Differential Equations for Engineers
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Lecture 35
Periodic and singular Sturm

Liouville Problems Analogous to matrix Eigen value problem which is a minus lambda i into x equal to zero so we have defined regular Sturm Liouville system for a differential equation ok, so you have given a differential equation so you will have an operator L ok you can put it in some nice form a derivative so with some reduction so we have defined kind of normal form so that is L, ly equal to lambda y, y is the solution.

So for this equation on a finite domain in the interval ab you have defined have given some boundary conditions so for which you provide these boundary conditions when the p of x coefficient of second order term second order derivative if it's non zero at the end points then it is called regular Sturm Liouville system for which you have given the boundary conditions this Eigen value problem ly equal to lambda y.

With these boundary conditions its regular Sturm Liouville system so so why we defined these boundary conditions in order to make L is Hermitian ok in order to make the operator L differential operator L to be Hermitian or self ad joint or skew symmetric we can also do in a different way when p of a equal to zero or p of b equal to zero or p of a or p of both of them are zero or when also in other case you can also do make it the operator may can make it Hermitian.

When p of a equal to p of b ok so these two cases we will see later so before I see them we will try to give the regular Sturm Liouville systems some examples ok we will just do some examples for the regular Sturm Liouville system so this is the regular Sturm Liouville system we have seen so this is a together so if you have given equation like this.

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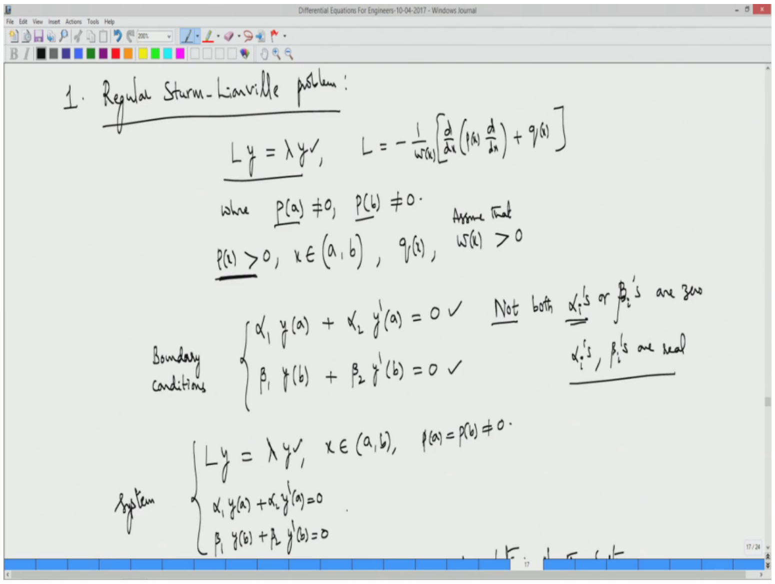

And you have these boundary conditions like this. So you can find the Eigen values and Eigen functions like I explained in the steps, steps to do find the Eigen values and Eigen vectors and once you see this if you see the all the Eigen values are distinct then corresponding Eigen functions are orthogonal with respect to the dot product that is defined based on your operator L.

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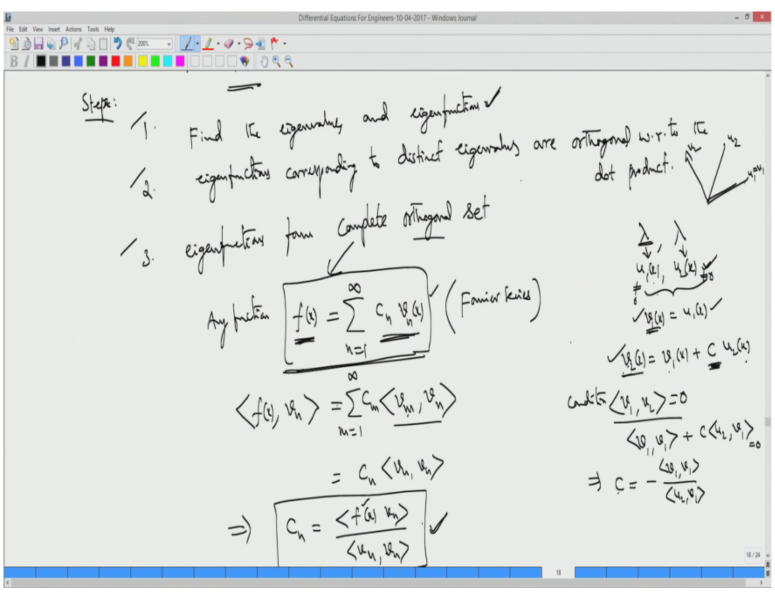

And once you have this once you get all these Eigen functions they form orthogonal which you know and they are complete.

That means any function f I can write in terms of this any actually square Integrable function you can write in terms of these Eigen functions ok so that is actually a Fourier series and then how do you these are arbitrary constants cn's are arbitrary constants those things you can find just by using the dot product so make a dot product both sides with b and so that you can get your cn (()) (03:14).

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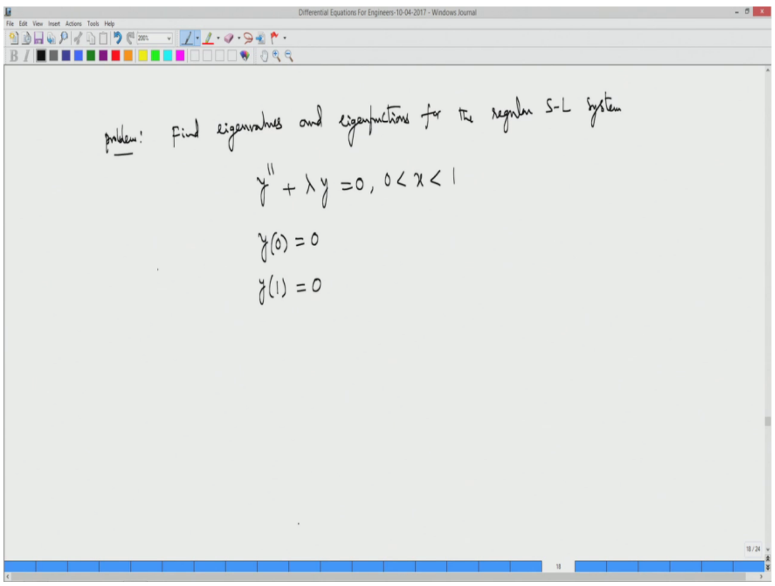

So we will see some examples how we do this . So what we do is of make a problem like problem find Eigen values and Eigen functions for the regular Sl system that is y double dash of plus lambda y equal to zero lambda is a parameter and your domain is between zero to one and the boundary conditions are given as y zero is zero y at 1 equal to zero so these are your boundary conditions so this is your system.

