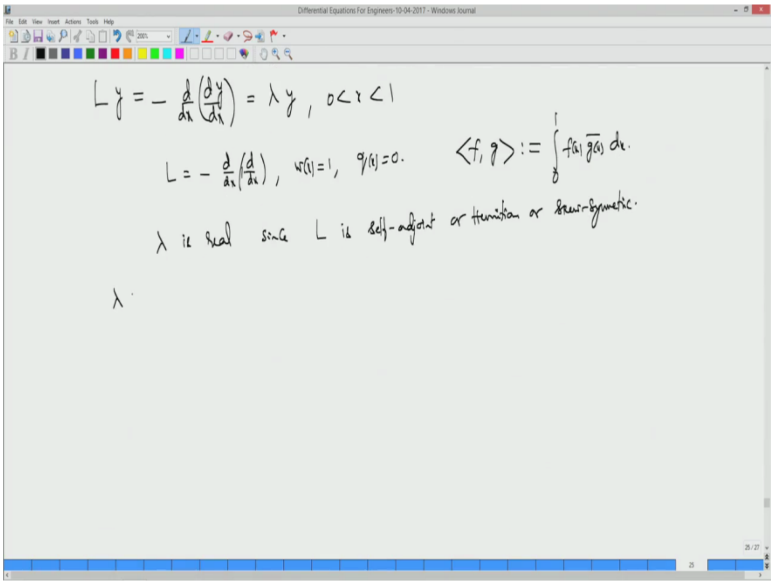
So you have e power x dot product sin nx mx e power x sin mx, okay. That is cm is integral 0 to pie, you have e power minus 2x is wet function, e power minus 2x fx, e power x sin mx dx. So this e power x and e power minus 2x will go, so you have e power minus x this divided by, now this one integral 0 to pie, e power minus 2x. E power x sin mx whole square, so when you will square, this is what you get. So this will go, so what you get is this, this quantity is simply, because integral 0 to pie sin square mx dx is. You can write 1 minus cos 2 mx divided by 2 from 0 to pie, that value is pie by 2.

Just the half that you put the limits 0 to pie, so you have pie by 2 so you have 2 by pie cm is 2 by pie 0 to pie, 0 to pie e power minus x fx sin mx dx. So that is what, that is what is your cm. So these are your fourier coefficients, fourier transforms. Fourier transform with this with respect to these eigen functions, this is your fourier series, fourier series, okay. So because something 2, some simple, some other example, not from, in terms of sines and cosines, so you can say that is generalised fourier series, okay. You can say generalised fourier series and generalised fourier transform.

This, for this particular example, this particular fourier type of transform and series, okay. And this way for a piecewise continuous function fx, this convergence of this series is actually point wise. You fix x as the series, the series of numbers, this converges to a function value at the point. Okay. So look at some other examples of regular sturm louisville system, in which case, so we may not be able to explicitly your eigenvalues. Okay. Let us see that example.

(Refer Slide Time: 20:19)

