Differential Equations for Engineers.
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Lecture-37.
Examples of Sturm-Louisville Systems.

We were looking at an example of periodic Sturm Louisville system, what we have seen is we, we have seen that operator, we just, we were trying to find the eigenvalues, we found the eigenvalues and eigen functions corresponding to lambda is positive. So lambda equal to 0 and lambda as negative, that is lambda equal to minus mu square, we have to check whether they are, they, there may be any eigenvalues in them. Okay, corresponding eigen functions, if there is any eigenvalue, okay. So we will try to see those 2 other cases. So we start with lambda equal to 0.

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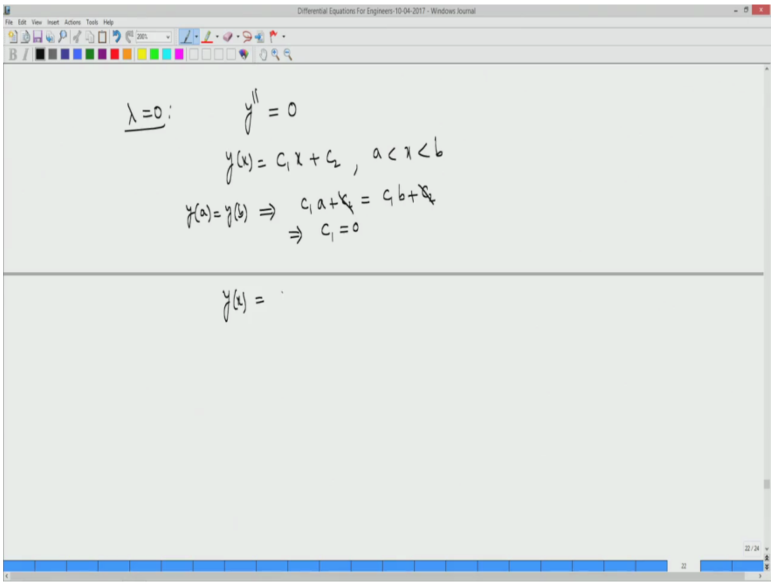

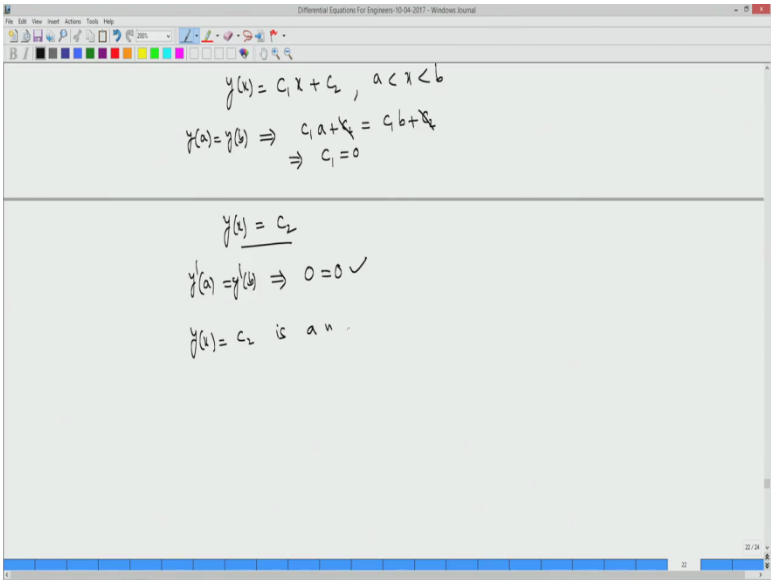

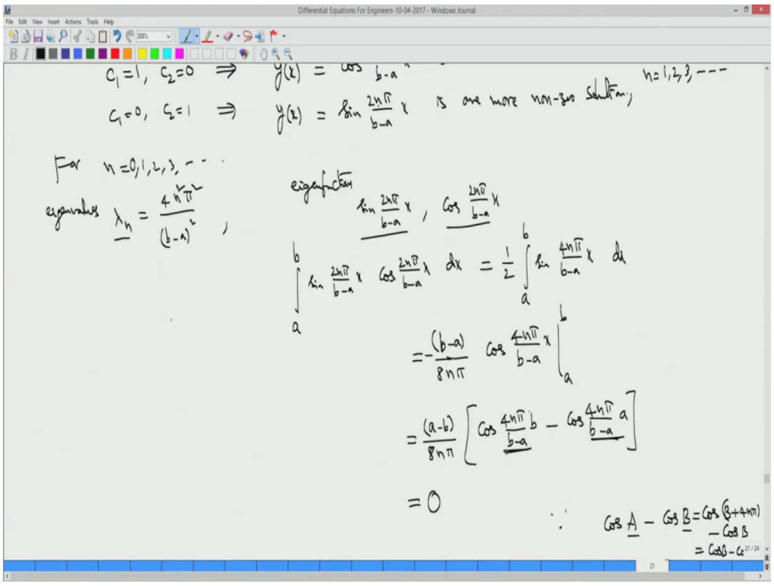

If lambda equal to 0 if you do, what is equation, said the equation is y double dash plus lambda y equal to 0, so basically equation becomes lambda is 0, that is why this is 0. So the general solution is y of x equal to C1 x plus C2, x is between a to b, okay. Now you apply the boundary condition bound, that is y at a is equal to y at b. This gives me C1 a plus C2 equal to C1 b plus C2, C2, C2 cancel, so this will give me C1 has to be 0, because a and b are different from So C1 has to be 0 if you want this to be same. Okay. So what is your general solution, then the general solution becomes C2, just a constant. Okay. and now clearly y dash of a equal to y dash of b, here if you apply for this general solution, this will give me 0 equal to 0, satisfied, okay.

So the eigenvalue 0 is, you get what is a solution, so y x equal to constant, that is a nonzero solution. If I choose my C2 is nonzero, nonzero, that is a solution, that is actually satisfying the equation when lambda equal to 0 and the boundary conditions, okay for them so we choose C2 equal to1, so is an eigen function because they have a nonzero solution, eigen function corresponding to an eigenvalue lambda equal to 0, okay. So this is simply constant, so which you can see from the earlier case when lambda n is equal to 4n square by pie square by v minus a the whole square. And when I put n equal to 0, if I include 0 here, lambda is 0, that is the 2nd case.

