Differential Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-38.
Examples of Sturm-Louisville systems (continued).

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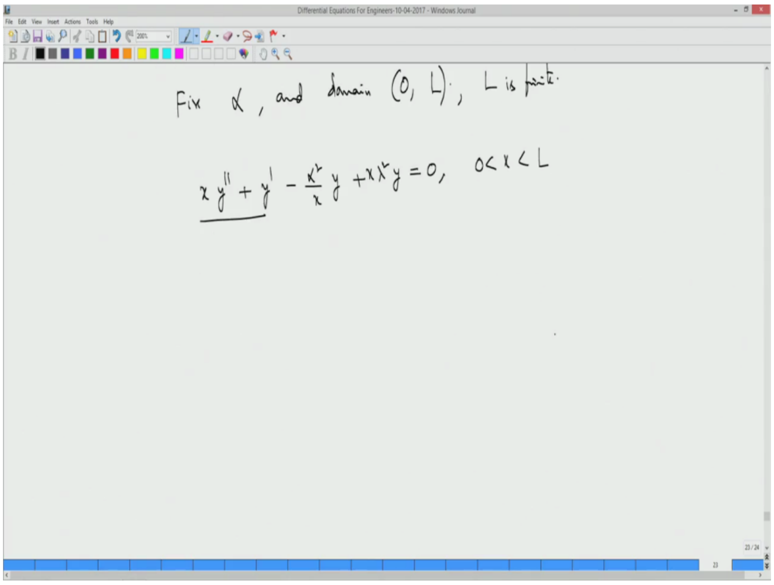

The last video we have seen an example of sturm louisville system, comes from legendre's equation, so we can have another example of singular sturm louisville system based on bessel equation. So we will see what it is, we can write this example of another example example, one more example, you can write mo, one more example of singular sturm louisville system. So that is basically we take bessel equation. So you can have x square y double dash plus xy dash minus alpha square, we use alpha square, so alpha square. Actually what you have is earlier plus x square minus alpha square y equal to 0. This is your bessel equation.

If you replace you can also have modified equation, that is what i have shown, when you replace x by lambda x, what you get is same, okay. So what you get is x square lambda square x square d square y by d square y by dx, lambda square x square, that lambda square, this lambda square cancels, these 2 terms will be either this. So you can replace lambda, x by lambda x, so we can have lambda square x square minus alpha square. This is also bessel equation type, okay. Where is the domain, so domain is from, for this x positive. So for the sturm louisville problem, system, it should have a finite domain, okay.

So that means, so what are the parameters, have, i brought in this parameter lambda, eigenvalue parameter lambda and this i have already one parameter in the bessel equation, okay. And my domain is unbounded domain, x is greater than 0. So i have to make it a domain as finite, okay, so what i do is, i make this domain fixed by keeping some l, i fix it and i fix this l, i fix this alpha which is the bessel parameter and this lambda i keep as eigenvalue. Okay. So eigenvalue parameter. So if you do that, what happens, so if we can rewrite, fix alpha and domain 0, 0 is only open and l, okay, this you fix it. This is what is your domain.

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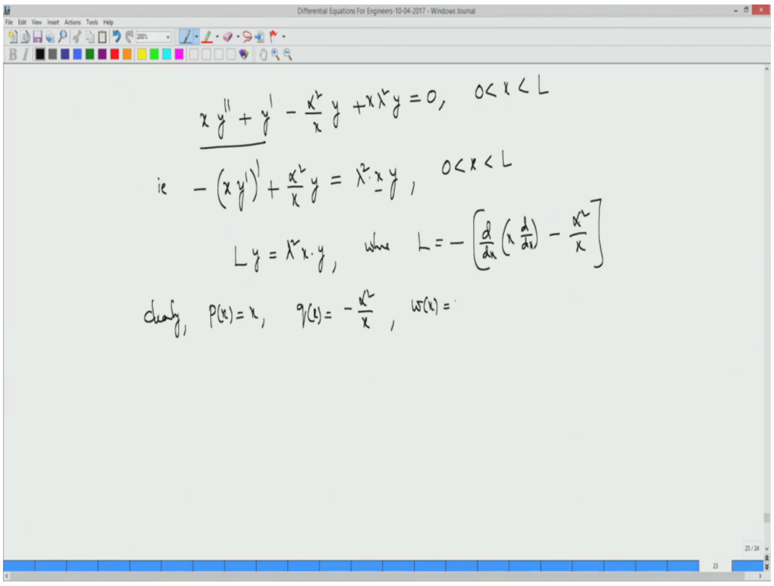