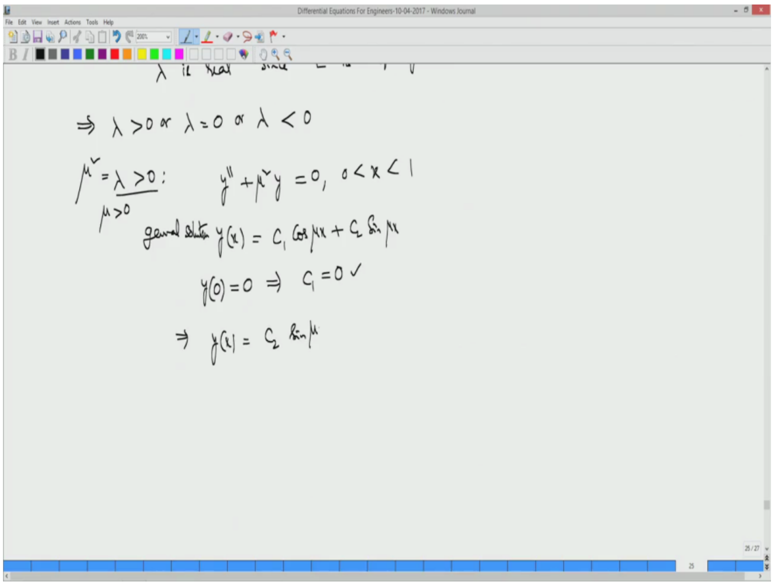




One more example, with that we will wind up sturm louisville theory. So you can have, so what you consider is now find eigenvalues and eigen functions for the regular sturm louisville system, regular sturm louisville system, that is y double dash plus lambda y equal to 0, lambda is a parameter and you have the domain is between 0 to 1, x is between 0 to 1 and you have y0 is 0 and y1 minus y – 1 equal to 0. So because it is a regular sturm louisville system you can give, this is simpler one but this is a combination of its derivative. So we have chosen combination of y and its derivative at 1, this is equal to 0.

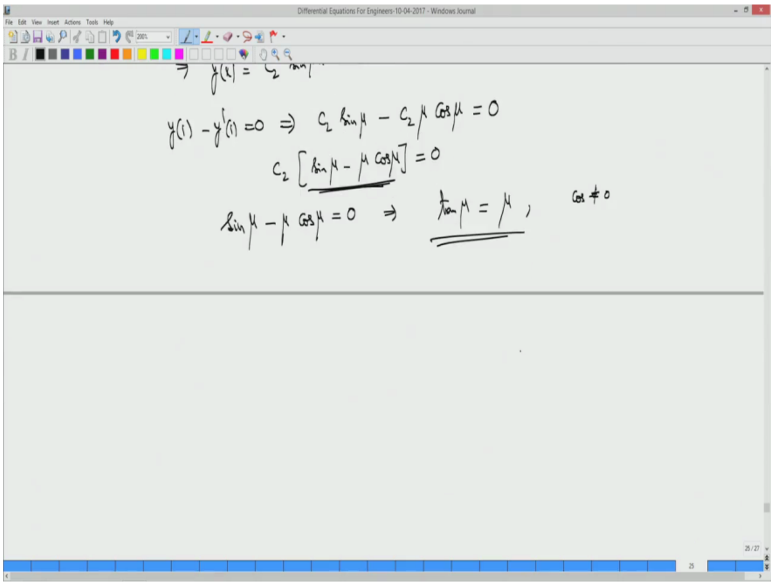
So if you have like this, always when you take the combination, you may not be getting the solution eigenvalues explicitly, we will see how it is. So we write this as y, dy by dx ddx minus equal to lambda y, this is my ly, ly, okay, so x is between 0 to 1. So it is already in the self adjoint form. So l is minus ddx of ddx, p is 1, 1 into ddx and w is 1, q is 0. So dot product of, i can define dot product here, this system is 0 to 1, w is 1, so you have simply fx gx bar dx. So what happens, now you find the, now it is in self adjoint form, so lambda is real, since l is self adjoint, l is self adjoint or hermitian or you can say skew symmetric.

(Refer Slide Time: 22:46)



Now you can look at all the different cases. So lambda is either positive, lambda equal to 0 or lambda less than 0. So we look at all these cases, okay, so start with lambda positive, if you take the 1, if lambda is positive, so we have y double dash plus lambda y equal to 0. So lambda positive means lambda equal to mu square, mu is positive, as usual we can think of like this, we can put lambda equal to mu square. So we will get general solution, general solution y x is c1 cos mu x plus c2 sin mu x, so this is the general solution. Okay, between, x is between 0 to 1.