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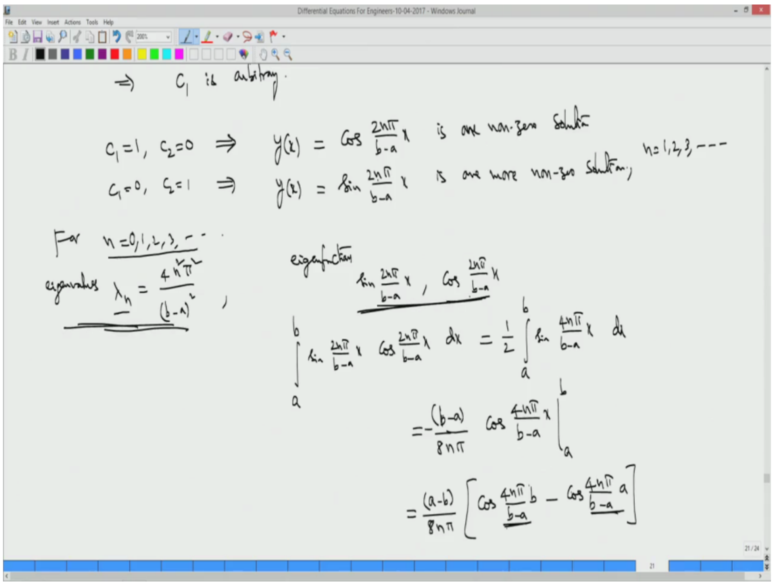

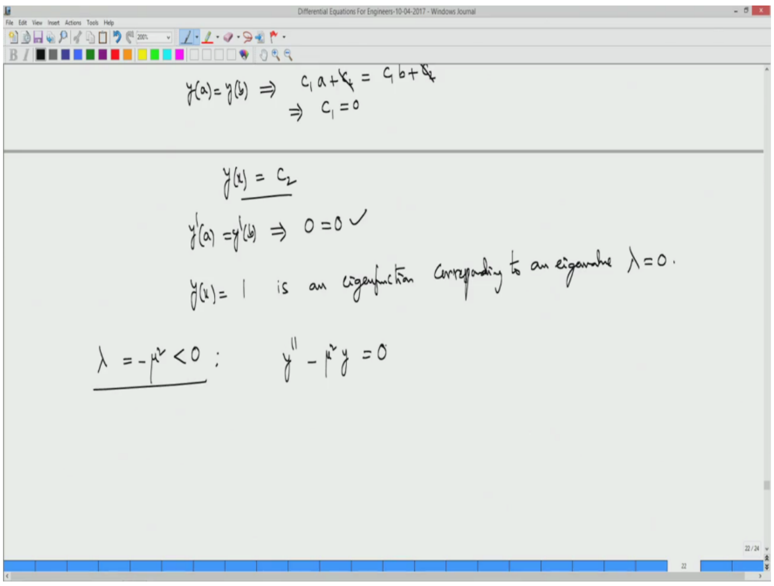

What happens to the eigen functions? When I put n equal to 0, this becomes 0, this is 0. and what about you, cos 2n pie by Vminus a into x, when n equal to 0, this is simply one. So I already have eigen functions. So I can include this lambda equal to 0 case into the earlier case by including n equal to 0. So I can make now n is from 0, 1, 2, 3 onwards, I have these are eigen functions, corresponding eigen functions, eigenvalues. These are eigenvalues and corresponding eigen functions are sin and cosine. but the only thing is n is equal to 0, lambda is 0, lambda 0 is 0 but eigen functions are not 2 but they are only 1, that is because sin of 0, sin n equal to 0, it becomes sin 0 is 0. So this function is 0 function.

We are looking for only nonzero solution, so eigen function should be nonzero, so that is only, that is when we put n equal to 0 into the cosine function, this becomes one, so I already have here. Okay. I can include this case into that. now we have to see what happens to this lambda negative, okay, that is minus mu square which is the negatives, okay. So this is the case, if you see the general equation is given differential equation is lambda square minus mu square y equal to 0. So its general solution, again if you look for general solution, K square minus mu square equal to 0, K equal to plus minus mu, so you have C1, e power mu x plus C2 e power minus mu x.

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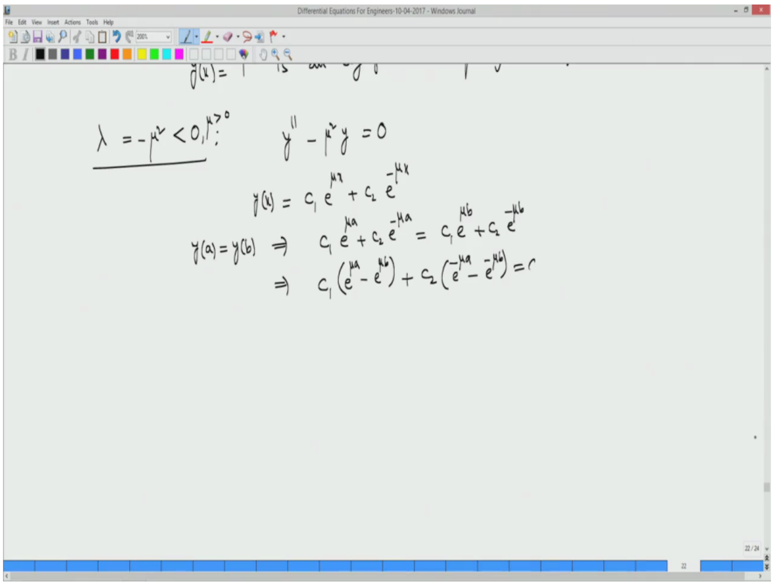