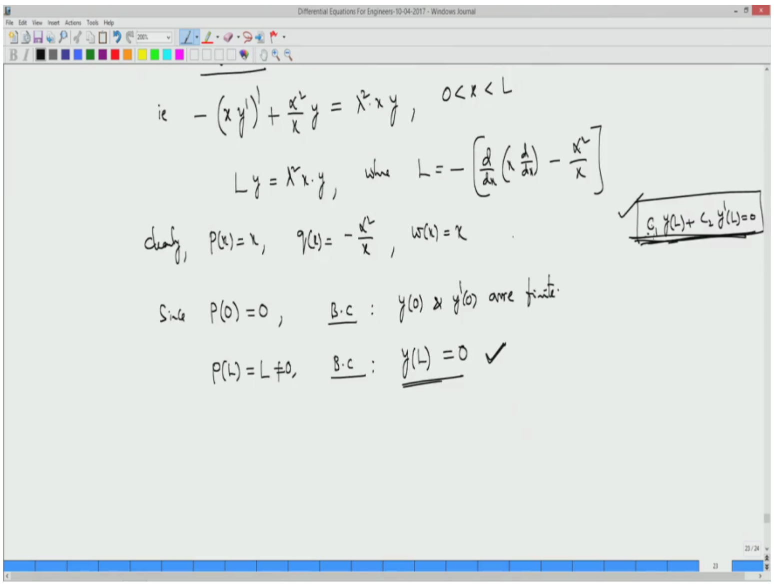

Domain you fix some finite quantity, l is finite. So if i do this, what is equation, so equation is x square, so you can have equation, i read, we write like this, i divide with x because x is anyways greater than 0, i can divide it so i can rewrite xy double dash plus, i simply, i simply cancel one x both sides, i divide with x, so you get y dash, so that these 2 terms i put it as self adjoint form minus alpha square by x into y, that is your qy plus lambda square y into x, okay, equal to 0. This is from this between this. So that is, you can rewrite this as pxy dash of dash, okay minus alpha square by x y, okay, minus n plus equal to lambda square xy between x less than 0z less than l, okay.

So what is this operator, this is like in the form ly equal to lambda, so lambda is already lambda square, okay, so this is a kind of lambda square, okay. Lambda square x into y, this is lambda mega y, so where lambda is, where l is the operator l is minus ddx of x ddx plus or rather minus here, if you put the bracket, alpha square by x. So this is your, this is like a self adjoint form, so you already have, so p is clearly p of x is x, q of x is what is q of x, minus alpha square by x and wx is now x, okay, so you have x, x is between 0 to l.

You see that p of 0, boundary points are 0, p of 0 is 0, since p of 0 is 0, immediately my boundary condition should be, boundary condition which i have to define as y of x, y of 0 and y dash of 0 are finite or bounded. What happens to p at l, which is, p at l is l, which is nonzero. So a boundary condition should be, one of the boundary conditions we can use either, this you can, because this is nonzero you can provide, you can provide regular sturm louisville system boundary condition, that is c1, y at l plus c2 y dash at l equal to 0, okay. So combination or, so you can choose any of this c1 and c2, not both are 0. So not both are 0 means, either c1 is 0, if c1 is 0 c2 is 0, c2 is nonzero, c2 has to be nonzero, that means you can give y dash of l is 0, y at l0 or a combination of them, you can also give, okay.