(Refer Slide Time: 23:58)

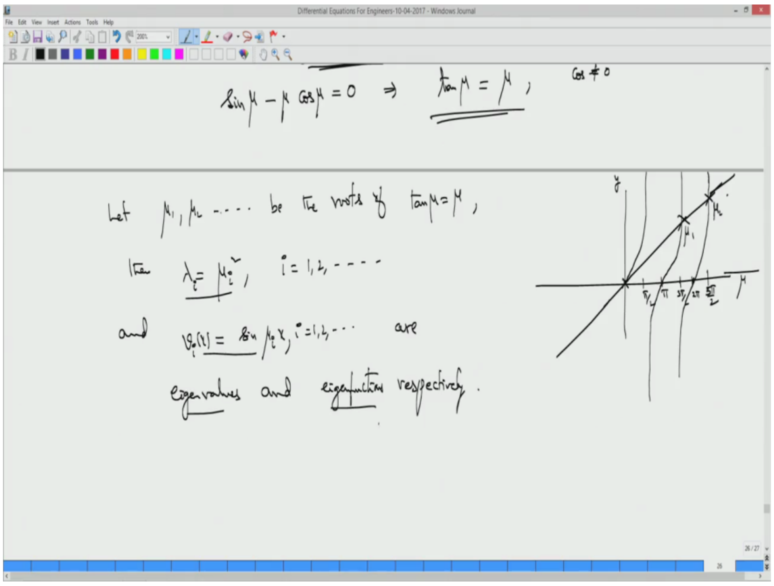


Now when we apply the boundary condition, like earlier we have, so you have y0 is 0 will give me c1 is 0, c1 is 0, that is 1, cos 0 is one plus c2 into 0, that is 0. So i have c1 which is equal to 0. So we have c1 is 0. So general solution becomes now c2 sin mu x. Now you apply other boundary conditions, that is y at 1 minus y – at 1 equal to 0. If you apply the boundary condition here for this, you have c2 sin mu minus c2 mu cos mu into 1. So that this, this has to be 0, so this is nothing but sin mu minus mu cos mu has to be 0. So i just have to see for those new positive values for this point is 0.

Is no such value satisfying this equation, that means c2 has to be 0, that means all new positive are not an eigen values. But we have to see this one, this is called dispersion relation, so sin mu minus mu cos mu, whether it has any positive mu solutions. So this is, this is actually tan mu equal to mu. Right. We can just divide, taking that cos mu, so we can rewrite this tan mu equal to mu. So sin mu divided by cos mu, that means cos mu should not be 0, right. When cos mu equal to 0, that is you know that these are pie by 2, 3 pie by 2 and so on, so these are the values, we know that this quantity is nonzero. Okay.

So this except, except that this has to be 0, so cos mu should not be 0 when you do this one, when you divide. Suppose it is 0, okay, suppose if it is 0, that is true only at these values. But these values you can easily see that this quantity is nonzero. So these mu pie by 2, 3 pie by 2, this quantity is nonzero. So you can think of other values, you can divide it and you can now find the roots of tan mu equal to mu. Okay.

(Refer Slide Time: 26:13)



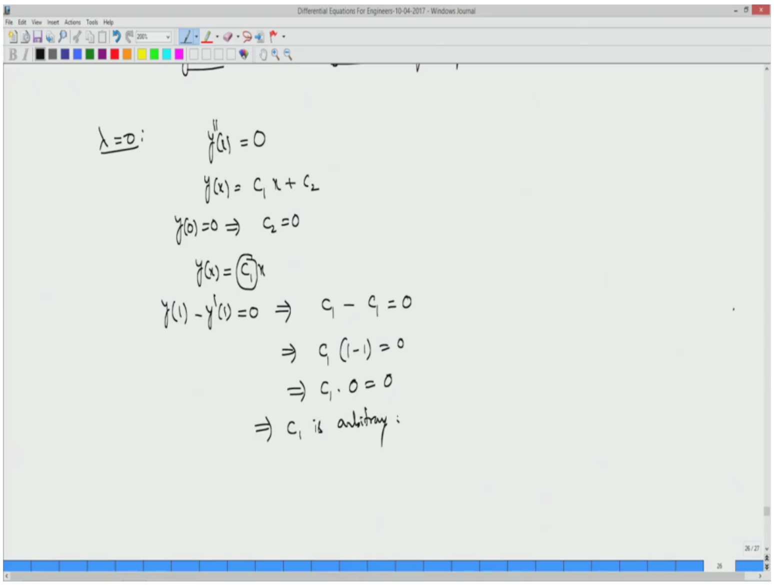
So this we do graphically, this is like let us say mu is here, this is my tan x equal to x, okay, this is your y. Y equal to mu x, y equal to x is something like this line. And you have tan mu, that is tan x, y equal to tan x, so tan 0 is 0, tan pie by 2 is, only you want positive side, it tan pie by 2, tan pie, tan 3 pie by 2 and 2 pie and so on. So for this you have, you have things like this, when you say pie by 2, it is going to infinity, so you have something like, going to infinity at pie by 2. Other things you will have tan, what is the other value, so that is tan pie, and what is tan pie, sin pie by cos pie, so that is 0.

