Differential Equations for Engineers
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Lecture 33
Sturm-Liouville Problems

So in the last video we were discussing about properties of Hermitian matrices, in this video we will try to prove the properties of this Hermitian matrices, Hermitian matrices are also called as skew symmetric matrices ok and the entries are complex number these matrices are called as skew symmetric when the entries of this matrix are real then it's a symmetric matrix so you have beautiful properties for this symmetric or skew symmetric matrices.

That all the Eigen values are real and you have Eigen corresponding n number of n linearly independent Eigen vectors you have you can make them (ortho) orthogonal once you have these Eigen vectors and they form so because (they are) you are in the n dimensional space or (any vector) any n dimensional vector you can write in terms of these n linearly independent vectors which are orthogonal, so we will see one by one all the properties that we listed in the last video.

So we will just try to prove that ok, so before I do that so we will just see the definition of skew symmetric or symmetric you can put in a different way more general way ok so we will see that

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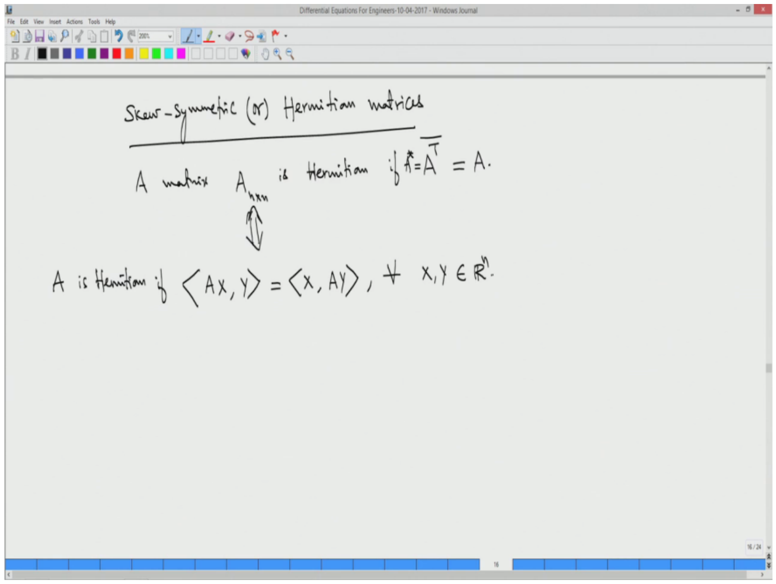

So let's put it so what is a skew symmetric matrices or Hermitian matrices so the definition is what we know is n by n matrix is Hermitian if is the definition ok so you have A bar or A transpose of it and you take a bar is actually is A itself.

So this you can call it A star so star denotes transpose and bar ok, so if this is the case A star equal to A then you we say that it's skew symmetric matrix so the same thing you can put it in a different way in an equivalent form but it is more general is if you take the dot product with you have as you operate this matrix A on any vector x and then you take a dot product with another vector y ok in order ok this should be same as x you bring this A here so if you bring it here that should be same so if I can put A acting on y and you take the dot product with x.

If this is the case ok so you can also say A is Hermitian if this is true this equality is true for every x, y in a vectors for all vectors (x and) you take any x y this is true then this is actually same as saying A star equal to A

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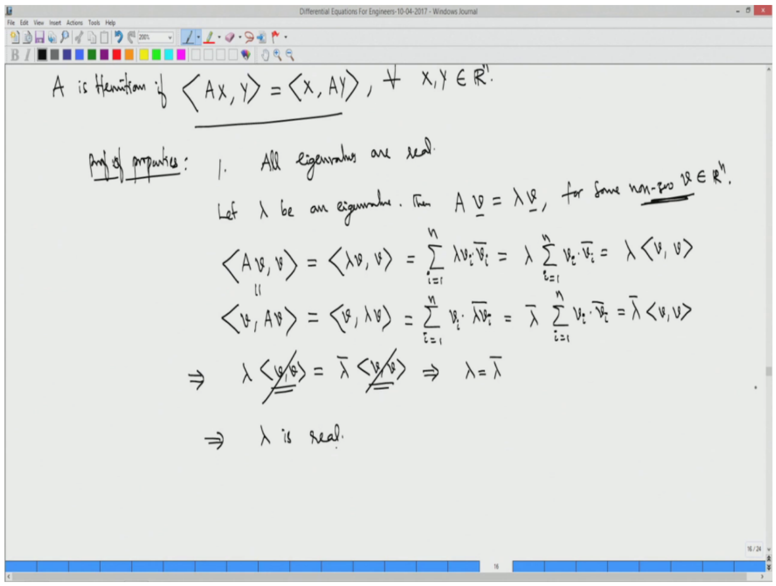