So what we do is because with this boundary condition which i give, i take c1 equal to1 and c2 equal to 0, that makes it y at l equal to 0, a simple regular sturm louisville boundary condition. Okay. You can actually take in general like this, you can choose c1 and c2, fix your c1 and c2 to make a combination and you give but the only thing is that, we did not know, we do not know, with this boundary condition we do not know, we need not solve the equation, it is not easy to solve this bessel equation, okay. But with this boundary condition we already solved, if you actually see, okay.

So if you look back, the orthogonal property of the bessel function j alpha of x, okay, x between 0 to l is actually satisfying this form, okay, from, because we know that, that is why i am fixing this one. You need not fixed this boundary condition, you can fix any boundary condition with the sum, not both c1 and c2 are 0, some nonzero c1 and c2 you can choose and you will make it 0, that you can write it here. This we choose because we have already solved it. Okay.

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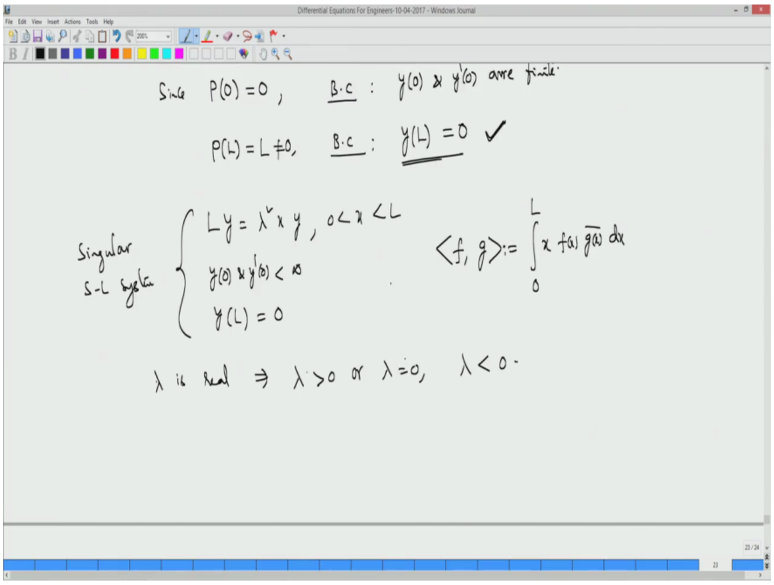

So, so the eigenvalue problem is ly equal to lambda square xy and your boundary conditions are y0 and y dash of 0 is finite and y at l equal to 0, x is between 0 to l, so this is our singular sturm louisville system. Okay. So immediately, because w is this, what is your dot product, dot product is integral between 0 to l, now i have a wx, that is x fx gx bar dx, that is your dot product, this is the definition of dot product, okay. So what are these solutions, now we know that this is lambda is positive, lambda is real, this is now, this operator is in the self adjoint form, so a skew symmetric form or hermitian form, so this lambda is real implies lambda is positive or lambda equal to 0, lambda is negative.

So in all these 3 cases you can see the solutions, okay. What are the solutions we have, what are these? And 1st of all you consider some any l, any lambda, l by, ly minus lambda square, ly, l y minus lambda square xy equal to 0. You consider the equation, that is, what is this ly minus lambda square y, this is actually this equation itself. That is xy double dash, okay, so that is nothing but your equation itself, okay, both are same. So if you have x square y double dash, i will read, this is actually x square y double dash plus xy dash plus lambda square x square minus alpha square y equal to 0, this is between 0 to l.

And we know that this has solutions, okay, this has solutions that has to be bounded at 0 and at l it has to be 0. What are those boundary, what are those, what are those values for which you have? Those are, we have seen already, so this now we have, so the solutions, solutions, the solutions of solutions of the equation, above equation satisfy the boundary conditions are j alpha, alpha are fixed, of lambda i, okay, they are x, right, they are x, these are the solutions. So solutions, these are the solutions where lambda i satisfies j alpha of lambda i l equal to 0. Okay.