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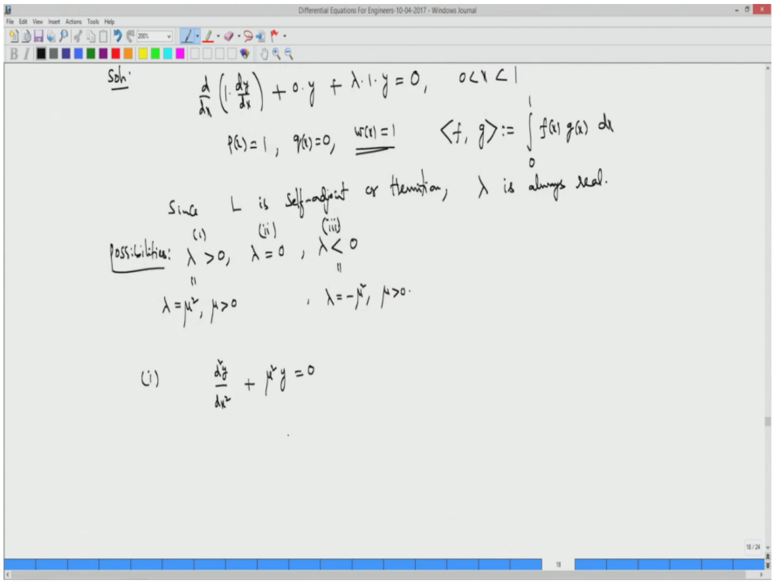

So how do I solve this I need to find Eigen values and Eigen functions so first we want to see put it as equation where lambda is a parameter this you put it as Eigen value problem, what is Eigen value problem? So you know so if for a matrix ax equal to lambda I the lambda x ok lambda x this is your Eigen value problem so try to look for x for each lambda values ok fix your lambda or find those values of lambda for which you have a non zero solution x those are your Eigen values and Eigen vectors so you do the same thing.

So try to put this equation in this form first that I can write like earlier so this is like my is zero ok my like my equal to zero m is the operator so m is actually same as l you can see that ok so the second order differential equation I will put it in a nice form so the form which looks like a self ad joint so that is y dash of dash so this is like ddx of dy by dx so this is one p of x is one os that is why we are putting so plus q is zero ok.

Zero into y plus lambda into one into y equal to zero ok so from this you can recognize what is your p of x, p of x is one, q of x is zero and w of x is one, moment you see wxc equal to one immediately you can define the dot product is dot product of any functions is simple zero to one that is your domain fx gx into wx it is one ok so dx so this is your dot product then is actually is easy to find the and you see that this you know that with these boundary conditions.

You have seen that this regular Sturm Liouville system is self ad joint so we know that since it is regular Sturm Liouville system 1 is self ad joint or Hermitian we have seen yesterday ok these kind of things are actually Hermitian since this is this lambda is always real that is the property of self ad joint or symmetric or skew symmetric matrices values of the Eigen values are real so that is what we are analog ally seeing this here.

Even here it is true but we are not proving it ok so you have this lambda is always real so if lambda is real we can think of three cases lambda is positive, lambda equal to zero or lambda is negative so these are the three cases you have possibilities ok so real these are the possibilities if lambda is positive I can write this as mu square ok with mu is positive, lambda is this is anyway lambda is zero as the same ok.

And here also you can make it like lambda equal to minus mu square if mu is positive so this is how you make three possibilities and now take the equations so that is actually d square y by dx square so look at the case 1 so this is a case 1 this is case 2 and this is case 3 so look at the case 1 where is you take the equation d square y by d x square q zero plus lambda is now mu square y equal to zero.

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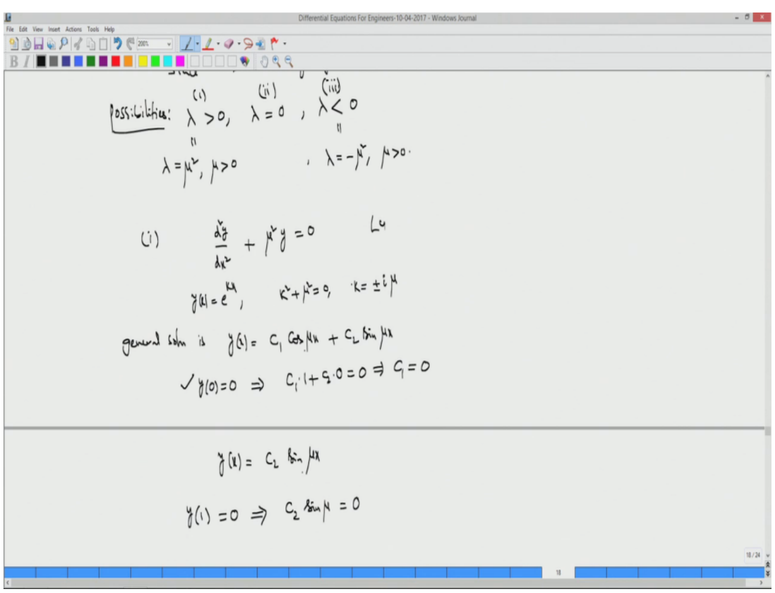

So this is a second order equation with constant coefficients. I know it's solution by earlier methods you look for yx as e power kx ok y of x you look for e power kx what you get is you get k square plus mu square equal to zero so that gives me k equal to minus mu square so that is plus or minus imu ok so your solution is general solution is y of x equal to some c1 e power imux plus c2e2 e power minus imux or c1c2 are arbitrary constants or you can also write c1 there is no real part.

