Differential Equations for Engineers

By Dr Srinivisa Rao Manam
Department of Mathematics,
Indian Institute of Technology, Madras

Lecture 36

Generalized Fourier series

In the last video we have defined periodic Sturm-Liouville system and singular Sturm-Liouville system. How did we do that? We just had, we had differential operator L, ly equal to lambda as an Eigen value problem. If you provide your boundary conditions in such a way that they are same at the end points and its derivatives are also same at the end point that you do only if p is the coefficient of the second order derivative that is p x. P x is either of this p x, p x p a is also periodic that means p (a) equal to p (b) and they are non zero.

In such a case that is when you define these periodic boundary conditions and with these boundary conditions the equation becomes the system differential equations with these boundary conditions. It becomes a periodic Sturm-Liouville system, ok. So and you have seen that l will be again hermitian or sulphate joint or skew symmetric in this case. So you also the inner product dot product will be same and you also have seen what is the singular Sturm-Liouville system.

And when p(a) is 0 or p(b) is 0 or both of them are 0. In each case each of these cases we have defined what are the boundary conditions we should provide so that the operator differential operator l is sulphate joint or skew symmetric or hermitian, ok. We will just give you an example in the case of periodic Sturm-Liouville system. So we will just start periodic Sturm-Liouville system. So what is your request so we will give an example of periodic Sturm-Liouville system.

Let me take the let us take the example of singular periodic Sturm-Liouville system, ok. So let us solve we will give an example of example of periodic Sturm-Liouville system. What we have is take the simple equation like earlier like y double dash plus lambda y equal to 0, ok. And what is your domain x is between general a b, ok. So you can also give 0 to 2 pi for the regular Fourier series, so this is what you take.

for regular Fourier series you have to take a equal to 0 and b equal to 2 pi, that we will see, we will see the special case later. So this is the general equation if you consider y double dash plus lambda y and boundary conditions are y at a equal to y at b because it is a periodic system it is a periodic system. So you have y at a equal to y at b as derivative at a is same as a derivative of b. So these are your boundary conditions.

With these boundary conditions so it makes this periodic Sturm-Liouville system. So what is the equation you want to write this as Eigen value problem. So l y equal to lambda y so what is your l? l is so d dx of minus d dx of minus d dx minus 1 by omega x is 1, ok. So w x 1 by w x is 1, so w x is 1 here minus d dx 1 into d d x. So p x is 1 and plus q x is 0 into y 0 simply 0 since simply here q x ok that is 0.

So you have this is your, this is your l. So L is this you can clearly see this one so clearly p(x) is 1. So and which is non zero and p(a) and p(b) both are same which is 1, ok. So it is periodic and you have these boundary conditions. You have already seen that this kind of operator ok is sulphate joint or hermitian. So the dot product as usual is same, so the the dot product is because that only you have to see what is your w, w is 1 here so you simply write from a to b that is your domain of the differential equation.

W(x) that is 1 and you simply have f(x) into g (x) bar so this is your dot product in this case, ok. Now you have to find the Eigen values and Eigen systems, ok. So again the problem is example find the Eigen values and Eigen functions of the of the system. So yeah so this is your problem, ok. So how do you find the Eigen values and Eigen functions? So you have to see you know that is because these are periodic boundary conditions and this is in this form.

This is Eigen value problem with this periodic boundary conditions l is hermitian implies lambda is always takes the real values ok since l is hermitian lambda is real. So that means lambda is positive or lambda is (neg) 0 or lambda is negative lambda is negative. So you can say lambda is negative means minus Mu square with Mu positive here lambda is Mu square with Mu positive. So this is you see the lambda is positive lambda is 0 lambda is negative.