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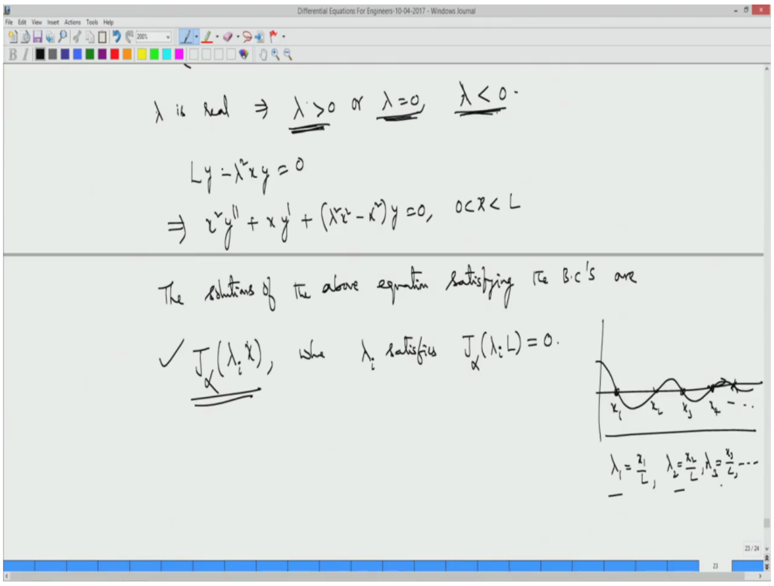

So based on these j alpha roots, okay, this j alpha roots, based on these j alpha roots, these routes, x1, x2, that is what we may ride, so you have x1, x2, x3, x4 and so on. I have infinitely many root for j alpha function. So you can choose your lambda 1 as xi by x1 by l, lambda 2 is x2 by l and so on, x3, lambda 3 is x3 by l and so on. So this is what you have, so all these lambda i's are nothing but roots of this j alpha divided by l, l is fixed domain, the finite number in the domain. So this is what you have, so these are, these are the functions that you have already seen that,, okay, either it is positive or negative, so basically you see only, they are defined only positive sides, so we have only, for positive side you have these eigenvalues. Okay.

These are icon functions corresponding to lambda positive because lambda when x1, x2, x3 are in the positive side, the lambdas are all positive. Lambda equal to 0 you do not have, so there is no root, j alpha will never be 0 at x equal to 0, okay. So this is not an eigenvalue, negative also is not an eigenvalue. Okay. Since j alpha of x has positive roots equal to, roots one right, equation you have to write, without equation you have to write positive zeros, positive zeros, lambda i's are all positive, okay. So immediately implies eigenvalues are positive, positive only.

So this implies eigenvalues are lambda i's, what are lambda i's, so eigenvalues are actually lambda square actually, so lambda square, so that is lambda square, lambda i square are eigenvalues, corresponding eigen functions are j alpha of lambda i x are eigen functions. Where is this from, i is from 1, 2, 3, onwards, this is simply a notation. If your roots are, roots of j alpha are x1, x2, x3 and so on, you can have i's, indx is also from 1, 2, 3 onwards. If i denote them as x0, x1, x2, then i can, my indx can have i is from 0, 1, 2, 3, onwards, okay. So these are simply eigenvalues and eigen functions directly which we calculated already.