So cos mux plus sin mux or two linearly independent solutions so you can write cos mux plus c2 sin mux ok so in this case I want so we have not applied the boundary conditions so far so what are the boundary conditions? Y zero is zero so you apply y zero equal to zero implies c1 cos mu zero that is one plus c2 zero c2 into zero into 1 ok this is one plus c2 into sin mu zero, zero equal to zero so this will give me c1 equal to zero.

So if I apply these boundary condition now my general solution becomes yx equal to c2 sin mux ok, now you apply the other boundary condition so y at 1 equal to zero so if you apply this for this now you have c2 sin mu equal to zero now c2 is if you take c2 zero then what happens you get a zero solution you are interested to find non zero solution for a Eigen value problem so here also when you write like this ly equal to lambda y ok.

So if you want for every lambda whatever lambda you take which here in this case I have taken minus mu square so that is plus mu square so I am looking for non zero solution y ok?

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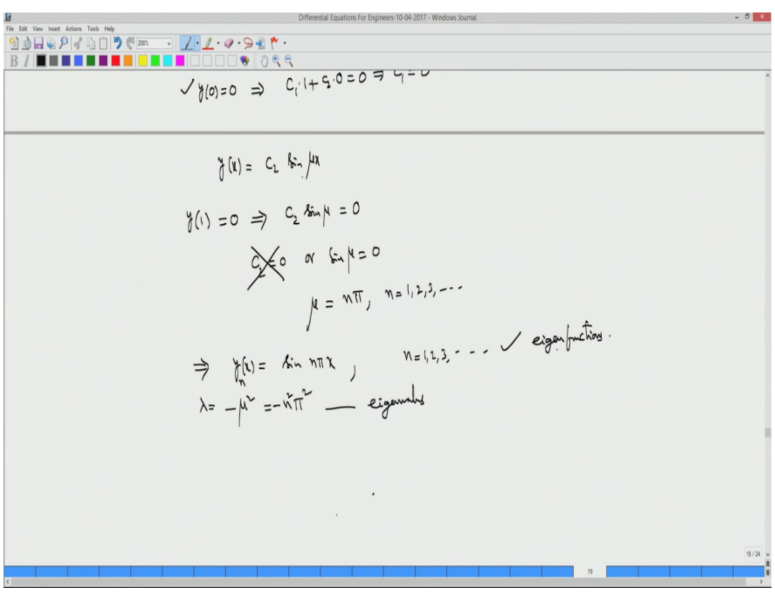

So if I want non zero solution c2 has to be non zero ok so if I take c2 zero I get only zero solution so that means actually I have only zero solution so but I am looking for non zero solution so do I have values of mu for which I get c2 non zero ok.

Still this quantity is zero so you have write sin mu is zero for certain mu values if no mu values satisfying sin mu equal to zero that means c2 is zero is the only option that means you don't have any non zero solution implies that is that corresponding mu all those mu values you considered ok are not Eigen values but here you have c2 is zero or sin mu equal to zero but this case we don't take it because if you take c2 equal to zero what you get is simply zero solution.

Which is not desirable ok so this will give me other option is n pi mu is remember mu is always positive strictly positive so n pi n is from 1, 2, 3 and so on ok because mu is always positive so this implies what is your solution yx equal to c2 you can choose c c2 sin mux is a solution of this homogeneous equation so any constant multiple is also solution you can take this arbitrary constant as 1 so you have a non zero solution is now replace c2 take it as 1 for each mu.

For each n ok, n is from 1, 2, 3 onwards what you consider is c2 is 1 so sin n pi x so these are your solutions ok what is your mu, mu equal to n pi for each if when mu is n pi I have this solution what is mu square mu is when mu is n pi what is actually lambda? Lambda is minus mu square ok so that is minus mu square that is lambda so that I have minus mu is actually n square pi square ok.

So these are your Eigen values corresponding to this that is mu equal to n pi I have corresponding solution yn you can call this yn because it depends on n ok so these are Eigen functions ok so what you know so what you have seen is you just found only Eigen values and Eigen functions for this regular Sturm Liouville system ok so what else you can do now we can also say that these Eigen functions no you have not found so far. So far these are the Eigen values and Eigen functions.

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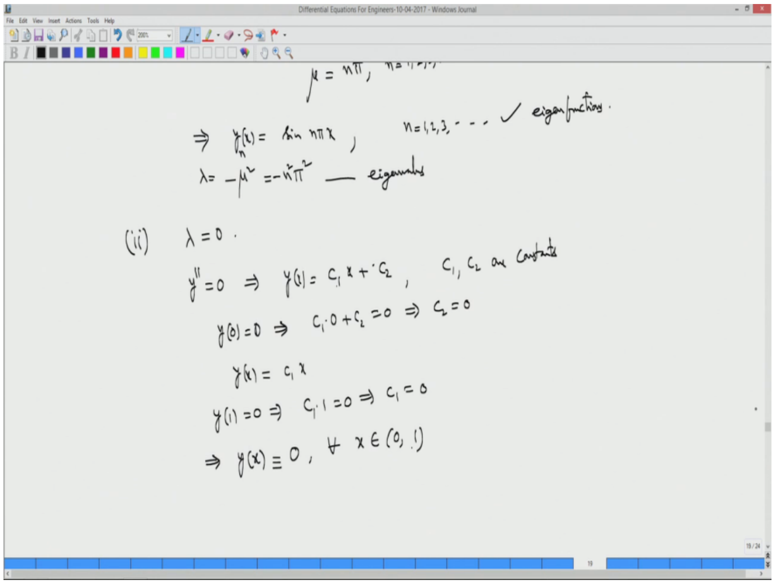

You still have to see two more possibilities that is lambda equal to zero and lambda is less than zero so you look at the case two, you quote some Eigen values and Eigen functions now look at the case two that is lambda equal to zero in this case what happens to your equation y double dash equal to zero so what is a general solution here? Is simply c1x plus c2 so you simply integrate you will get this arbitrary c1 c2 are.