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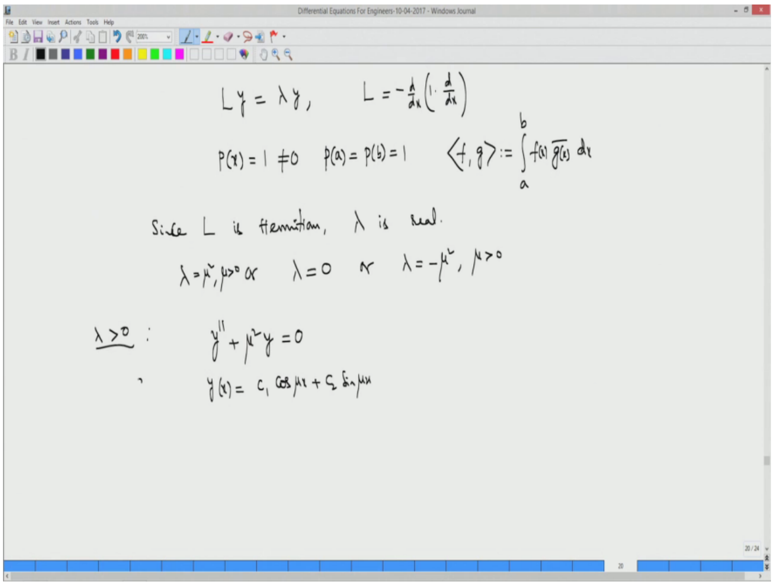

So now consider the equation y double dash in this case y double dash plus lambda is case so when lambda is positive case we are dealing, ok. So y double dash plus lambda Mu square into y equal to 0 and this what are these general solutions? So this general solution is c 1 cos Mu x plus c 2 sin x like we have seen in the last video if you look for solutions y(x) equal to e power k(x).

Then it will become k square plus Mu square is equal to 0 so k satisfies k square plus Mu square equal to 0 k equal to plus or minus i Mu. So the solutions are e power i Mu and e power minus i Mu. These are the two linear impendent solutions. So if you sum and difference are nothing but cos Mu x and sin Mu x they are also linearly independent implies this is the general solutions.

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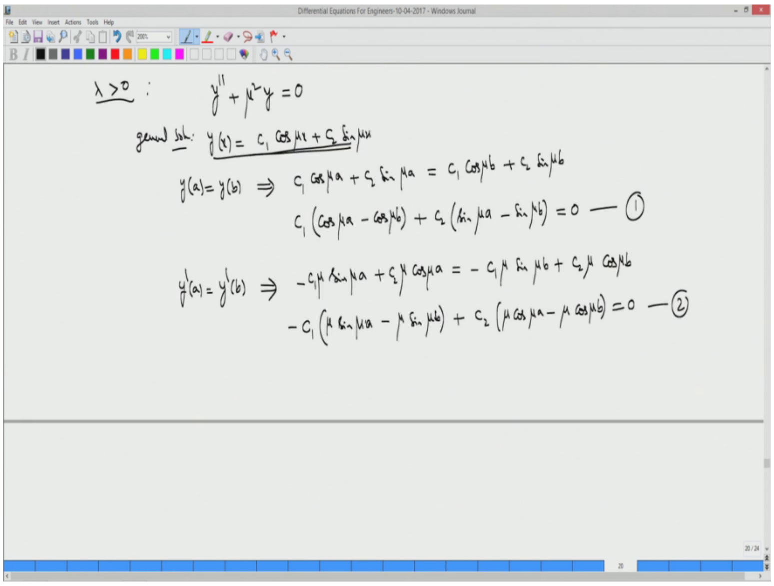

So this is general solution, now you apply the boundary conditions what you have so y(a) equal to y(b). So if you apply this boundary condition so what you get is c 1 cos Mu a plus c 2 sin Mu a equal to c 1 cos Mu b plus c 2 sn Mu b. Otherwise do the same thing now apply the other boundary (cond) so you you may not get anything out of this, ok.

So what you have is what you have is this, ok. So this you can rewrite as c 1 cos Mu a minus cos Mu b this is the coefficient of c 1 plus c 2 sin Mu a minus sin Mu b equal to 0, so this is 1. So this will first boundary condition will give you equation 1. Now you apply the other boundary condition y(a) equal to y dash(b) y dash(a) equal to y dash(b).

This boundary condition if you apply y dash is minus c 1 sin Mu a plus c 2 cos Mu a into Mu, ok. So cos Mu minus sin Mu x into Mu so you have a Mu here c 1 Mu c 2 Mu, ok equal to again same thing you write minus c 1 Mu sin Mu b plus c 2 Mu cos Mu b. So this if you rewrite so you bring it this side so you have minus c 1.

So what is the coefficient of c 1, c 1 is Mu sin Mu a plus or rather minus Mu sin Mu b plus c 2 Mu cos Mu a minus Mu cos Mu b equal to 0 so this is equation number 2. So you are looking for non zero solution to get Eigen value Eigen function, ok. So that means c 1 c 2 0 both are zero as a vector solution c 1 c 2 should not be 0.