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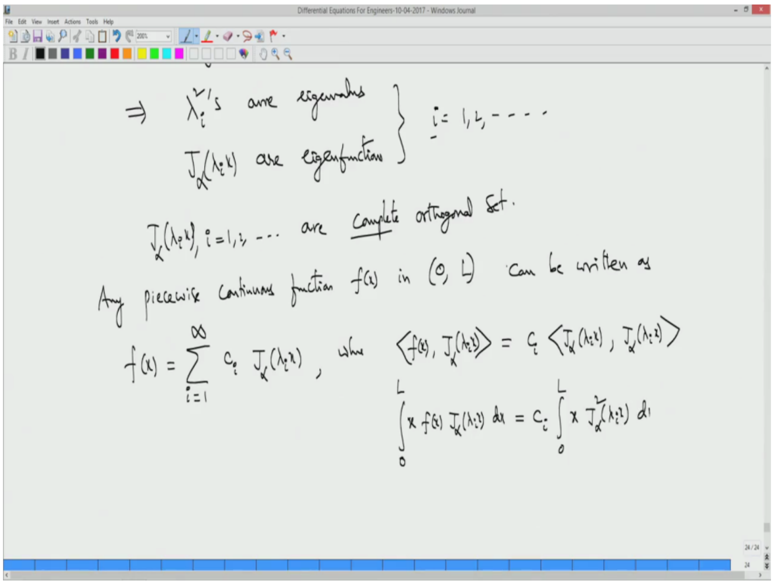

We have already done, basic groundwork we have already done while solving the bessel equation, so that now you have already seen putting in this setup, and seeing that lambda i squares are eigenvalues and j alpha of lambda ix are eigen functions, just like the legendre case. So immediately, so from the properties for skew symmetric or hermitian operator l, you see that j alpha, these eigen functions are orthogonal and complete, okay. From the properties, j alpha of lambda i x, i is from 1, 2, onwards a complete orthogonal set.

So meaning of completeness is any piecewise continuous function fx in 0, 1 can be written as, as the fourier series, you can have a fourier series which is i is now, i have this 1 to infinity and you can have cn j alpha of lambda i, x, okay where ci, ci's i can get it, ci's, how do i get this, take a dot product, so dot product with this s, with this eigen function j alpha of lambda i, x which is equal to ci integral this is also dot product which is j alpha of lambda i, x, j alpha lambda i, x. What is this dot product? Integral - 0 to lx fx j alpha of lambda i, x dx.

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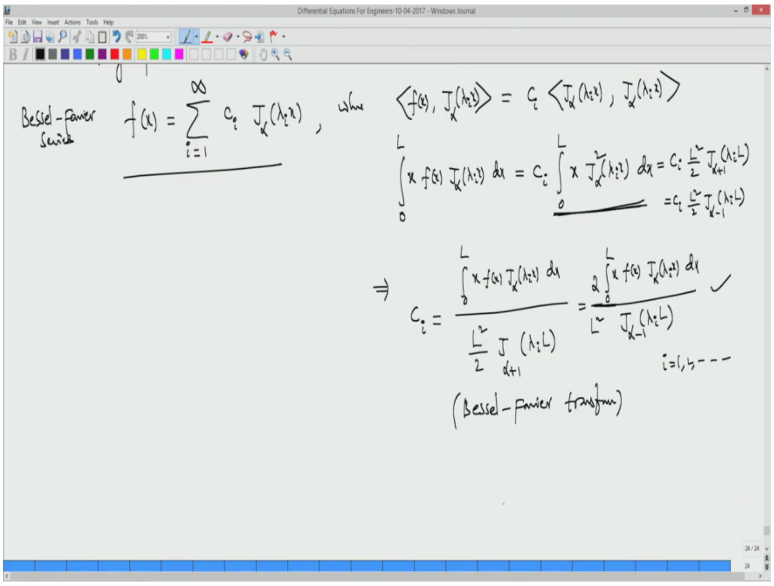

J alphas are all real valued functions, so bar is immaterial, so you have ci integral 0 to lx into j alpha of lambda i x this square, dx. Okay. So we know already what this is, okay, we just calculated it. Look back at labour bodies of bessel functions, we know this value, okay. So by using this you can get your ci finally, but this is what is the fourier series with these other fourier coefficients as i, so i can write ci as 0 to lx fx j alpha of lambda ix dx divided by this integral value l square by 2j alpha plus1 of lambda i l, okay. Or it has a value, this is actually equal to ci times l square by 2 j alpha plus1 of lambda il or ci times l square by 2 j alpha minus1 of lambda il, both are same, okay.