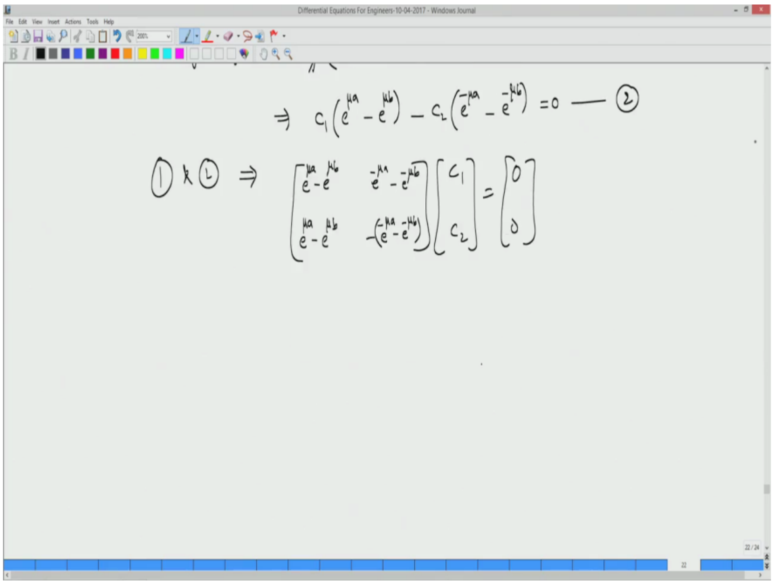

So this is negative, mu is always positive, okay. So that is how it is. So you apply the boundary condition, y at a is equal to y at b, if you apply, so you get C1 e power mu a plus C2 e power minus mu a equal to C1 e power mu b plus C2 e power minus mu b. So what you get the equation is C1, this minus this, so you get e power mu a minus e power mu b plus C2 Times a power minus mu a minus e power minus mu be equal to 0. Okay. So this equation number-one, if you apply other boundary condition, other periodic boundary condition, what you get is mu into C1 e power mu a minus C2 into e power minus mu a, that is what if you differentiate and put x equal to a, which is same as mu times C1 e power mu b minus C2 e power minus mu b.

So mu, mu you can cancel because mu is positive, so nonzero and you get C1 e power mu a minus e power mu b and you have minus C2 e power minus mu a and this when you bring it to this side, it becomes plus and you have finally minus, so minus minus plus, minus mu b equal to 0, so this is equation number 2. If you actually see this, if you substitute, so if you want to have a nonzero solution, okay, this 1 and 2 you rewrite, 1 and 2 actually gives me a system, actually if you are putting as a system, we have a matrix e power mu a minus e power mu b, e power minus mu a minus e power minus mu b, similarly here, e power mu a minus e power mu b, minus e power minus mu a minus e power minus mu b, C1, C2 equal to 0, 0.

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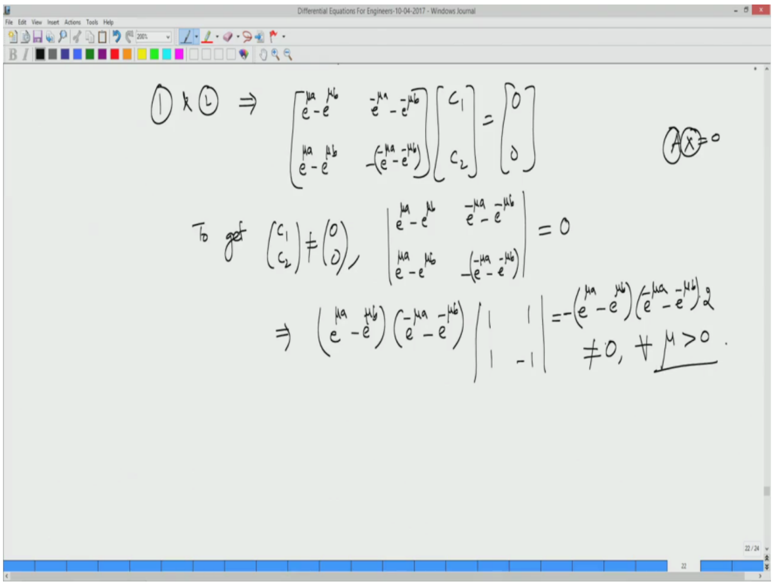

So you want to get a nonzero solution here. To get C1, C2 nonzero solution, okay, this determinant has to be 0, that is the determinant of a power mu a minus mu b, e power mu a minus e power mu b, e power minus mu way minus e power minus mu b, this determinant, minus e power minus mu a, this has to be 0. Okay. because it is like a x equal to 0, to get a nonzero solution if you are looking for, the determinant of a has to be 0. So this is if you, the determinant means e power mu a minus e power mu b, take it out, similarly e power minus mu a minus e power minus mu b you can take it out from the 2nd column, so what you get is 1, 1, 1, -1 which is nothing but, this is simply, this is nonzero and this is nonzero, then what you have the determinant is simply the power mu a minus e power minus mu b, e power minus mu a minus e power minus mu b.

This is simply -1 -1, that is going to be -2, so modulus is -2, so -2. This is never be 0 for every mu positive, you can easily see. for every mu positive and a is not equal to b, this quantity is nonzero and this quantity is nonzero, so 2 is nonzero, so it is never be 0 for any mu value. So you do not have, this determinant is never be 0, okay. This determinant cannot be 0, this determinant is actually nonzero since, okay, since, since this is actually, the determinant is this nonzero. So we see that, that means you do not have, mu, if there is no nonzero solution, that means C1, C2 has to be 0.