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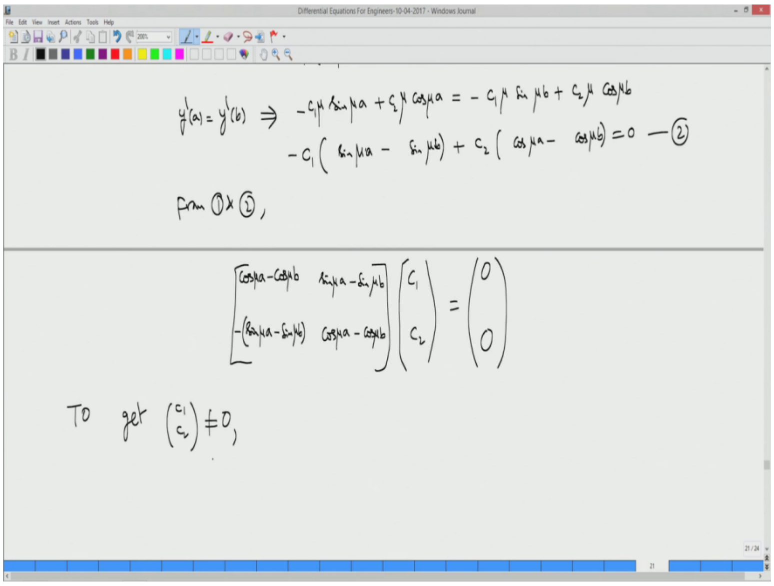

So from 1 and 2 we write this equations 1 and 2 as a system, ok. 1 and 2 we can make a system so we rewrite that 1 and 2 we write like this. this is matrix and you write like this unknown as c 1 and c 2 equal to 0 0. So these two equations I am writing in a matrix form that is cos Mu a minus cos Mu b sin Mu a minus sin Mu b so that is your first equation.

So this multiply with this vector is your first equation 1. And similarly you get minus Mu so any way so Mu is a common both sides you can cancel it ok. So you can cancel Mu both sides Mu is anyway positive so you can cancel it. So you have minus sin Mu a minus sin Mu b, here cos Mu a minus cos Mu b, ok. This is what it has become system so if you want a non zero solution to get this c 1 c 2 non zero that is what for which if you if you if you have a non zero solution then the corresponding solutions are Eigen functions, ok what ever y(x).

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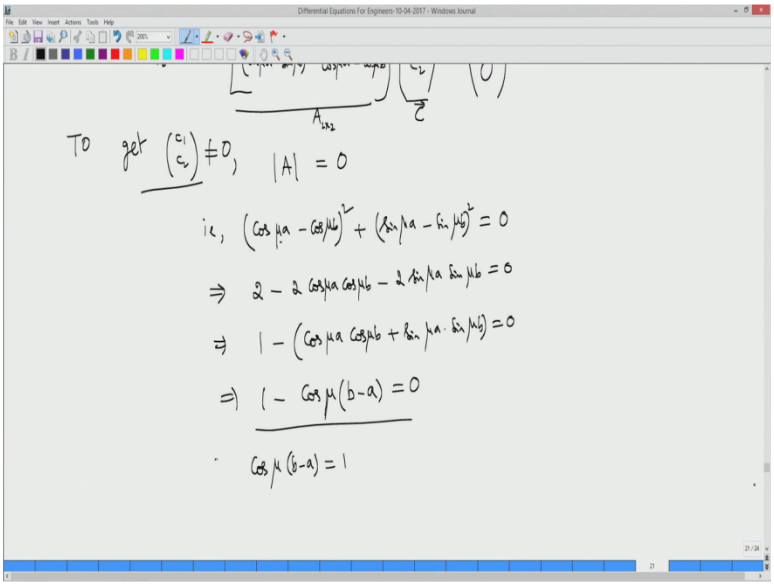

So to get this the determinant of this matrix should be 0, ok. So the determinant of this matrix you call this a, c , c is a vector this vector this is the definition if you think of the a as this matrix this is your c, c matrix c c vector this is a matrix of 2 by 2. So 2 by 2 cross this 2 by 1 so if you do this you get this so the determinant of I can write to get this we need to get determinant of a as to be 0.