So with this definition we can prove the properties for a Hermitian matrix or skew symmetric matrix so first property we will try to prove the properties ok proof of properties one so what is that all Eigen values are real.

So how do you prove this so what is the Eigen value so let lambda be an Eigen value if you have an Eigen value what is the meaning definition of an Eigen value is there exist there is a non zero solution some v ok v non zero such that a v is actually equal to lambda v ok for some non zero v belongs to R n. So that is the meaning, if you have a lambda as an Eigen value that means you have a non zero solution for the system a minus lambda I into x equal to zero.

So the solution is v for some non zero v if you have then you say you have this is an Eigen value and v is the corresponding Eigen vector. So how do I show that so what we do is all the Eigen values I want to show it's real so you take you consider so given that a is a Hermitian, so this is A a v, a into v ok, so this if you consider this, this is equal to because a v is now lambda v so lambda v into v. So this will give me, so lambda is a constant.

So you can take it out so as a lambda so what is the definition of this dot product, dot product is I is promount to n lambda v I ok into vi, vi square right so is what is the meaning so this is equal to lambda comes out now i is promount to n vi into vi so this is actually use your lambda into this is now dot product of v and v ok, now you consider now this is same as now if you can see that this is actually same as because a is Hermitian.

Now I use this definition of Hermitian v into v dot product with av both are same because a is Hermitian now I replace av with lambda v so you have v lambda v now this product is sorry this you have bars ok so you have bars i is promound to n and you have vi dot product with lambda vi bar full bar so that will give you lambda bar comes out what you have is i is promound to n vi dot with va bar so this is nothing but lambda bar of dot product with v.

So these are same so this implies lambda so what you have is all are same right you can see that this all are same so lambda vv is equal to lambda bar vv dot product so you know that this is a non zero v so this cannot be this means this is simply a non zero this is non zero quantity so you can cancel because this non zero point you can cancel that will give me lambda equal to lambda bar that means lambda is real so if you take any Eigen value of a Hermitian matrix then Eigen value has to be real.

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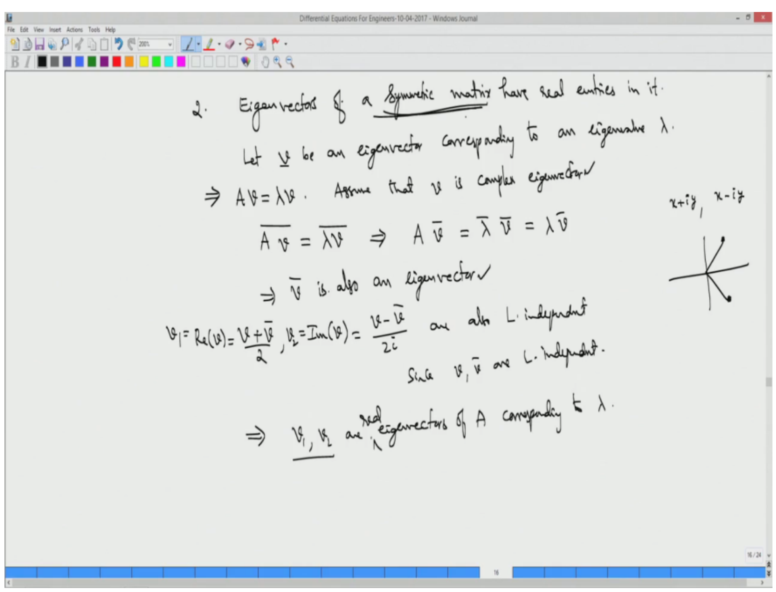

What happens to the Eigen vector so and you can make them so Eigen vectors let Eigen vectors of a Hermitian matrix as real entries how do I show this you can make them ok so Eigen vectors of a Hermitian matrix you can make them to to have a real entries in it so they have real entries in it so how do I show that so you say lambda is suppose the lambda is an Eigen value correspondingly you have an Eigen vector.