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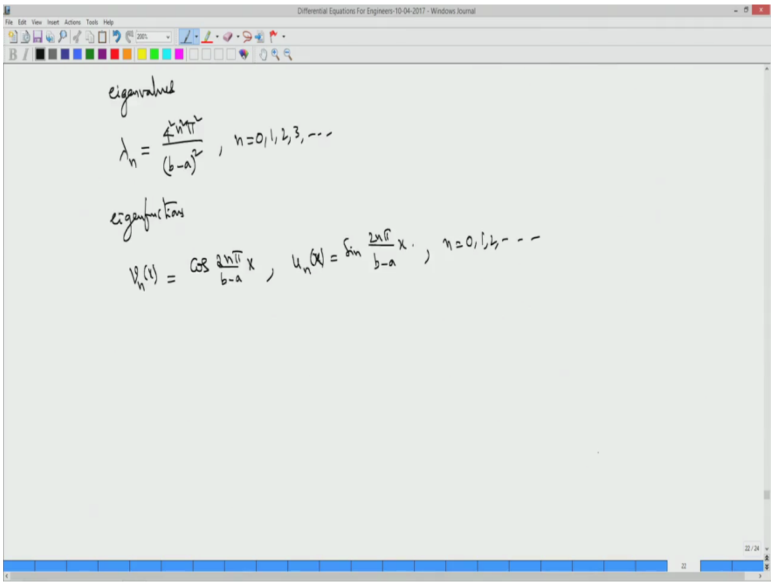

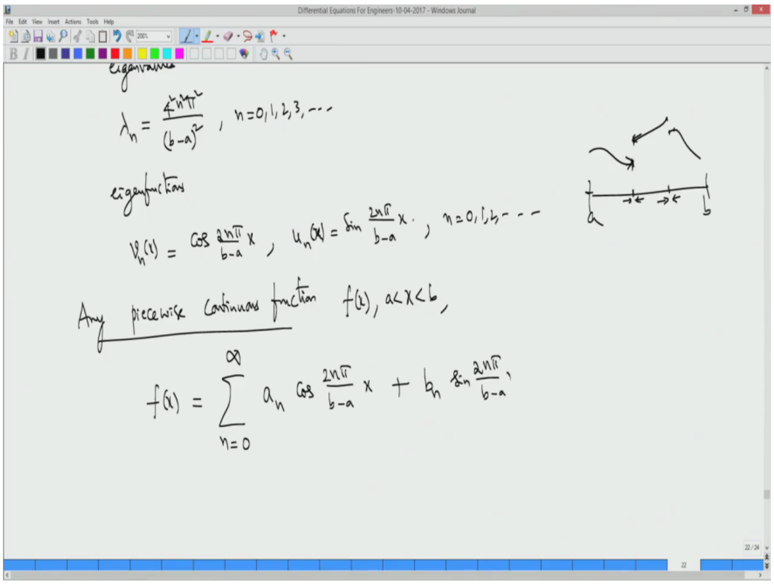

Is C1, C2, both are 0, that means y of x is the 0 solution. So completely 0 solution you will get in this case, that implies this mu equal to, this lambda equal to minus mu square for any mu positive is not an eigenvalue. Okay. Implies no eigen function. So what you have finally, all the 3 cases lambda positive, lambda equal to 0, lambda equal to minus mu square, you have eigenvalues are eigen functions, okay. So you can now put it together, all the eigen functions were eigenvalues eigenvalues 1st of all. I have only lambda n which is for square n square pie square by b minus a whole square. now for n is equal to 0, 1, 2, 3, onwards, okay, eigen functions of, you can call them bn of x and which are cos 2 n pie by b minus a into x and U n of x which are sin 2n pie by b minus a into x. again n is running from 0, 1, 2, 3 onwards. Okay.

So these are your eigenvalues and eigen functions, they are orthogonal, okay. and they are actually form complete orthogonal set, complete means I can write any square integrable function or any piecewise continuous function, okay. any piecewise continuous function, a piecewise continuous function this is actually a theorem, so which you will know, any piecewise continuous function fx, we will take is fx, x is between a to b, okay. Any piecewise continuous function, so you can have from a to b you can have some pieces, this kind of function, okay. So any finite domain, that means outside if you want to see, it is like a repeated everywhere like this.

So piecewise continuous function, that means you should have a jump discontinuity, it is not continuous but it should not be, it should not have, it is ready, to not go to infinity, okay, at these values, it should be jump discontinuity, this limited exist, it should be finite, okay. As you when you go from this side to this side, the value of the function should be finite. Okay. Similarly here, so you should have here and here, that means at this point if you take the limits from this side to the site, the value should be finite. Such a piecewise continuous function I can write as in terms of, as a linear combination of this.

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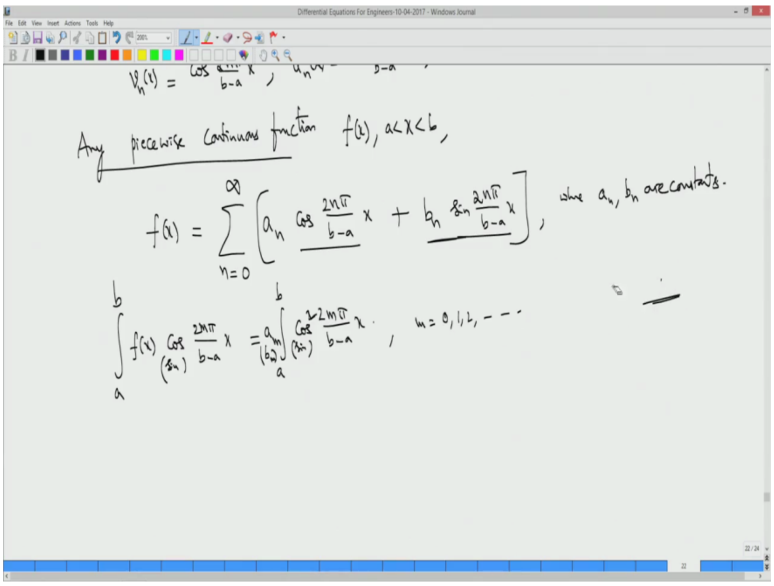

I have now running from 0 to infinity, what I have, what are the eigen functions? I have an, arbitrary constants with these eigen functions cos 2n pie by b minus a into x plus bn, another set of arbitrary constants corresponding to with this, for this sin 2n pie by b minus a into x. So this I can write fx in terms of this eigen function and this eigen functions, coefficients are simply arbitrary constants, okay, where ans, bns are arbitrary constants, okay, there are constants. How to find these constants? These constants you know, you have not product, you can make the dot product fx with cosine with the bns, okay, 2n pie by b minus a into x, this you integrate, take the dot product left-hand side, that will give me here a to b, an, okay cos square 2n pie, so 2n pie by b minus a into x.