Arbitrary constants so now we apply y zero equal to one, y zero is zero will give me c1 into zero plus c2 equal to zero that is nothing but c2 is zero now what happens to your general solution now, simply c1x now you apply y at 1 equal to zero for this you get that is c1 into 1 equal to zero that is c1 is zero, so this implies what is your yx now what you got yx is simply c1 c2 both are zero so this is zero identically zero for every x in zero 1 ok.

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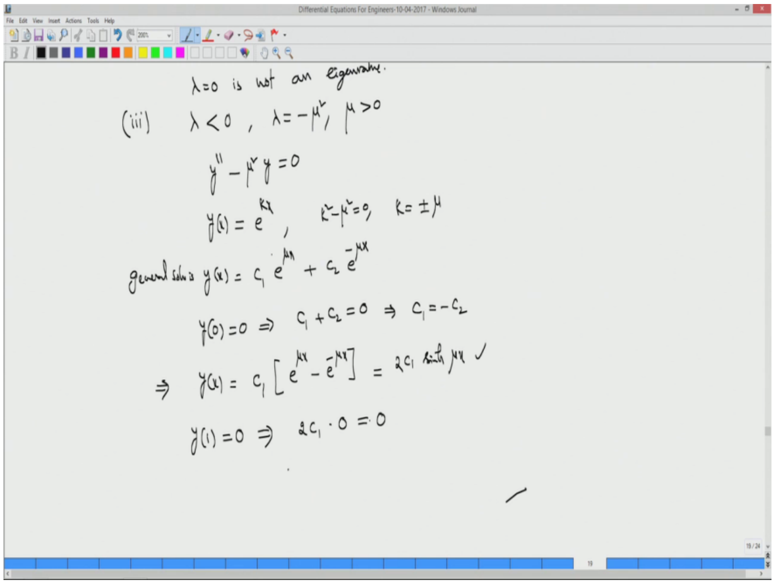

So that means lambda equal to zero is not an Eigen value ok so you don't have Eigen values, you don't have Eigen functions corresponding to lambda equal to zero because you don't have non zero solution there, now look at the third case lambda is negative so that means lambda is minus mu square mu is positive so in this case y double dash what is your lambda y lambda is minus right so lambda is minus mu square so you have minus mu square y equal to zero.

So what is the general solution here you write again you look for y e power kx kind of solutions you get k square minus mu square equal to zero that gives me plus or minus mu, mu is positive ok so you have a general solution is yx equal to c1 e power mux plus c2 e power minus mux now again you apply these boundary conditions whatever you have y zero is zero will give me c1 plus c2 equal to zero ok, so if this will give me c1 equal to minus c2 so the general solution becomes.

C1 is minus c2 so you have c1 c2 I replace with minus c2 c2, c1 c1 c2 you replace with minus c1 so what you get is e power mux minus ok so this is equal to c1 by 2 and another two times c1 into this is sin hyperbolic mux that is your general solution, now you apply other boundary condition y at 1 is zero will give me now y at zero will give me 2c1 sin hyperbolic zero that is zero right this is zero into zero is zero right so c1 is arbitrary constant y at 1 right.

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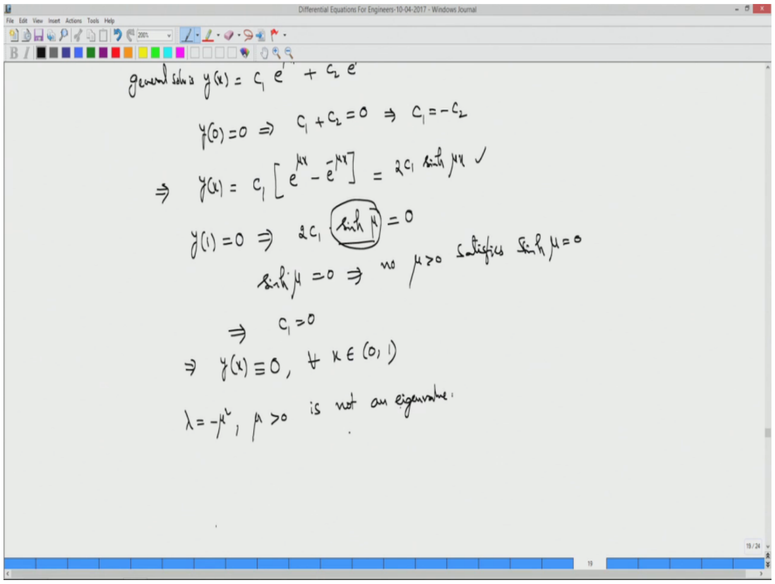

Y at 1 sorry this is y at 1 so it should be sin hyperbolic mu equal to zero so again either c1 is zero if c1 is zero you don't want because you want to see whether any because if c1 is zero then both c1 and c2 are zero that implies you have zero solution that means that you don't want you will just looking for non zero solution whether any possible mu you have a non zero solution ok so you have to see sin mu hyperbolic mu equal to zero.

This will give me what are the values for which mu is and I know that mu is positive ok so there is no solution no mu positive satisfies sin mu hyperbolic mu equal to zero that implies what? So you don't have any mu, mu positive satisfying this one because lambda equal to minus mu square mu is strictly positive so there is no so this cannot be zero so that implies c1 has to be zero that implies a general solution becomes completely zero for every x in zero implies lambda equal to minus mu square for mu positive is not an Eigen value.