Let us say v ok so let v be an Eigen vector corresponding to an Eigen value lambda so this all is corresponding so if you have an Eigen vector there is corresponding Eigen value lambda ok if you say something is some values values and Eigen value that means there is a non zero solution vector that is called Eigen vector ok satisfying av equal to lambda v so if you have an Eigen vector so what you have is immediately lambda av is lambda v ok.

Suppose v is complex ok so let us say complex Eigen vector assume that (lambda) v is having complex Eigen vector then you can take a bar ok what happens top v bar, v bar a av bar so av bar equal to lambda bar v bar so I just take the bar for this ok if you take a bar and I know that a is having real entries so you consider all the matrices with real entries ok that are skew symmetric so this is our real Hermitian matrix.

So this properties is not true further if A is having complex entries if A is Hermitian matrix that means for skew symmetric matrix ok if A is skew symmetric matrix with the real entries that means a is symmetric matrix for a symmetric matrix you consider all the Eigen vectors so if the Eigen vector is complex you can actually make them real ok that's what we are showing so if you consider Eigen vectors of a real Hermitian matrix ok.

That means basically symmetric matrices you can say symmetric matrices of a symmetric matrix, symmetric matrix have real entries in it so let's consider a non zero v be an Eigen vector corresponding to an Eigen value lambda so you have this that means this implies this now assume that v is having v is complex Eigen vector now if I take this bar so what happens this implies because A is a real symmetric matrix that means a is real is having real entries.

So this simply a v bar equal to lambda bar v bar so that means and this is as same as but I know that if it is symmetric implies this is a Hermitian also ok just way the property one and the lambda is always real so this lambda bar is actually lambda v bar so what you found is av bar equal to lambda v bar so if v is non zero Eigen vector because it's an Eigen vector it is non zero v bar is also non zero so that implies v bar is also an Eigen vector ok.

So you have two Eigen vectors corresponding to if v is complex Eigen vector v and v bar are linearly independent ok so if x plus iy x minus iy you take a division if y, y is non zero if you take the division that cannot be a constant ok, so they are linearly independent vectors so it can easily see right so x plus iy is here x minus iys x is same y is here so somewhere here so these are two linearly independent vectors so this is also an Eigen vector.

So now what I choose is (you can) now you have two Eigen vectors v and v bar what I do is I consider this sum and then I consider their difference divided by 2i you take the half ok now it is a real part of v and here these are imaginary part of v, v minus bar divided by 2i ok so if you choose this if v and v bar are linearly independent real part of v and imaginary part of v this is the sum and difference are also linearly independent.

Since v and v bar are linearly independent if they are complex yes they are linearly independent so that means if you have an Eigen value corresponding to a symmetric matrix so it may have a complex if it has a complex Eigen vector that means bar is also Eigen vector so that means it has (2 Eigen vectors) 2 linearly independent Eigen vectors corresponding to the lambda and you take the real part and imaginary part they are also linearly independent.

So what you have is real part of v you call this v1 ok, this is v2 v1 and v2 are Eigen vectors of a corresponding to lambda ok and you see that v1 and v2 are having real entries these are all real Eigen values real Eigen vectors that's what you have same. So if you have a symmetric matrix, symmetric matrix means there is also Hermitian so I have a real entries so a bar a transpose bar a equal to a so bar doesn't matter right.

Because you have a symmetric matrix that means real entries that means this Hermitian from the property one you see that the Eigen values are real corresponding to that one real Eigen value any Eigen value you have let us say complex Eigen vector so then this bar is also an Eigen vector so you have two linearly independent Eigen vectors and you take the real part implies immediately real part and imaginary parts are also linearly independent.

So that implies they are the real Eigen vectors of a corresponding to Eigen value lambda that's what is a property 2 so you see that (if you have a) if you will have a symmetric matrix with you if you have a matrix with real entries and it is symmetric all Eigen values are real corresponding Eigen vectors are also real Eigen vectors ok that's what you have seen.