You actually multiply with 2n, so let us use 2n, okay. So that you have, what you have is, you multiply cos 2n pie, 2n pie cos 2n pie cos square. So corresponding to n equal to m, that will be together cos square 2n pie by ba, b minus a. Other things will be cos 2n pie by b minus a into x into cos 2n pie by b minus a into x. That integration, the dot product is 0 because n is not equal to m, only n equal to m, that is cos square 2n pie by b minus a into x into an, this is what you get. So and if you do for sin, you get the same thing. So you do cosine or sin, you get the similar thing.

So you get sin here, instead of cos you get here bn instead of an. Okay. So this is running from, including m is, m is running from 0, 1, 2, 3, onwards, okay. Of course b0 is 0, a 0 only will contribute. a0 will be nonzero, okay. What is the reason? Because m equal to 0 sin square that is 0, 0 into, that is 0, right, 0 divided by, here also seen 0, 0 by 0, that is actually, what you have is bn is anyways 0, bn into 0, whatever, bn maybe arbitrary, arbitrary constant. So we can always write, I am simply writing corresponding to n equal to 0 to infinity for cosines, sines from 1 to infinity and that 0 part you can write like the 0 into 0 so, it does not matter, so b0 is arbitrary, it is anyways 0.

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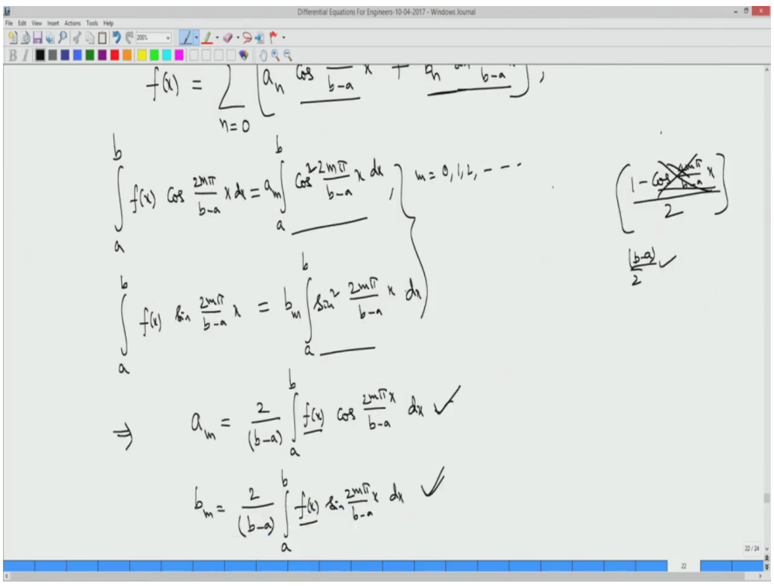

So you can have this kind of expression, okay. So to write it separately, so a to b fx sin 2n pie by b minus a into x equal to bn, a2 b, so you have now sin square, sin square 2n pie by b minus a into x dx. So you should not forget this dx. So this is, this is what you get for n is running from. So this will give me those fourier coefficients an equal to, you can now calculate this integral, these integrals you can calculate, that is one plus cos 4n pie by b minus a into x, okay by 2, right. This sin, once you put sin that will become 0 again, like earlier we have seen cosine at a minus cosine at b, that is 0. Okay.

Cosine at b 4n pie by b minus a into b minus cos for n pie by b minus a that is 0. So this will not contribute, finally what you get is an is 1 divided by, what you get is here, so this integral value is 1 by 2 integral a to b, so that is b minus a by2. So what you get is, if you bring it to the other side, an will be 2 divided by b minus a, this integral a to b fx cosine 2n pie by b minus a into x dx, this is known, okay. and similarly you get bn as, again you get the same thing, instead of plus you have a minus, that is 2 sin square, right. So this is again, so this will not contribute when you evaluate and so what you get is the same, so you get 2 divided by b minus a integral a to b fx sin sin 2n pie by b minus a into x dx. So these are your fourier coefficients.

Given a signal, time signal fx, you can have these frequencies, discrete frequencies, okay with these amplitudes and you, and you can combine it. So these are actually your fourier transforms, fourier transforms. Given a time signal fx you can have these fourier transform coefficients and then to get back your signal using these frequencies, using the discrete frequencies, you combine is a fourier series, this is what is the fourier series. from the fourier series you can get back your signal based on discrete frequencies. Combine all the discrete frequencies, you can get back your fourier series. Okay.

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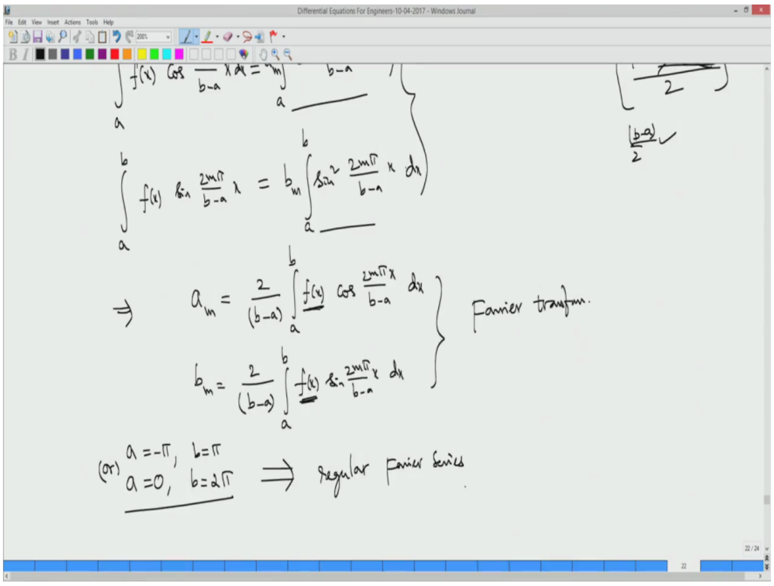

So this is why, this is what you have seen. So now you can use anything, so what you might see is a is 0, b equal to 2 pie is a regular fourier series, what you see in the text books, okay. 0, sometimes some, others may write minus pie, b equal to pie, these are this, okay. but taking this, what you get is regular fourier series, fourier series of time period 2 pie. Otherwise general fourier series between, with the period b minus a you can have this fourier series, okay general fourier series. So this is how regular periodic Sturm Louisville system will give you your fourier series, regular fourier series which you study in the engineering.