So what is that, what is that determinant of 0 means that is cos Mu a minus cos Mu b whole square this is minus minus plus sin Mu a minus sin Mu b whole square has to be 0. So what is that this implies cos square plus cos square so cos square here and sin square Mu a here that will be 1. Similarly cos square Mu b plus sin square Mu b that will be another 1.

So you have 1 plus 1 2 what you have is 2 minus 2 cos Mu a cos Mu b here minus 2 sin Mu a sin Mu b equal to 0. So this is nothing but 2 2 2 minus what is this one 2 into 2 you can cancel both sides if you want. So you have 1 what you have is cos Mu a cos Mu b plus sin Mu a into sin Mu b equal to 0.

So this is nothing but 1 minus cos Mu a minus b or b minus a because b is bigger cos does not matter minus plus so b minus a you can write into that is alright so this is b equal to 0. So to get this this one so I can see from this relation you can find I have to look for what Mu values I can get this relation so that is same as you want cos Mu b minus a equal to 1, ok.

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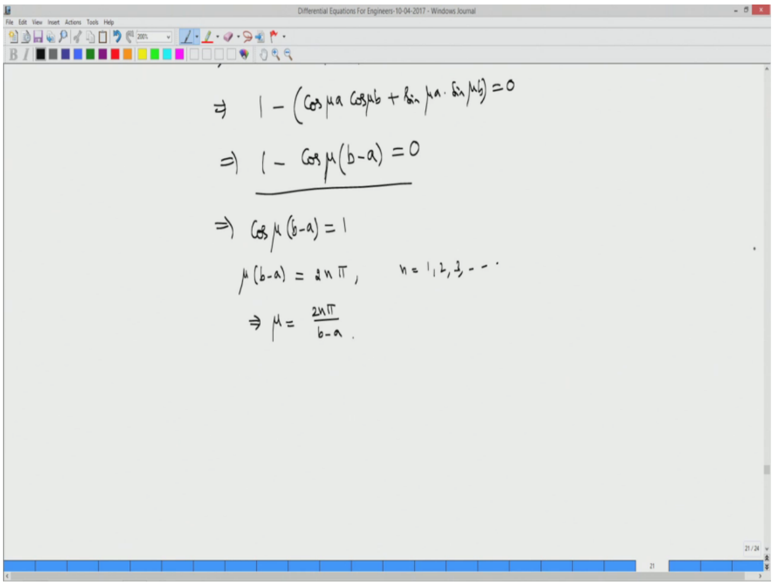

So what should be the Mu value for which this determinant is 0 is so Mu into b minus a should be equal to cos value should be 2 n pi that will give the value 2 n 1 2 n pi n is running from 1, 2, 3 onwards. Because 0 is also included but your Mu is actually positive value Mu is greater than 0 that is why I am taking 1 is n is from 1, 2, 3 onwards so this gives me Mu as 2 n pi divided by b minus a.

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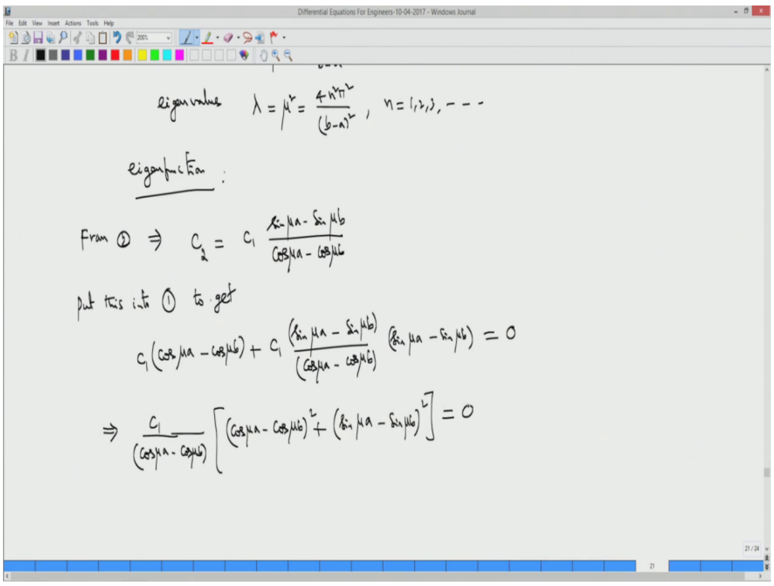

So once you get this Mu so what is your lambda so the Eigen values, Eigen values are lambda equal to Mu square so which is 4 n square pi square (b minus a) whole square. So this is your Eigen value what is the corresponding Eigen function.