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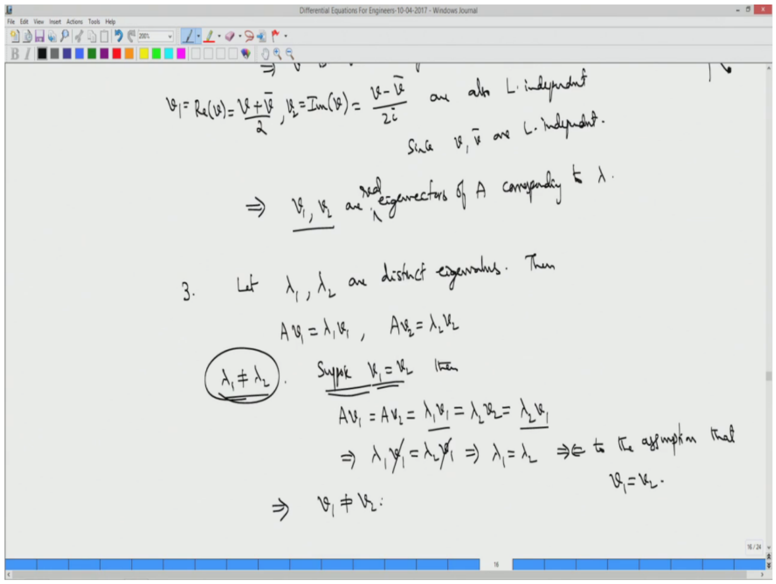

So another property which we have seen earlier what are the things we have so this is what we have seen distinct Eigen values Eigen vectors of (())(16:47) distinct Eigen vectors if Eigen values are all distinct so that is easy to see ok let (v1) that lambda1 lambda2, lambda1 is not equal to lambda2 ok or distinct then distinct Eigen values ok immediately from the definition of Eigen value a lambda a v1 corresponding Eigen vectors av1 equal to lambda1 v1 and av2 equal to lambda2 v2 because v2 is corresponding Eigen vector ok.

V2 is Eigen vector corresponding to Eigen value lambda2 so I want to show is given that lambda1 is not equal to lambda2 ok so I have to show that v1 is also not equal to v2 so how do I show that v1 is not equal to v2 so what you do is because lambda1 is not equal to lambda2, lambda1 minus how do I show that Eigen vectors all suppose ok so let us say suppose v1 equal to v2 then what happens av1 equal to av2 which is equal to av1 is I already know.

That lambda1v1 ok which is equal to lambda2 av2 is lambda2v2 so what does it mean so because now v1 equal to v2 right so v1 equal to v2 so both are same so this is same as lambda2 into v1 ok lambda v because that is what we assume if we assume and this is true right, so if you consider these two the implies you have lambda1v1 equal to lambda2v1 v1 is an Eigen vector which is same so non zero so you cancel both sides.

That will give me lambda1 equal to lambda2 but we know the given there it is non zero they are not same so that means in a contradictory way you can show that Eigen vectors are also different so that means this assumption is wrong this is a contradiction to the assumption that v1 equal to v2 so that means v1 cannot be equal to v2

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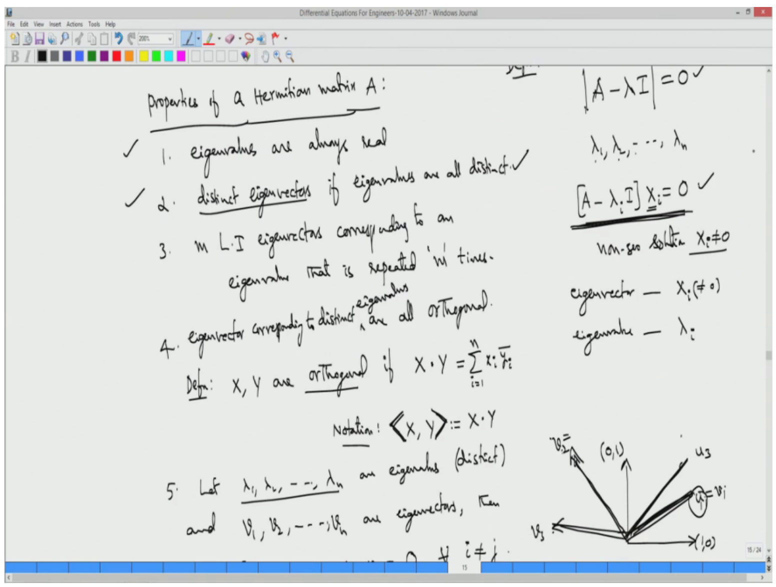

So just now you see the property here which we listed in the last video that distinct Eigen vectors. You will find corresponding to distinct Eigen values ok, and you can also show that they are orthogonal (corresponding to) corresponding Eigen vectors distinct Eigen values let us say you have distinct Eigen values.