So it should be 0 here and here at pie by 2, the negative side, it should go to, it should nicely go to, eventually asymptotically this should go to minus infinity. At 3 pie by 2 it should go to asymptotically minus infinity. Again at 2 pie by 2 and 2 pie plus pie by 2 is 5 pie by 2. And again here you have something like this curve. Okay, this is at 2 pie, so like this you get. So you see that this curve and this car is touching at these points, okay. When this is 0, so we do not consider, so this is your, because mu is positive, mu is positive, so you have to worry about only this 1, this is your mu 1, this is your mu 2 and like this and so on.

You can denote them like that and say then let mu 1, mu 2 and so on be the roots of, you have roots, that is clear from this graph, tan mu equal to mu. Then lambda is mu i square, y is from 1, 2, 3 and so on, that is your lambda i, these are your eigenvalues for which c2 is arbitrary. Once c2 is arbitrary, so you have yx is sin mu x, sin mu x, those are your vi of x, these are eigen functions. Sin, sin mu i, mu ix. So these are eigenvalues, then these are eigenvalues. So these are, this and this are eigenvalues and eigen functions respectively. This corresponds to eigenvalue and this corresponds to eigen functions.

So you do not know explicitly exactly what are your mu i's, some values, okay. Wherever, so numerically you can find out all these values mu 1, mu 2, 3 and so on, there will positive, okay. So i form certain eigenvalues are eigen functions corresponding to the case mu positive, that is lambda positive. Now look at the lambda equal to 0 case.

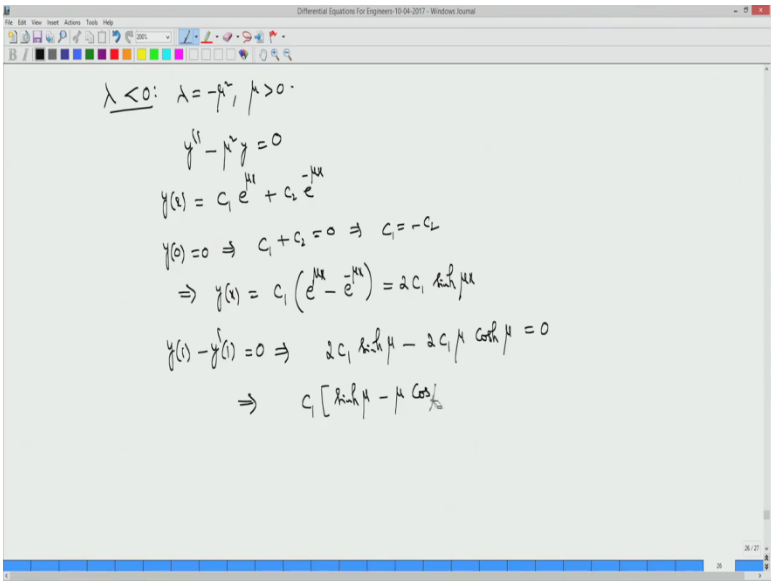
(Refer Slide Time: 29:52)