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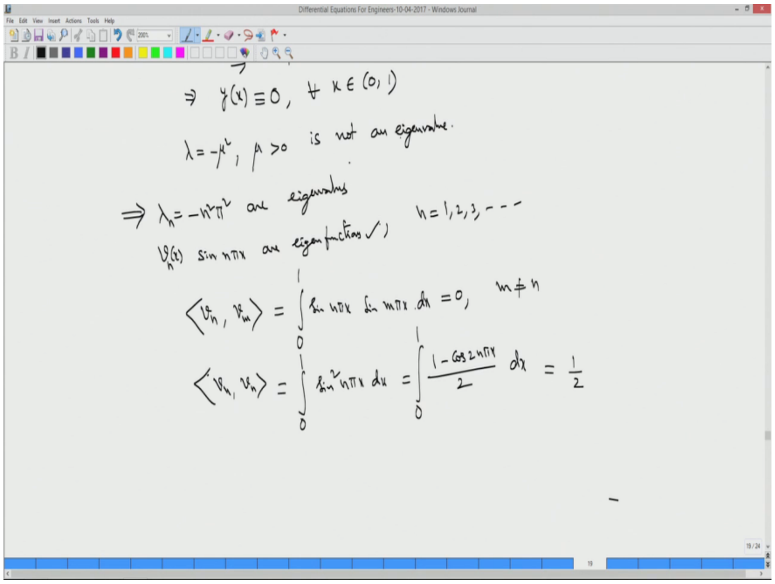

So what you have seen in three cases lambda negative is not an Eigen value lambda equal to zero is not an Eigen value only here you get Eigen values ok, lambda so case 1 that is for lambda positive you have an Eigen value lambda equal to mu square or Eigen values so you found what are the Eigen values and Eigen functions minus n square pi square and sin and pi x.

Or the Eigen values and Eigen vectors Eigen functions respectively so the answer is solution finally minus n square pi square or Eigen values and then sin n pi x or Eigen functions corresponding to n is from 1, 2, 3 and so on because n equal to zero is not an Eigen value and n equal to zero you see that this becomes zero so these are your Eigen values and Eigen functions so you found only first step and so what is the second step?

So we can see that these Eigen functions you can actually verify ok so you can actually verify these Eigen functions if you call them as vn of x and these are your lambda n's corresponding to lambda n is from 1 to infinity this vn and vm dot product which we have which we define based on what is your differential operator zero to one sin n pi x sin n pi x dx ok so actually true this is this should be zero whenever m is not equal to n.

So you can easily verify that these actually zero when m is not equal to n but when n equal to m that is vn vn vn dot vn ok sin square n pi x dx this is one by two ok so this is one minus one minus cos 2n pi x divided by two this is zero to one right so this is what is integration so dx ok so this is simply because this is sin 2 n pi x will get that is between zero to one is zero so only this will contribute one by two this will get simply one by two is the answer.

So you see that these are these functions are Eigen functions they are orthogonal you can easily see and the dot product when they are take the same function so this value is half.

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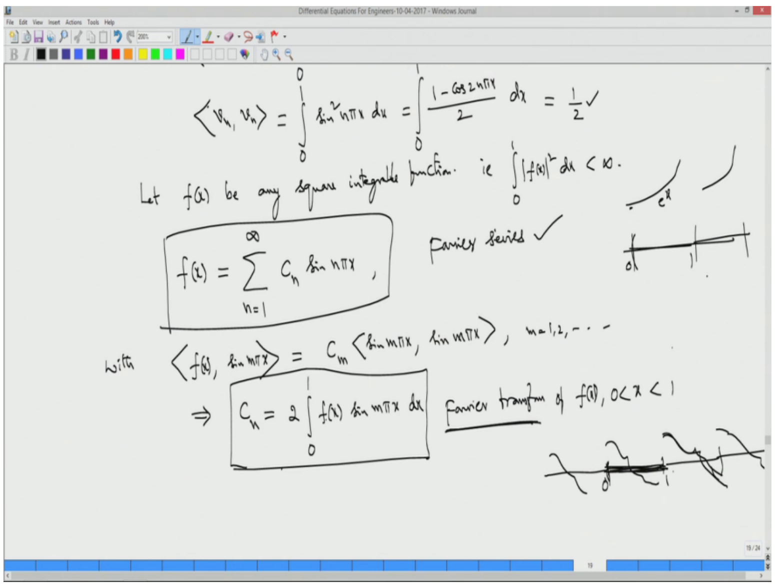

So what is the third step which we have seen we have told ok so that any function you take any let fx be any square integrable function that is integral zero to one that is your domain modulus of fx whole square dx is finite so any such function any f satisfying this ok.

I can write this is what you have to take that is the meaning of complete orthogonal Eigen functions means any function f square integrable function f I can write it as a linear combination of all these Eigen functions so what is the linear combination you have infinitely many Eigen functions so linear combination is the sum with some arbitrary constants cn sin n pi x n id from you have from one to infinity so this is what is the meaning.

So fx I can write like this ok so where cn how do I find my cn if I can write like this but I still don’t know what is my cn so cn if you apply the dot product both sides fx with cn will be dot product with vn that is sin n ok so you can say with some vm of x equal to you have cm n is only when n equal to m with little contribute that is actually sin m pi x and you are multiplying vm sin m pi x so here also you can put it as sin m pi x ok.

So this will give me this is true for every m is form one to infinity this is true so this means cm or cn now you can change m as n both are same so cn this value we have seen here so there is one by two ok so it means two times integral zero to one now this you can write fx sin m pi x dx so this is exactly your fourier transform ok and this is your fourier series, fourier transform of fx x is between zero to one any periodic function you can say fx is defined between zero to one.

And let us say you take any function like this signal is given between time zero to one like this and one to two is also its repeated same thing same profile is repeated like that you can go on so its periodically it's the such a signal you can also use you are given this between zero to one you can define you can get this discrete frequencies these are the frequencies this is I will give you the fourier transform and use once you know the all these discrete frequencies.

The time signal you break it into discrete frequencies and finally you can get back your signal itself in terms of this discrete frequencies these frequencies cnc this number into with these Eigen functions will form will give you as this a fourier series ok is actually fourier series will give you inverse fourier transform of this it's a fourier series here inversion of so you are given a time signal here you break this signal into discrete frequencies.

That is giving fourier transform is giving that if you want (if it's a) suppose you are given all these cn's you want to get back your signal so that is given by this fourier series ok so that is what is you are seeing here so as a regular Sturm Liouville system here you find the Eigen values and Eigen functions and you see that they are actually they form complete orthogonal functions now immediately that means any function.

I can write as a in terms of them linear combination of these Eigen functions that is actually fourier series so if I ask you to find here find the fourier series of given a function say e power x you can take e power x fx as e power x you can write fx e power x in terms of all these Eigen functions sin and pi x what is actually that, that is actually fourier series for e power x, e power x is given between zero to one as a signal which is periodically repeated.