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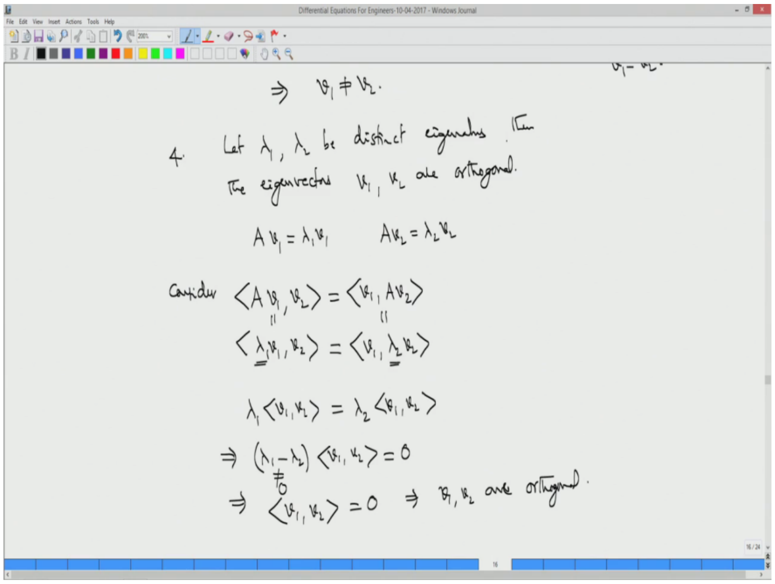

So this v1 and v2 are orthogonal so I want to show that v1 v2 are orthogonal 4 let us say write let lambda1 is not equal to lambda2 be distinct Eigen values then the Eigen vectors v1 and v2 are orthogonal.

That means the dot product should be zero so how do I show this so these are just write the definition of Eigen values and Eigen vector so v1 is Eigen vector corresponding to lambda1 so implies av1 equal to lambda1v1 and v2 similarly you have av2 equal to lambda2v2 is what you have now I consider what I do is I consider av1 v2 ok this dot product is I consider ok consider this, this by definition because a is Hermitian ok.

And I don't say that is real symmetric that means is actually Hermitian it may have complex entry (21:37 you may have complex entries) so by definition this will give me v1av2 ok now what is this left hand side, left hand side is actually equal to this is equal to av1 I can write lambda1v1v2 which is equal to this I can write v1 av2 is lambda2v2 ok lambda1 lambda2 are real that we already know ok this thing (Eigen values of a) any Eigen values is real.

For a Hermitian matrix so these are real (so if you write this) so if it is here if it is a real if it just comes out lambda1 you can just verify make a dot product and take this lambda1 out what you get is v1v2 here if the value is here if the value is here this will come out as a bar because it's a real no issue so you don't have problem here so you don't even so when it comes out bar because it's real it is same as lambda2 into v1v2.

Now v1v2 are Eigen vector so they are non zero if they are non zero and lambda2 is not equal to lambda2 so that implies lambda1 minus lambda2 so this is same so this into this dot product v1v2 as to be zero

Now this cannot be zero ok because lambda1 is not equal to lambda2 so that means v1v2 has to zero, dot product has to be zero implies v1v2 are orthogonal this is how you show ok.

So you can just remember if any distinct Eigen values corresponding Eigen vectors.

Are always orthogonal if they are not orthogonal that is possible when they are not distinct so when they are not distinct then the Eigen values are not distinct that means you have a repeated Eigen values when it is repeated m times you have m linearly independent Eigen vectors ok that we don't show it's not that simple so you just it can assume that so if it is repeated m times you will have m linearly independent Eigen vectors so those you can make those m vectors.