So what we have seen is a regular Sturm Louisville system also gives you some kind of fourier series and this periodic Sturm Louisville system is actually giving the series, fourier series which you studied in your engineering. Okay. So what you actually study is this periodic system, periodic system Sturm Louisville system that gives the fourier series. now let us see the 3rd case, singular Sturm Louisville system, okay. I will give you an example of singular Sturm Louisville system that from which, for which if you find eigenvalues and again functions you can get a fourier series there as well.

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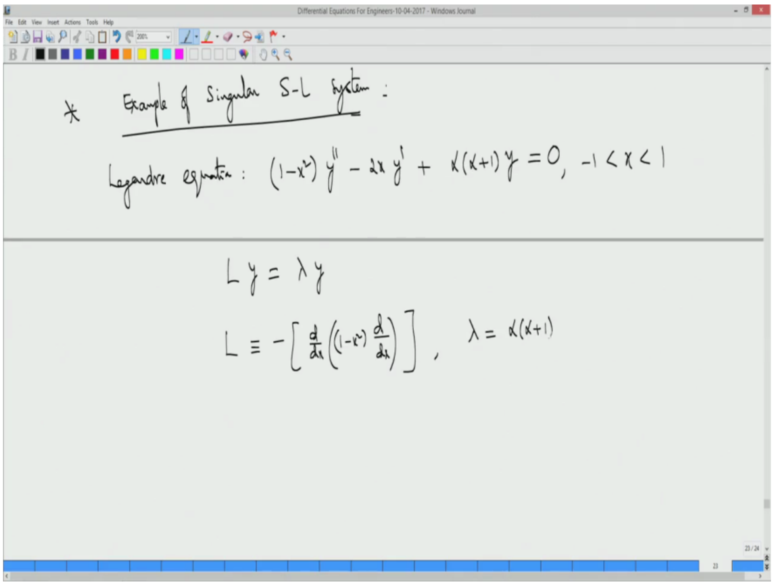

So there I will try to see the example in the 3rd type, singular Sturm Louisville system, example of singular Sturm Louisville system. I do not really do something new here, so which already know. I will just give you an example of, 2 examples of 2 differential equations, examples of 2 differential equations which you have already studied. One is Legendre equation, other one is Bessels equation, that is what we study here in the, we will give an example here for the singular Sturm Louisville system. So let us take this Legendre equation, Legendre equation, what is the Legendre equation?

If you remember 1 minus x square y double dash -2 xy dash plus alpha into alpha Plus1 into y equal to 0. So this is what is our Legendre equation, right. So this is equation, it is defined between minus1 to1. and you see you can rewrite this, put it like Ly equal to lambda y, like self adjoint form or skew symmetric form, where L is skew symmetric, if you put it in this form, what is my L, L equal to is actually 1, 1 by minus, 1 is, 1 by W is 1, so what you have is P is 1 minus x square d dx of, okay and d dx of this whole thing. So you have ddx of 1 minus x square into ddx. Okay.

and Q is 0, Q is 0 so you have, there is no Q here, this is your L, what is lambda, lambda is simply alpha into alpha Plus1. Okay. So because what is P, that means Px is 1 minus x square which is actually 0, 0 at x equal to1 or minus1. That means it is a singular self adjoint equation, singular Sturm Louisville type of equation. So boundary conditions should be, what are the boundary conditions to prescribe what are the boundary conditions to prescribe in this self adjoint, singular Sturm Louisville system? When both P of a, P of a is P at 1 and P minus1, both are 0, boundary conditions should be y of x, rather y at a, here y is 1, y at 1, y dash at 1 are bounded.

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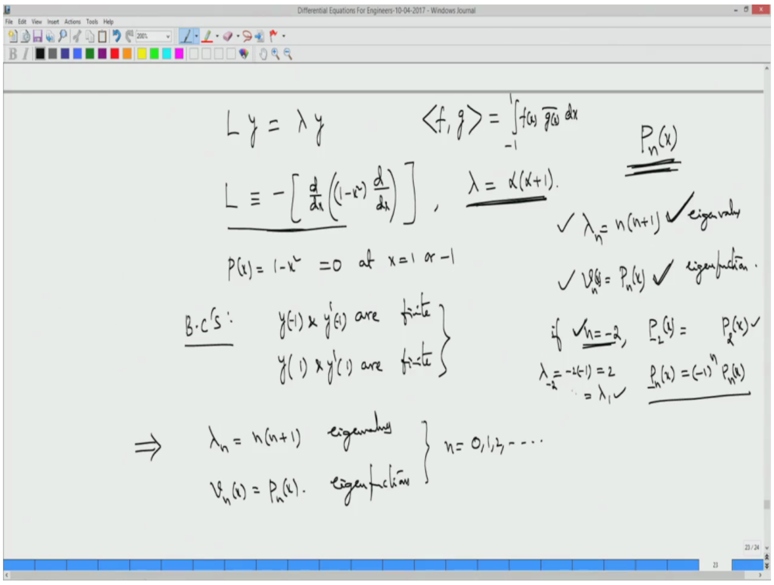

They are finite, similarly y at minus1 and y dash at minus1 are finite. So actually minus1 is a smaller one, so a is this, b is simply one. So these are the finite bounded R finite. So these are the boundary conditions. So you have already studied, when do you have your bounded solutions, you have, it has a solution Pn of x, these are the bounded solutions, bounded solutions for the Legendre equations. All other solutions are series solutions which are unbounded plus minus1. So the bounded, only bounded solutions are Pn of x. That means lambda is equal to, when lambda is n into m Plus1, okay, I have a corresponding solution Pn of x which is nonzero. Okay.