Now you know that general solution of the equation is y double dash is 0 is equation 1 lambda equal to 0. So the general solution is c1 x plus c2, you apply the boundary condition y0 is 0 will give me c1 into 0 + 2, so that will give me c2 is 0, okay. So y x becomes c1 x, now you apply the other boundary condition y at 1 minus y – at 1 is 0. So this will give me c1 minus c1, this derivative, that is also c1, which is 0, it is satisfying. So you see that, you take any arbitrary value of c1, so that means nonzero value of c1, it is actually satisfying the 2^{nd} boundary condition which is satisfied, okay.

So that means, so this is something like, see what you have is c1 times 1 - 1 equal to 0. So you have c1 times 0 is 0. That means c1 is arbitrary. Once you have c1 arbitrary, the general solution is this one. So you have nonzero solution, so this implies lambda equal to 0 is an eigenvalue. And we call this lambda 0 corresponding to v equal to 0. So v0 of x, corresponding eigen function i am denoting as v0 which is, we can take c1 as 1, so you have this is x, is corresponding eigen function.

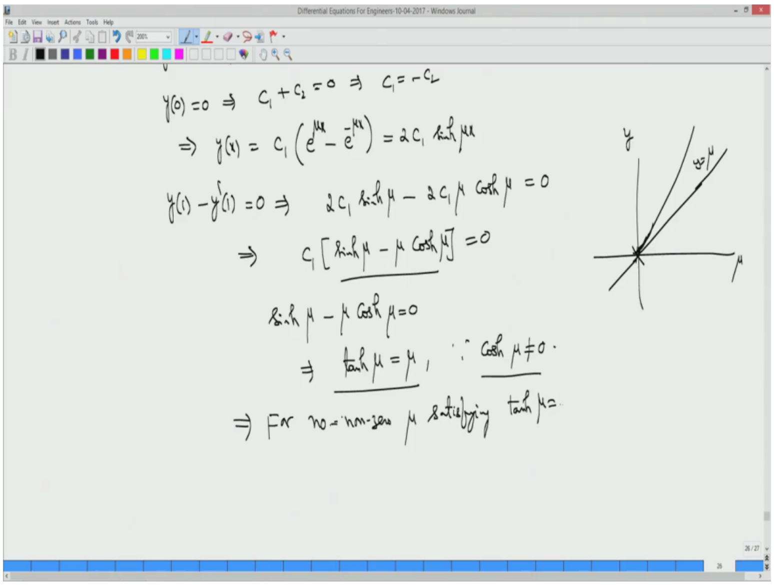
(Refer Slide Time: 31:37)



Now look at the case lambda negative, that means lambda equal to minus mu square with mu positive. In this case, what is your y double dash, minus mu square y equal to 0, this is how the equation becomes, general solution of this equation is c1 e power mu x plus c2 e power minus qx. You apply the boundary conditions now we will give me c1 plus c2 equal to 0, that will give me c1 equal to minus c2. This implies solution becomes, general solution becomes c1 times e power mu x minus c2 i am replacing with minus c1 so you have e power minus mu x. Okay.

So now you apply, this is actually equal to c1, 2 c1 and this is sin hyperbolic mu x, mu is positive, okay. So now you apply the other boundary condition, y at 1 minus y – at 1 equal to 0 will give me for this, we are 2 c1 sin hyperbolic mu - 2 c1 mu cos hyperbolic mu equal to 0. So this will give me 2 c1, 2 cannot be 0, so this has to be c1 times sin hyperbolic mu minus mu cos hyperbolic mu has to be 0. So this i just have to check whether for some positive mu values this quantity is 0. So again we do the same thing, so to check this sin hyperbolic mu minus mu cos hyperbolic mu equal to 0, that is we can do if this is tan hyperbolic mu equal to mu if cos hyperbolic mu is not equal to 0, because you are dividing with it.