If you want it is not actually e power x everywhere e power x between zero to one only given so one to two is also repeated same e power x one to two so whatever profile you have between zero to one e power x, e power zero is one, e power one is like this ok so something like this it will be right so same piece it will be repeated at every one two like that, (such a signal) such a function if you want you can write .

So such a function between zero to one which is e power x you can write in terms of cn sin and pi x where cn's are e power x into sin and pi x dx so because this is n so it should be n ok this is from n is from one to three and so on ok.

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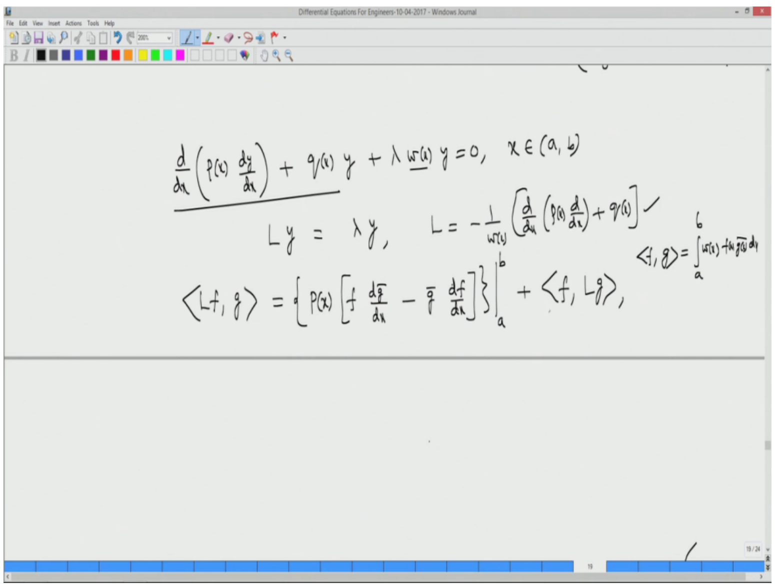

So this is how you solve or you say you can have given if you see the differential equation if you know the differential operator l ok? If you are given a second order linear differential equation you can put it in this form get the l based on that you define your dot product and then make get the Eigen values and Eigen functions you can make a Fourier series ok so this Fourier series whatever you have see the Eigen values and Eigen functions so we will use it to solve partial differential equations ok in the future (so let's get back to) so we will do some more examples like regular Sturm Liouville systems.

So now let us go back to general case second case when what we had is if you are given a differential equation like this any second order differential equation you are putting it now px dy by dx into ddx ok plus qx into y plus lambda into wx y equal to zero x belongs to ab ok so this is actually your ly this is ly plus lambda y so ly equal to minus lambda y so actually put it in this form where l is minus one by omega x and this one ddx of px ddx plus qx.

So this is what we have seen x is between zero to one x is between a to b so for this you have shown that if you have shown that any two functions f and g you have lfg equal to you have that special thing that px times ok so what you have that you have seen earlier I can write without saying so this px times fx let me see exactly take exactly what we got f into g bar dash fg bar dash f g bar dash that is dg bar by dx minus g bar df by dx.

And this form this whole thing you take this whole thing you take the limit from a to b plus and here we get lg so this is what you have seen only from this ok and the dot product is dot product between f and g this means here this is actually from a to b wx that is a weight function whenever you have this w you have to use that, that is positive fx gx bar dx this is what is a dot product which you use here this is what you have seen so y one to make this special thing to zero, if I give certain boundary conditions ok?

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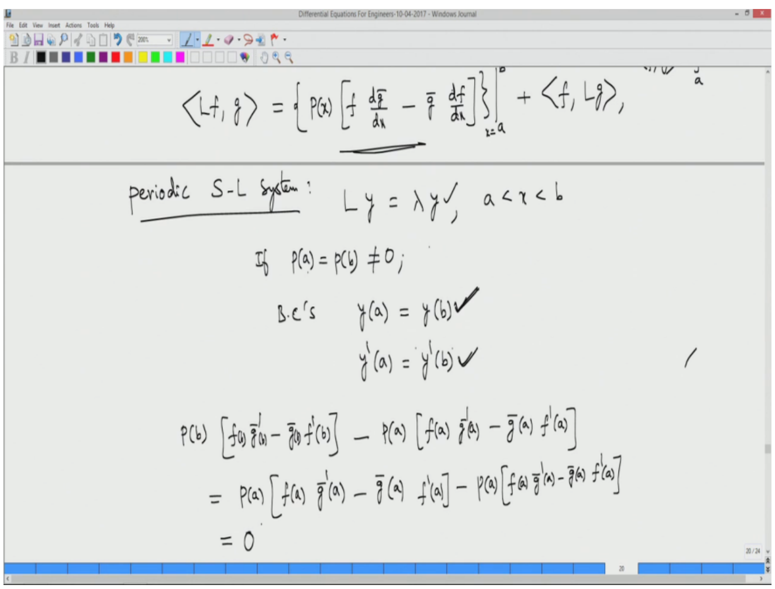

So let us see in this second case ok second that is called a periodic system, periodic Sturm Liouville system so I define this as ly equal to lambda y ok so x is between a to b now I give certain boundary conditions based on if p of a equal to p of b then what should be the boundary conditions ok it is not regular it is something p of a equal to p of b but they are not zero and which are not zero.

So that is what is the case in this case I give the boundary conditions are like this boundary conditions are periodic system you get y at a equal to y at b you expect the solutions also periodic so it's like p is periodic you expect your solutions are also periodic ok and then y dash at a is also same as y dash at b like this you can give your boundary conditions ok these are the boundary conditions.

So if you give this boundary conditions this and this together with this you can make this term so now you can see pb and f g bar dash ok g bar dash minus g bar f dash all are at b ok so this is one minus now you apply the limit at x equal to a that's why we have a minus, minus p at a f at a g bar dash at a minus g bar at a f dash at a so this is what you get that special term becomes this, this value you can easily see now we put pa equal to pb.

So pa is common now what you have fb, fb is also same as pb right fb equal to fa if you apply this boundary conditions let fb the solution of this system, f and g are two solutions of this system ok this equation with this boundary conditions then you have first boundary condition will give me I can replace fb with fa similarly g bar is also another solution so is g bar is another solution g bar at a g bar dash at a is same as g bar dash at b.