These are the eigenvalues corresponding to this lambda I have a nonzero solution Pn of x which is satisfying the boundary conditions. Okay. So you call this Vn of x as Pn of x and this you call as lambda n. So these are eigenvalues and these are corresponding eigen functions. They that into check, okay, lambda is positive, negative, equal to 0, all those things are already checked, verified, what are the solutions, okay. So your eigenvalues and again functions are these in this case, okay.

Immediately implies, what is the dot product, so you can see the dot product, so there is no W is 1, so the dot product is, you can, you can also write the dot product fg is actually integral minus1 to1 because determinant is from minus1 to1 fx gx bar dx. Okay. So, so for the sake of completeness you can write lambda n or n into n Plus1 eigenvalues, corresponding eigen functions are Vn of x which are Pn of x eigen functions, corresponding to n is from 0, 1, 2, 3 and so on. Okay.

but n is from 0, 1, 2, 3 onwards, you have this P0, P1, P2, and so on. What happens if I take Pn equal to -1? If n equal to minus1, I have still P minus1 of x, this is also polynomial, okay. What is this actually, this is actually you have shown that it is actually minus1 power 1, okay, minus1 power minus1 into P1 of x. you have seen P n of P minus n of x equal to minus1 power n into Pn of x. Using this relation which we proved earlier, so we can say that P minus1 and P1, they are actually linearly dependent. So corresponding to this you have same solution.

So if we think that this is your eigenvalue, okay, if n is, corresponding to n equal to minus1, so what is that eigenvalue, you will lambda minus1 which is equal to minus1 into 0. So this is actually lambda 0 which you already have here, okay. If you take lambda equal to -2, okay, lambda equal to -2, so if you write lambda equal to -2, so if you think that lambda -2 is also an eigenvalue, lambda -2, because you have p -2, okay, P -2 is P2, so which is nonzero solution. I have eigen functions but I want to see whether this lambda mod is lambda -2.

Lambda -2 is -2 into minus 2 +1 into 1, minus1. This is nothing but simply 2, right. -2 into -2 Plus1, so this is simply -2, so this is 2. 2 is corresponding to lambda equal to1, so lambda 1. So lambda 1 is already here corresponding to n equal to1, okay. Lambda 1 is also 2, so like this all lambda minus ns, they are already here, they are same, same eigenvalue, okay. So let us not bother about this lambda is negative, negative natural number. If lambda is minus n, okay, if we choose n equal to negative values, negative integers, you, the eigenvalues are already here, you do not have to, they are not, they are not different from these eigenvalues, that is what I mean to say, okay.

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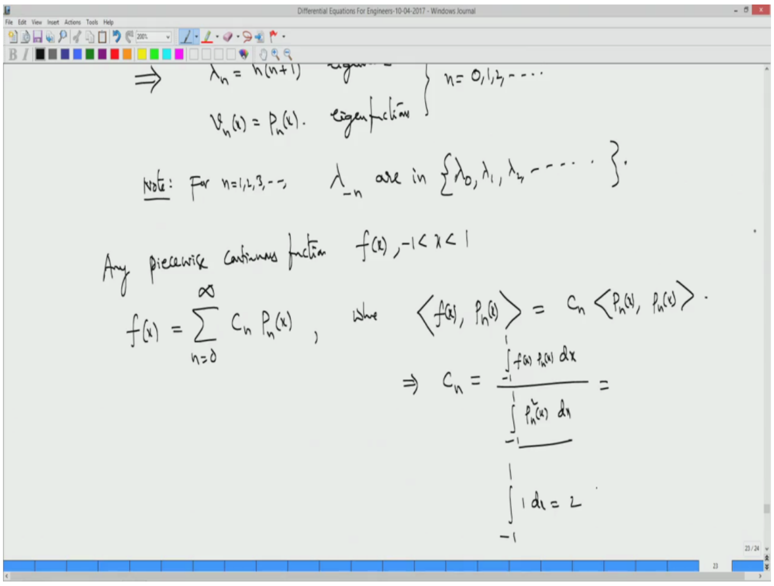

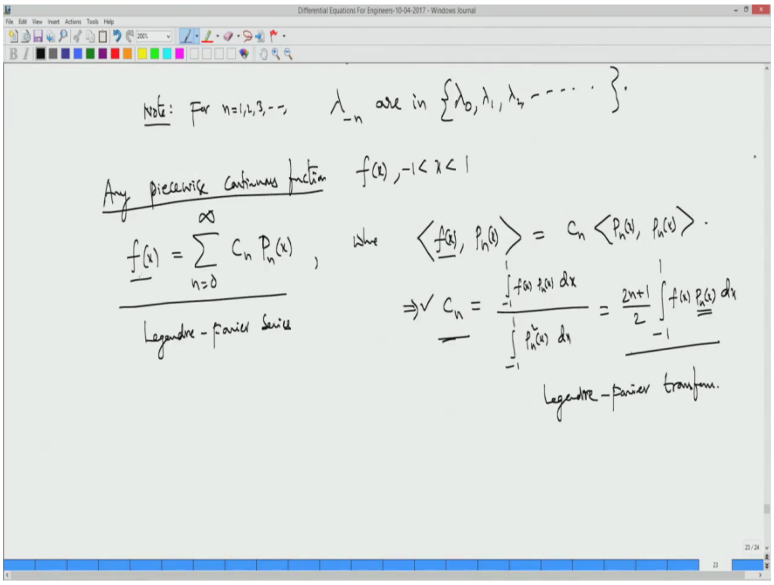

So you can say as a note, if, since lambda ns, lambda minus ns or 0, 1, 2, 3, there actually 00, lambda 1 lambda 1, lambda 2 and so on. for n is from 0, 1, 2, 3, onwards. Okay. Lambda minus n for, if you choose lambda minus n, n is from 1, 2, 3 onwards, okay, you can rewrite. So lambda n where n is from 1, 2, 3, so if you consider this, nothing but lambda 0, lambda 1, lambda 2 and so on, these are here. So you can say note, if lambda n is this, for lambda ns, these are, I should properly I should write for n, for n is equal to 1, 2, 3 onwards, lambda minus n are in one of these eigenvalues.