So how do I get the Eigen functions essentially you have to see what is the non zero solution, ok. So what happens to the solution so if you actually use try to solve this 1 and 2 before without solving to get a non zero solution that Mu should be of this form, ok. That means the Eigen value should be this, ok. So Eigen value should be this n is from 1, 2, 3 onwards.

To get the Eigen function so you should have the particular solution non 0 solution so what you find this we will try to solve these two equations whatever you have so this is 1 and this is 2. So from 2 you can get either c 1 in terms of c 2 or c 2 in terms of c 1, ok. So we can see that c 1 will be in terms of c 2 or you write c 2 in terms of c 1 or substitute into first equation to get to see what happens ok.

So just pick up first one so you can write first one c 2 in terms of c 1. So let us write c 2 equal to c 1 times ok what is the c 1 times sin Mu a minus sin Mu b sin Mu a minus sin Mu b divided by cos Mu a minus cos Mu b this you substitute from 2 we get this ok we get this put this put this into to warn to get what you get if you substitute into the equation 1 that is c 1 cos Mu plus c 2 this thing.

So c 1 cos Mu a minus cos Mu b plus c 2, c 2 I am replacing with this one. So that I have c 1 sin Mu a minus sin Mu b divide by cos Mu a minus cos Mu b into this is my c 2, c 2 into sin Mu a minus sin Mu b equal to 0. So this all thing is equal to 0 this is what you have. So this implies you have c 1 divide by cos Mu a minus cos Mu b, ok.

And what you have is if you do this if you take this out ok so you get (cos Mu a minus cos Mu b) whole square plus (sin Mu a minus sin Mu b) whole square equal to 0.

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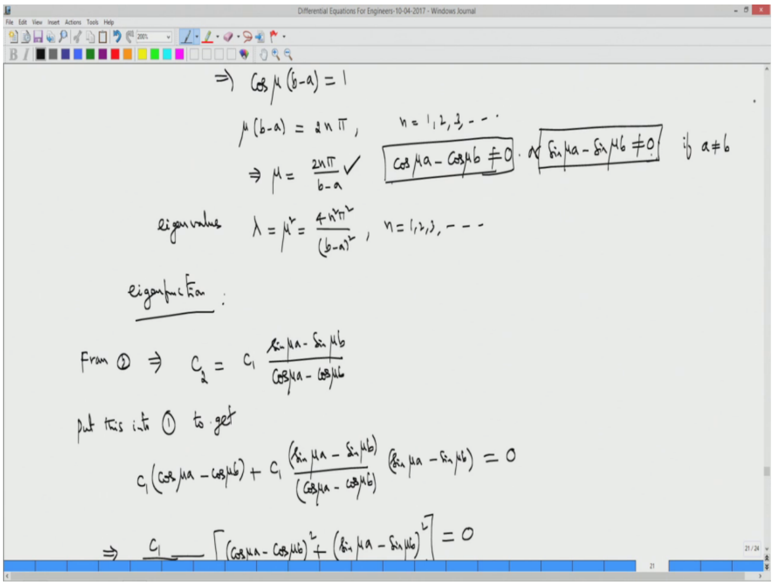

So what is this one this is actually equal to c 1 divide by and first of all for all the values of Mu these values of Mu this cos Mu a minus cos Mu b is actually non zero.

But you can see or sin Mu a minus sin Mu b is actually non zero for these Mu values ok you can see that these quantities are never be 0. If a is different from b, ok. If a is not equal to b so this is always true. So first note that this is this quantity is never be 0 and this quantity is never be 0.