So you apply the second here so you get g bar dash at a similarly here so because g now for g bar is a solution and you apply the first condition you get g bar at a and similarly for f dash for f, f is a solution of this system you apply the second condition for f, f dash at a and this is same as a second one so this is repeated p of a this you write as it is so both are same so this is actually equal to zero, ok?

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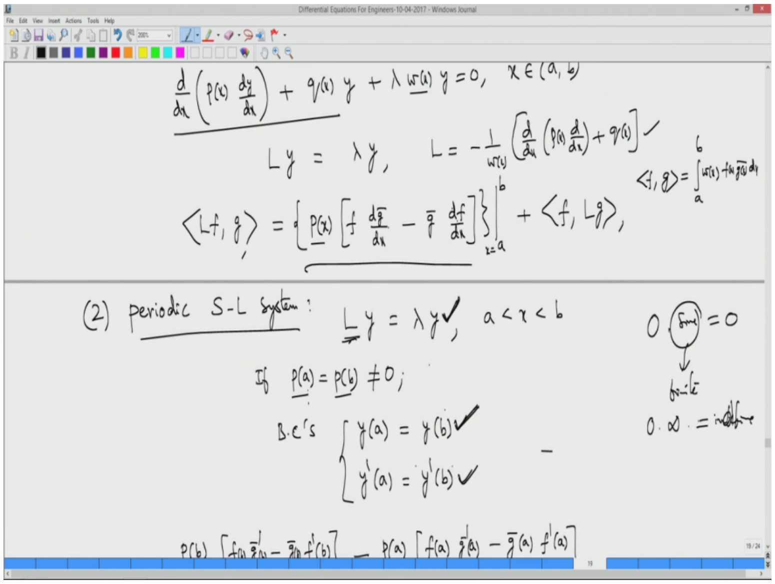

So that means you made this this term zero. Again if I provide these boundary conditions so this is periodic Sturm Liouville system so if you can give to your differential equation this Eigen value problem with these boundary conditions that also make this operator l Hermitian or self adjoint or skew symmetric ok so this is a another case this is case two so this is periodic system so what else in what are the other forms you can so what is left is now if p of a is zero or p of b is zero.

Or both are zero in that case what boundary conditions you have to give to make this special terms this boundary term zero ok we will see that, that is singular Sturm Liouville system so what is actually p, p is actually here so p if p is zero that is a coefficient of second order derivative highest derivative if p is zero p at a is zero that means a is a singular point right if is singular point so b is also singular point if p of b is zero.

Both are zero ab are similar point like you have seen for legendary equation ok legendary equation is defined one minus x square y dash right y double dash so you can see that p of x is one minus x square there you see that zero minus one and plus one both are zero they are is like that so that is an example you can see that both there is a possibility for certain equations for both p of a p of b both are zero so in such a case.

So you see that p of a is zero p of b is zero or one of them is zero and this quantity when can you say that zero into something you want to zero is it zero, zero into something is zero this something if you want this to be zero this something has to be finite ok because zero into infinity you cannot say is indefinite ok indefinite this can be anything ok this can be even for some non zero quantity so you have to make sure that this quantity is finite ok.

So how can make sure so that is that you are giving through your boundary conditions if p of a is zero ok then you make sure that this quantity at a is also zero that means f or any solution f and its derivative this one or g its derivative has to be bound at a that is a condition right f and g bar are solutions f and f dash or g and g bar, g bar and its g bar dash they are the solutions right g and g bar if they are boundary.

If you say that a solution and its derivative is bounded at x equal to a that make sure that this quantity is finite ok so the finite into p at a is zero that is known so zero into finite quantity is zero that makes this term is zero at x equal to a if p at b is also zero ok then you have zero here and you can make this zero ok so this again you have to make sure that f one its derivative a solution and its derivative are also bounded at b.

Then together it will become zero ok normally you see either both of them are zero so it’s rather that one side it is zero and the other side is non zero ok what happens if pb is non zero pa is zero pb is non zero let us say pb is non zero you have to make sure that this quantity is zero ok if you want this quantity is zero either you have to give the second boundary condition like because pb is non zero ok then either f is at b is zero or its derivative at b is zero one of these you can give you can fix your boundary conditions, ok?

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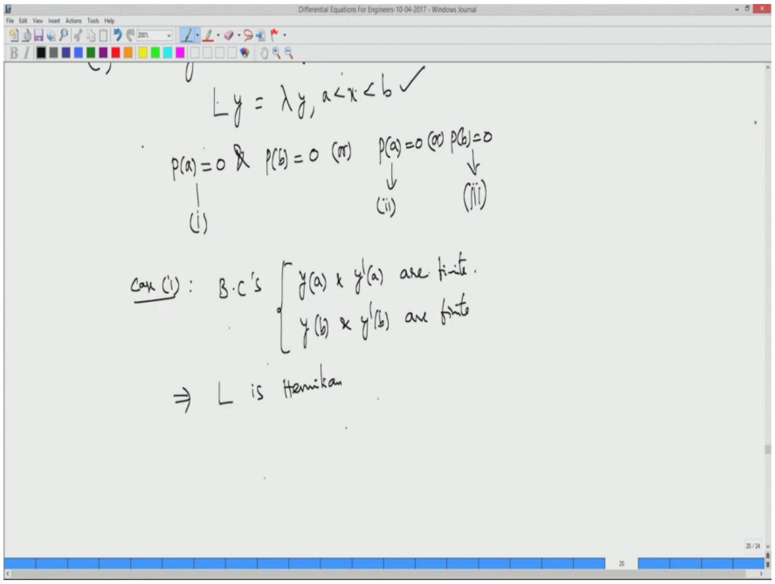

So we will see this as a singular Sturm Liouville system ok again you have ly equal to lambda y x is between a to b and what we will see three different cases here p of a equal to p of b sorry p of a is zero and p of b is zero so you have three cases this or p of b is zero or both p of a equal to zero and p of b equal to zero so this is case one this is case two this is case three ok.