So we need not consider negative discrete numbers, so negative integers. So you have these ones, so once you have this, if you use the properties of these eigen functions, self adjoint operator or skew symmetric operator, any skew symmetric, any, any piecewise continuous function, piecewise continuous function fx, I can write x is between minus1 to1, I can write in terms of Cn Pn of x, n is from 0 to infinity, that is what we have. So where Cn is, how do I find my Cn, you take the dot product with fx. With P n equal to you get Cn times Pn Pn, all other things will be 0, okay. So that means Cn is, you can rewrite, so you can write integral minus1 to1 fx Pn of x, these are all real valued, so there is no bother, does not matter.

So divided by minus1 to1 Pn square of x dx. you know the value of this, this is actually 2 divided by 2n Plus1, so you have 2n Plus1 by 2, okay. So this value is 2 divided by 2n Plus1, okay. So P0 is 1, P0 is 1, 1 is simply, one is 2, right, so 2 divided by 2n Plus1 is the integral of minus1 to1 dx is equal to 2, okay. One Plus1, so 2, so 2 divided by, this is actually 2 divided by 2 into 0 Plus1. So 2, so this value is 2 divided by 2n Plus1. So which we know. So this is equal to 2n Plus1 divided by 2 integral minus1 to1 fx into Pn of x dx. This is my Cn.

So I have a fourier series bessel, this is called Legendre fourier series. and these are fourier Legendre fourier transform you can say, these are fourier transforms, Legendre fourier transform. So given a time signal I can split, I can make it discrete frequencies with these functions P on and get the discretes call, discrete means 0, 1, 2, 3, onwards, okay. So I can have these frequencies and I can combine with these functions, instead of sines and cosine I can have these Legendre functions, I can combine them discreetly.

I can combine is the discrete sum, with these discrete frequencies I can combine them as a linear combination, I can get back my signal fx, this is what is fourier series is all about. So you can have this Legendre fourier series, if you consider this singular Sturm Louisville system, okay. So why I choose your piecewise continuous function. So if I do this, any fx I can write in terms of this. So that means this sum is converging to fx, you fix any value x, that is, that is point wise convergence, that is called quite wise convergence.

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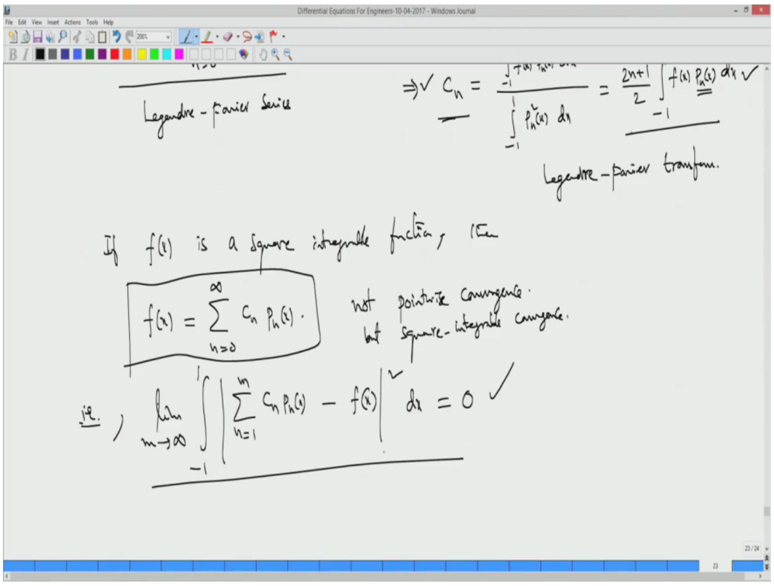

That means the series, you fix x, the convergence is to corresponding fx, okay. This has a series of numbers once you fix x, the series converges to Sx at that point x, okay. Once you fix x, this is point wise convergence. but there is a, but in all the 3 cases, if f of x is that means regular Sturm Louisville system or periodic Sturm Louisville system, all these singular Sturm Louisville systems, if fx is a square integrable function, square integrable function, then, then this fx, still I can write this fx as sum, n is from 0 to infinity. So for example, just for the sake of example I can do in this case Legendre fourier series, I can write this Pn of x where Cns are same here, okay.

Cns are same but the convergence here is not point wise, not point wise convergence. That means this convergence means I take n is from 1 to n, Cn Pn of x, okay minus fx, okay, you director minus1 to1, okay. This square dx equal to 0, when you take this limit, limit m goes to, this limit goes to 0. That means as in the square integrable sense, on an average this converges to fx, okay, that is this, okay. So this convergence is different, so if it is not quite wise convergence but it is a square integrable convergence, it is called square integrable convergence, then this means, this is same as this means not you fix your x and then see that this number series converges to the particular value, okay.

It is actually is this meaning of this one, so this, this you need not worry, so, that is why you learn only piecewise continuous functions, you can have this fourier series, it is point wise convergence, okay, but square integrable, square integrable convergence, that is this one, this is the meaning of square integrable convergence. This is true even in the regular Sturm Louisville system or periodic Sturm Louisville system which I missed to explain last time, okay. So if you take square integrable function fx, then still this fourier series converges to that f but in the square integrable convergence, okay.

but if you take piecewise continuous function fx, this convergence, this series is converging to fx point wise, okay. We have studied bessel, bessel equation, okay, earlier. So we can give operator as a bessel equation operator and we can have an operator of a singular Sturm Louisville system, another example, that is bessel equation, bessell Sturm, bessel Sturm Louisville system. So the operator L is the bessel type, so that we will see that example in the, so we can give one more, one more singular Sturm Louisville system with certain boundary conditions, okay. So we will see that in the next video, thank you very much for watching this.