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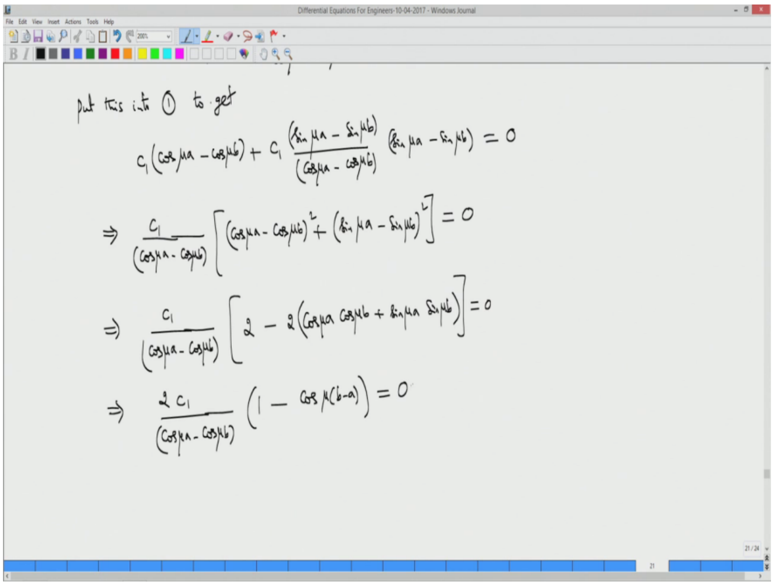

So keeping this in mind you can see that c 1 divide by this is a non zero quantity.So Mu a minus cos Mu b what you have is cos Mu square cos square Mu a plus sin square Mu a that is

1 plus cos square Mu b plus sin square Mu a that is 1 plus cos square Mu b plus sin square Mu b here. So that will become 2 plus 2 then you have minus 2 cos a cos Mu a cos Mu b here it will become you will get sin Mu a into sin Mu b. This is what you get equal to 0.

This is actually equal to c 1 divide by cos Mu a minus cos Mu b and 2 comes out. So what you get is 1 minus this is actually cos b minus a into Mu ok cos Mu times b minus a equal to 0. And actually Mu satisfying this equation this exactly equation. So this is equal to 0, sorry ok.

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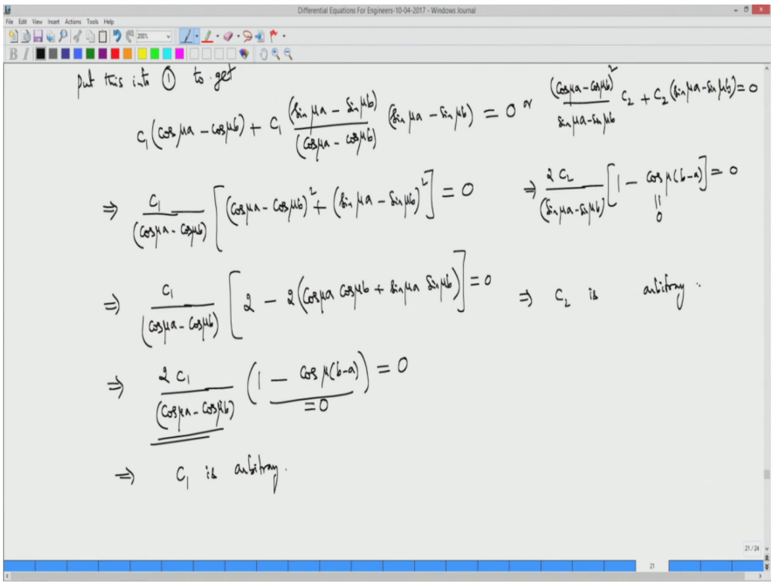

So this is actually equal to 0 for those Mu values. So if this is 0 right? This cannot be 0 right? This is non zero and 2 is non zero so you have c 1, c 1 can be that implies c 1 is arbitrary c 1 can be anything so essentially that arbitrary constant that c 1 you want to find out. So c 1 can be whatever value of c 1 this is this relation is true ok. Because this quantity is 0 for these Mu values and for those Mu values this denominator is non zero and c 1 has to be arbitrary. You can take any c 1 value.

And similarly you can get I will just try to get this same thing here so you can parallelly you can have c 1 in terms of c 2 that will give you cos Mu a minus cos Mu b from equation number 2 you will get this either this or this, ok. Sin mu a minus sin mu b into c 2. This you substitute into the equation 1 to get c.

Now I am replacing with this, so that will give me cos Mu a minus cos Mu b by sin Mu a minus sin Mu b into c 2 that is for c 1 into cos Mu a minus cos Mu b. So that if I combine it with this it will be square, ok plus c 1 plus c 2 sin Mu a minus sin Mu b that is the equation.