So in this case first case ok let's look at that case three as first one ok so if you want this you write it as this and this both are zero this you put it as r so that this is your case one and this is your case two and this is your case three so let's look at the case one where both p and pb are zero ok case one so you want that special term that is if you look at this so that is this one p of b so what are the boundary conditions you give.

So boundary conditions for case one when p of a p of b is zero or y and y at a and y at p y dash at a are finite ok if y is finite and y is defined between a to b and at the boundary it is finite its derivative when it exist it is also finite ok so that is how you can say that both will be together so will be true if ya is finite ok and other boundary condition is yb and y dash b are finite so these are your boundary conditions if with these boundary conditions.

This equation you can make this quantity because now this pa pb both are zero, zero into your make because of this boundary conditions y and p (f and g are) f and g bar are satisfying these two boundary conditions that make this quantity and this quantity finite so zero into finite zero minus zero into finite finally zero so you see that again that special term here is a boundary term is zero so what you have seen is lfg equal to flg dot product.

So again you see that this is Hermitian so you can see that now once you get here immediately say that l is Hermitian ok in the earlier case now in the singular Sturm Liouville system case also l is Hermitian

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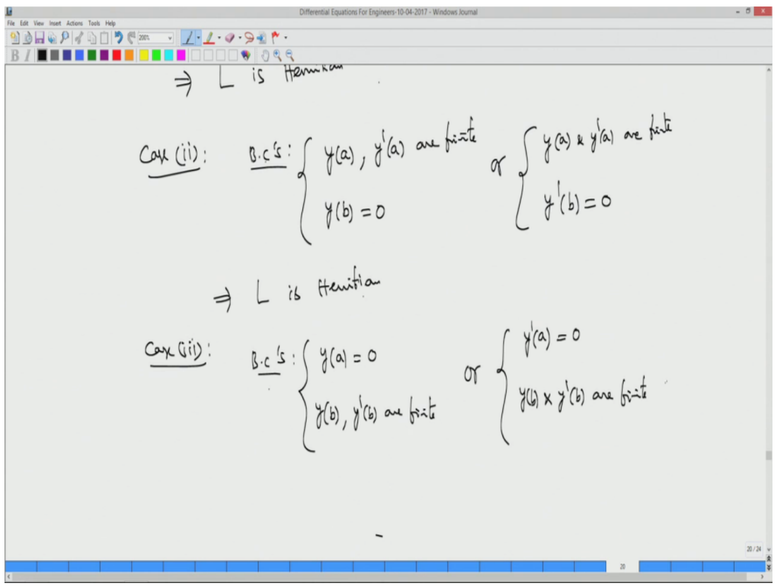

What is a case two? Case two or case three we will see boundary conditions will be because p of a is zero I want to make sure that this quantity is zero so you have to give the same boundary conditions this y at a y at b and y at y dash at a are finite.

That is first boundary condition one ok, a second boundary conditions because p of a is only known ok p of b maybe non zero that means p of b is non zero if p is non zero though I have a non zero so p of b is non zero this quantity so when this will become zero together so this becomes zero this is non zero into if you want to make this zero f at b f g bar at b either this or a derivative value at b has to be zero so your solution you can give two boundary conditions.

Y at b equal to zero ok or same boundary condition for the first one y of a and y dash at a are finite or y dash at b is zero so either of this boundary conditions we will work we will make it make this special term zero ok so that you have lfg that is l will be self adjoints so this again implies make it l as Hermitian, case three is simply repetition of case two, case three is simply replace boundary condition simply instead of a you interchange a and b ok.

You simply write for the sake of completeness so you can give this boundary conditions for ly equal to lambda y, y at a equal to zero ok and y at b y dash at b are finite ok or these boundary conditions y at y dash at a equal to zero and y at b and y dash at b are finite.

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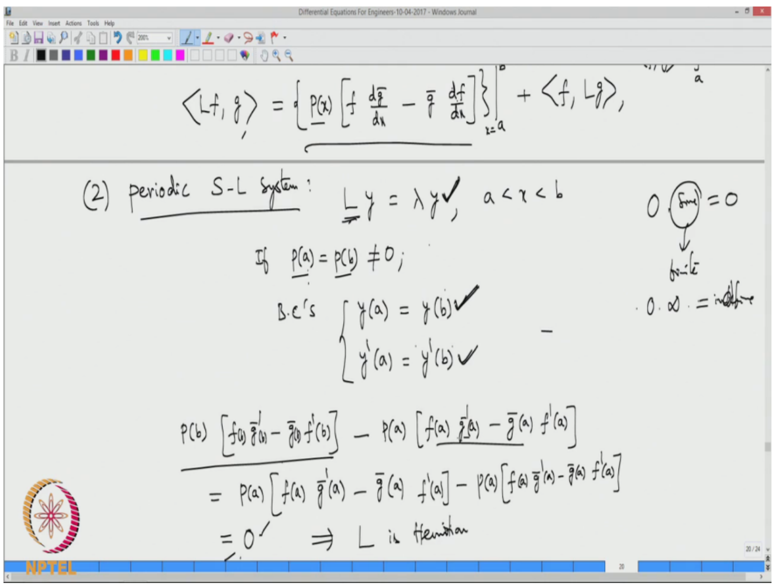

With these boundary conditions you make this boundary terms will be zero so that makes the differential operator as self adjoint or Hermitian or skew symmetric ok.

So once it is Hermitian so we can just find the Eigen values and Eigen functions ok so the inner product is same sorry dot product is here the dot product is periodic system, what is the dot product here? So whatever once you have the Hermitian (you can) you have seen the dot product is same so that is a to b ok we have written already so the dot product is same so this is what you have always ok it doesn't change here.

So we will see the examples in each of these cases for the periodic Sturm Liouville system and singular Sturm Liouville system in the next video, we will see these two special Sturm Liouville systems periodic Sturm Liouville system and singular Sturm Liouville system so when you are given a differential equation as a Eigen value problem with one of these boundary conditions so you can see that it will be either regular.

Or periodic or singular Sturm Liouville systems so I have given an example for the regular Sturm Liouville system and in the next video we will have examples for periodic and singular Sturm Liouville systems ok.