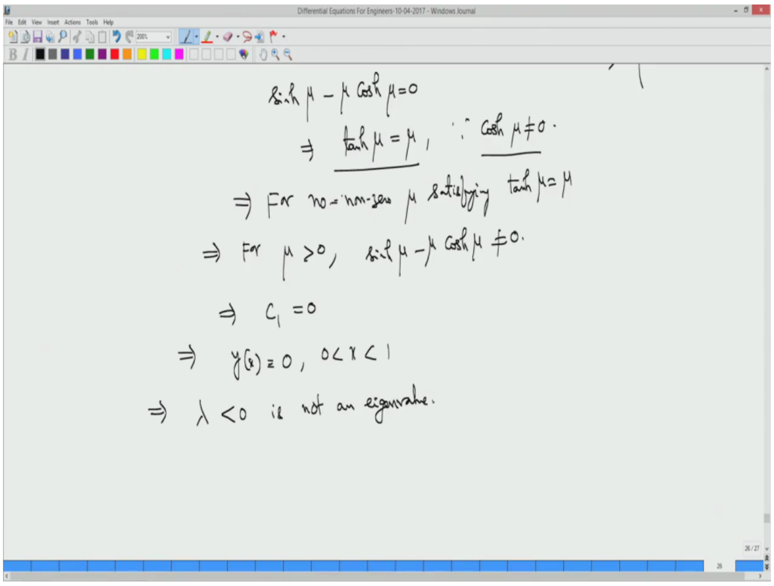
(Refer Slide Time: 34:16)



But this will never be 0, this will never be 0 for any mu positive value. Actually you can do it, okay, it is not just is the cos since this is nonzero, you can always divide. Do you look at this tan hyperbolic plot and this y equal to mu plot. This is y equal to mu and you plot tan hyperbolic mu, they never touch anywhere, okay. You can just do it, they do not touch any other place. So you can plot it and see tan hyperbolic mu do not have any solution, nonzero solution. Okay, positive solution, strictly positive mu, you do not have any solution.

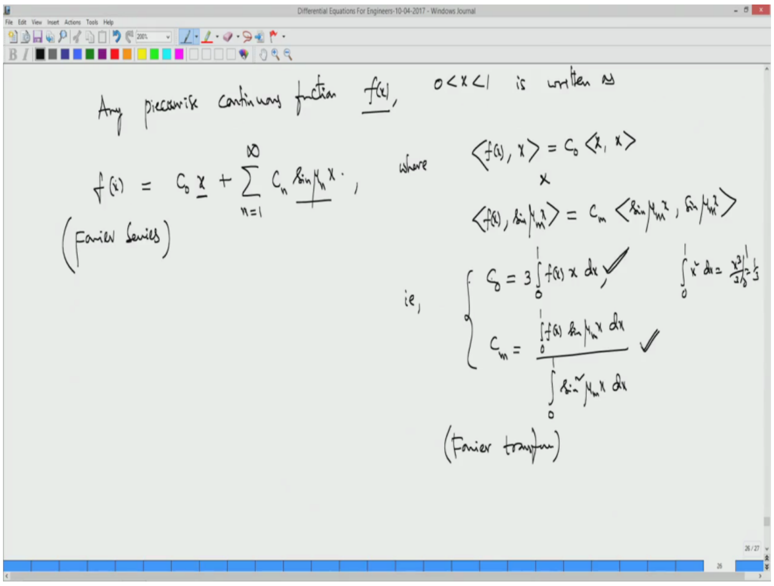
Actually touching and above it goes, simply goes above, only touching at 0, this is the only route but that is mu equal to 0. So implies for a nonzero mu, no nonzero, no nonzero mu satisfying tan hyperbolic mu equal to mu. That means, that implies for mu positive this quantity sin hyperbolic mu minus mu cos hyperbolic mu is nonzero. That means by looking at this c has to be 0, c1 has to be 0.