So this equal to 0. So if you see this also here you get c 2 again you do the same calculations you get here you get sin Mu a minus sin Mu b, ok.

And here you get 1 minus cos Mu times b minus a equal to 0. So now this is 0 for these Mu values implies and this denominator is non zero implies c 2 is also arbitrary, ok so c 2 is arbitrary. So what you found is c 2 and c 1 both are arbitrary in your general solution. So that means this is your general solution.

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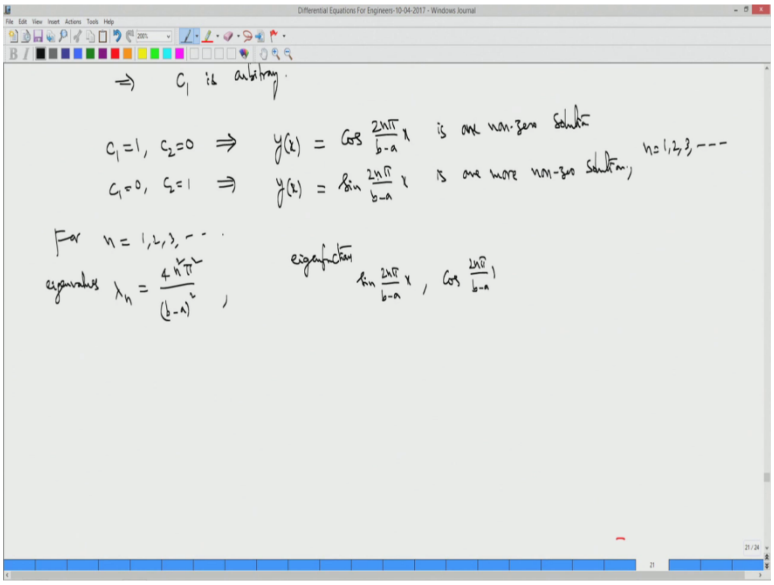

So if you choose c 1 equal to 1 so this is arbitrary so can choose c 1 equal to 1 c 2 equal to 0. And the general solution gives me y(x) equal to cos Mu, Mu is what is Mu? Mu is 2 n pi by (b minus a), 2 n pi by (b minus a) x this is one solution is one non zero solution. If I choose c 1 equal to 0 c 2 equal to 1 because there are arbitrary I can choose one is non zero and the so you get other solution so the another solution is you will get sin 2 n pi by b minus a into x is 1 non this 1 ok so one more non zero solution.

Non zero solution and linearly independent the moment you say non zero solution it is a linear independent and these are cos cosec and sin they are actually both are linearly independent solutions, n is from 1 2 3 onwards. So what you have is Eigen (function) Eigen values lambda square lambda which lambda n which I denote for each n for n is from 1 2 3 onwards.

So you have Eigen values that is 4 n square pi square by (b minus a) whole square so you have Eigen functions, so these are your Eigen values and you get Eigen functions. Eigen functions are nothing but non zero solutions you have two linearly independent Eigen functions here so those are sin 2 n pi by b minus a into x and cos 2 n pi by (b minus a) into x.

So these are your Eigen functions corresponding to this Eigen value lambda m. So what you have is, so what you have is for one single Eigen value, for one single Eigen value you have two linearly independent solutions, ok. So two Eigen functions, so two Eigen functions so you want to see whether these are they are orthogonal or not. If they are orthogonal then well and good it is well and good, ok.

If they are not orthogonal you can make them orthogonal by Gram smith process. And so happens here that in this example they are actually automatically they are orthogonal so that you can verify with this dot product.

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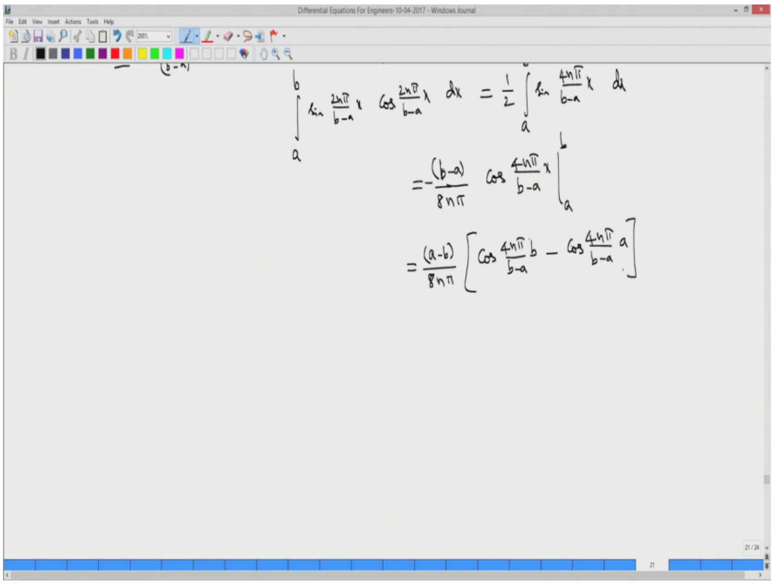