Either this or this, both are same, so you can use any one of them, okay. These are 0 to l x fx j alpha of lambda ix dx, then you have 2 goes up, 2 by l square divided by this integral, so 2 by l square is divided by j alpha minus1 of lambda i l. So these are your fourier coefficients, i is from 1, 2 onwards, 1, 2, 3 onwards. So these are fourier transforms and these are your fourier series, okay. So this is a bessel fourier series, bessel fourier transform, okay. So these are, so you can have like this you can have many fourier serieses, okay. You can break that time signal into in terms of these orthogonal constants, complete orthogonal functions j 1 of lambda ix, i is from 1 to infinity.

Instead of sines and cosines have get back your signal in terms of this orthogonal functions as a discrete sum, so that is 1 to infinity, take linear combination of this discrete frequencies, if you add them up, you can get back your signal. So this is what we can get. So these are these are the only 2 examples we do for this singular sturm louisville system. You can pick up many things but the only thing is you have to do the groundwork to solve, to find the eigenvalues and eigen functions that involves a lot of work to find the solutions. For example if you consider hermite equation, you can hermite polynomials okay, deform complete set and get your hermite fourier series and so on like that.

Legendre log array polynomials, these are the all the special equations, special functions or eigen functions, with that you can get your fourier series. So a periodic system is unique which have done mostly general setup, it is between the domain a to b and we have seen only one example for the regular sturm louisville system. So we will have one more example for the regular sturm louisville system, we can solve and try to find the eigenvalues and eigen functions.

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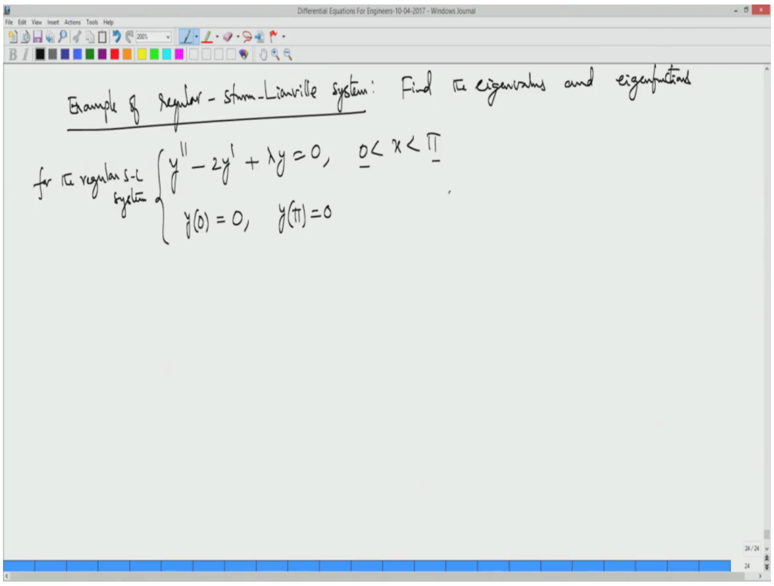

One special type of regular sturm louisville system, okay. So an example of regular sturm louisville system. We have this example of regular sturm louisville system, we will do this now, so we have equations are y double dash minus 2y dash plus lambda y equal to 0 and x is between 0 to pie, let us take this. Because regular sturm louisville system, you see that coefficient of y double dash is nonzero, so that means you can expect p of x is nonzero at, at those endpoints 0 and pie, so the boundary conditions i give, it is given that these are y pie and 0, take the simpler boundary conditions, these are the boundary conditions.

So we will try to find eigenvalues, so basically the problem is to find the eigenvalues and eigen functions for the regular sl system, this, okay. So we will try to find these eigenvalues and eigen functions, just like what we had earlier for the regular sturm louisville system. Only difference is you will not be having explicit eigenvalues, you may have to denote them based on some equation, it is called dispersion equation. So we will see this, this example, how to get these eigenvalues and eigen functions, okay. That will see the next video, thank you for watching.