(Refer Slide Time: 35:03)



That implies the general solution becomes, earlier after applying the 1st boundary condition over general solution is this, now that you found c1 is 0, that means this is completely 0 between 0 to 1. That means lambda negative is not an eigenvalue. Any lambda negative is not an eigenvalue. So what you found is finally you have only, you have this is one eigen function corresponding to 0 and the other eigenvalues and eigen functions which form mu i's sin mu i's, mu i is satisfying than mu equal to mu. So now you know what are your eigenvalues and eigenvectors, as usual you can write the fourier transforms and fourier series in this case.

(Refer Slide Time: 35:59)



So any piecewise continuous function fx which is defined between 0 to 1, is written as fx which is equal to, what you have, it is running from, i is running from 0 to infinity, 1 to infinity is what you have for lambda positive and i lambda equal to 0 corresponding to i equal to 0. So you have c i and what is your functions, vi, so those are sin mu ix and x, you have 2 such things. So you have, you can rewrite like c0 into x, that is one corresponding to lambda equal to 0, and these things you can sum it up, with n is from 1 to infinity and you have cn sin mu nx. Okay.

So you have these are your eigen functions sin mu ix, so i will just change the indx as n. Okay. Where ci, so how do you get your c0, c0 you should get it from, by, this is your eigen function, this is your eigen function, so by multiplying this eigen function x and take the dot product with f you can get your c0 x, x. Okay. And this if you take, your dot product with sin mu nx, later mu nx, you write mx, you can get cm got up with sin mu mx, sin mu mx, okay. So what are these, so you can think of c0 is integral 0 to 1, i do not have, so the weight function is only 1, so it is usual dot product fx into x dx, between 0 to 1 into the real part, so does not matter.

So you have to divide by integral x square, so you have to write 0 to 1 x square dx. That is x cube by 3 between 0 to 1, so which is 1 by 3. So you have 1 by 3, so you have total 3. So you finally get c0 as this one between this. And cm has integral 0 to 1 fx sin mu x dx diverted by this, because you do not know what is exactly your mu m, we cannot, may not be able, you can actually find sin square mu mx dx. This you can calculate and put it, this is how you find this. So these are your fourier transform, generalised fourier transforms and this is your fourier series. You can think of fx as the signal, you can split it into discrete frequencies, at these frequencies you can have these solutions, okay.

These are your discrete frequencies, so we have 1 to, 0 to infinity. So divide by frequency, you split the signal, since we have this fourier transform, you can get back your signal by combining all of these discrete frequencies, okay, in these functions. So this is how you can get eigenvalues and eigen functions. So here you got implicitly, so you would not find explicitly these eigenvalues and eigen functions, right. So this is the example i will tell you story. So this is our sturm louisville theory, so what you will learn from this sturm louisville theory is it is just the property of second-order linear differential equation, okay, ordinary differential equation.

So using this you can actually develop what is the fourier transform and fourier series of a, of a function defined on a finite interval. So if you think of this as a real line which is a periodic function, okay. And that finite interval is whatever is defined repeated everywhere as periodically. So such a thing you can represent as the fourier series in terms of eigenvalues and eigen functions. Okay. So another use of this sturm louisville theory is you try to extract these sturm louisville problem when you solve partial differential equations in a simpler domains, that we will do in the future videos, okay.

So when you are solving these partial differential equations, in a simpler domains, such as rectangular or circular or elliptical domains, you may have to convert, you may have to extract, the main idea is to how will you solve this partial differential equation, idea is to extract, if you can extract this sturm louisville problem out of the boundary value problem, whatever you have for the partial differential equation, then you get all the solutions, eigenvalues and eigen functions of the corresponding whatever you get extracted sturm louisville problem, using them you combine them, you make a general solution of the pde and get back your unknowns. And take a linear superposition because it is a linear equation, you can only solve linear partial differential equations on a simpler domains using this sturm louisville problems. That we will see in the next videos.