So you take the dot product between these two that you take it as sin 2 n pi by (b minus a) into x and cos 2 n pi by b minus a into x d, ok.

This is your dot product right? This is your dot product which you can see from your the way you define based on your linear operator. So this is your operator so the dot product is this, ok. Bar these are all real valued functions so you need not worry about the bar, ok. Since the bar with bar it is always same, so this is you can rewrite as half of integrant you can rewrite half of sin sum plus sin difference, ok.

Sin sum is sin 4 n pi by (b minus a) into x plus sin difference both are same so the difference is 0 sin 0 so you have this is the one, so this is equal to 1 by 2 (b minus a) by 4 into 2 8 and pi cos we have minus sign cos 4 n pi by (b minus a) into x from a to b, ok. This is this what is the integration so this you will get minus b minus a that you can write it as a minus b by 8 n pi and what you have here is cos 4 n pi by b minus a into b minus a cos 4 n pi by b minus a into a.

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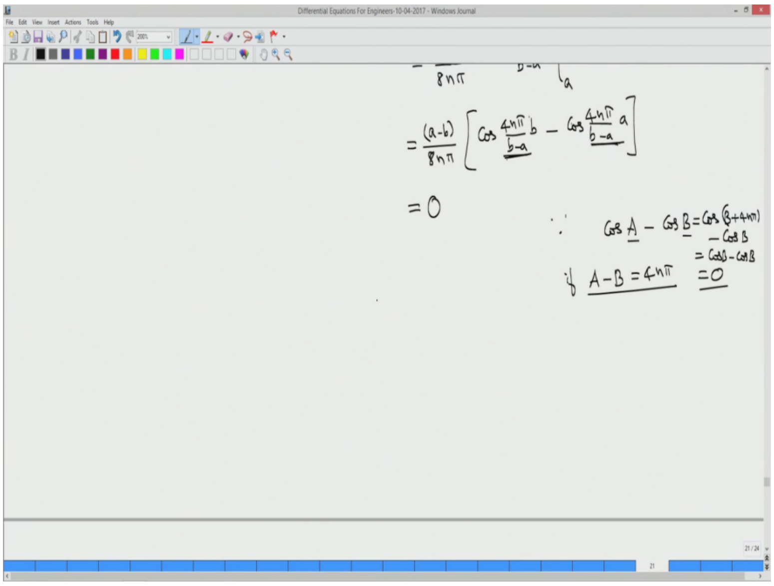

So you can see that this quantity and this quantity so cos of something and cos of something else so you have, so we use this just degration so cos let us see a minus cos b what is this value so this value what is A and B so you calculate A minus B what is A minus B is actually equal to this is A, A is 4 n pi by B minus A into B is your A capital B is 4 n pi by b minus a into a.

So the difference is actually equal to 4 n pi. So what is the value in the fifth this is what is the case. What is this value? This value becomes cos A minus B I am replacing with or A I am replacing with cos B plus 4 n pi minus cos B and 4 n pi plus B cos B is itself. Cos B minus cos B so that is 0. Because this cos ordinate and cos ordinate here both are there difference is 4 n pi this difference cosine of A minus cosine of B cosine of this minus cosine of this quantity it becomes 0.

So this makes it 0 ok since using this. So clearly these functions are already orthogonal automatically orthogonal. So what you have Eigen functions; Eigen functions are now we can happily so we can say that these are your Eigen functions for each lambda n and they are all orthogonal. So once you have orthogonal what you have so if you use the property any function.

No, not yet so these are the Eigen functions corresponding to the case lambda is positive ok that is what we have seen. We will see the other cases in the next video.