They are linearly independent and you can make them orthogonal by the graham smith process which I explained in the last video ok

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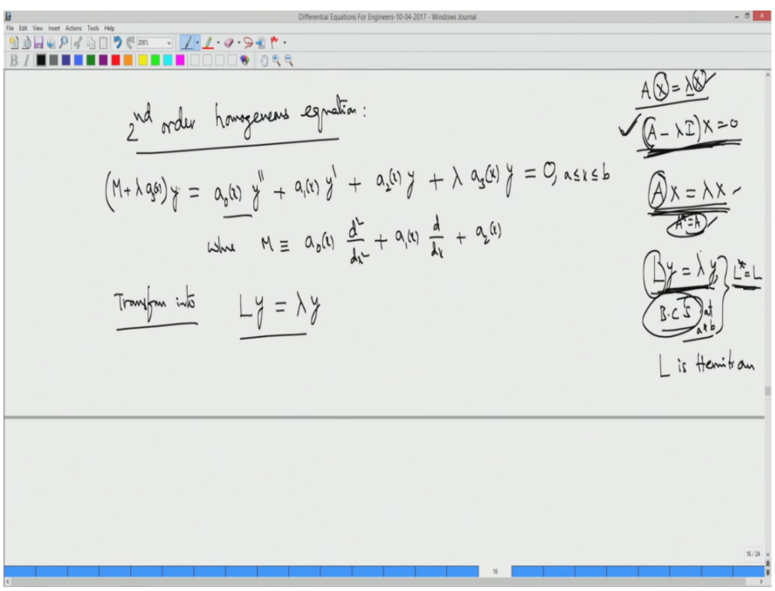

so with this we will try to imitate the whatever properties we have for this Hermitian matrices we will just take a second order homogeneous equation ok let's start with a second order homogeneous equation so let's consider this we will imitate whatever we have ok so if we have a not of x this is what we have.

A not of x y double dash plus a1x y dash plus a2xy ok and then you include some parameter lambda ok so because we are dealing with this Eigen values ok ax equal to a earlier you have for the matrix ax equal to lambda x right this is what you look at if you want to see the something whether we have a non zero solution for for this ax if there is a non zero x then lambda is an Eigen value ok corresponding (Eigen) that non zero solution is Eigen vector.

So the similar lines we consider the equation so you have consider a minus lambdai as matrix ok this is what we consider so the same way when you consider the homogeneous equation differential equation also what we do is a2 this is where normally ends is the differential equation so I want to consider this kind of equation so you have you include a parameter lambda so lambda y so if you say lambda a3 of y ok a3 of x into y.

Equal to zero so this is the general second order equation which is of this type ok, analogous to the matrices matrix operator so this is the matrix operator x equal to zero I have this differential operator ok m of y is this rather m plus lambda a3 of x so this is what is you're a minus lambda x ok so let's write this m minus m plus lambda a3 of x acting on y so were m is actually defined as a not of x d square by dx square from this term plus a1x ddx plus a2x.

(26:55 that's it) so that is your my so if you add put y here that is actually this 3 terms and plus lambda a3y is as it is so that is equal to zero so that is the equation so I can always put this in the form of in a Hermitian form so let me see what that Hermitian form (Ly) I make this as I'll put it in a nice form so that is for the what I do is I will try to put this in this form ly equal to some lambda y so this I transform ok I bring it into transform into this type.

So that this is like ax equal to lambda x ok, so this is a general equation so if it is a matrix directly you have already in this form both are same this is already symmetric form if a is a general matrix (you will all the properties) so a is not if a is a general matrix though it is in this form it is not symmetric or it is not skew symmetric given that a is skew symmetric it is already in this form ok, so in the infinite dimensionally for the differential equation.

What happens though you consider this a is a general matrix so I have I try to put ly equal to lambda y L is a operator L is like a matrix here and along with some boundary conditions I make this L is Hermitian ok so here for the matrix you don't have any boundary conditions there is nothing so you A is Hermitian means A trans A star equal to A ok, if you know that A star so you put it then you have all the properties, but for this equation already in this form.

So you first (in this) put in this form then you show that L star equal to L I will tell you what exactly it means ok this is actually symmetric provided you have some boundary conditions so that means y is y of x write so y is actually y of x so x belongs to some domain a to b finite domain ok so you have boundary conditions at a and b, ok? So we will see that,

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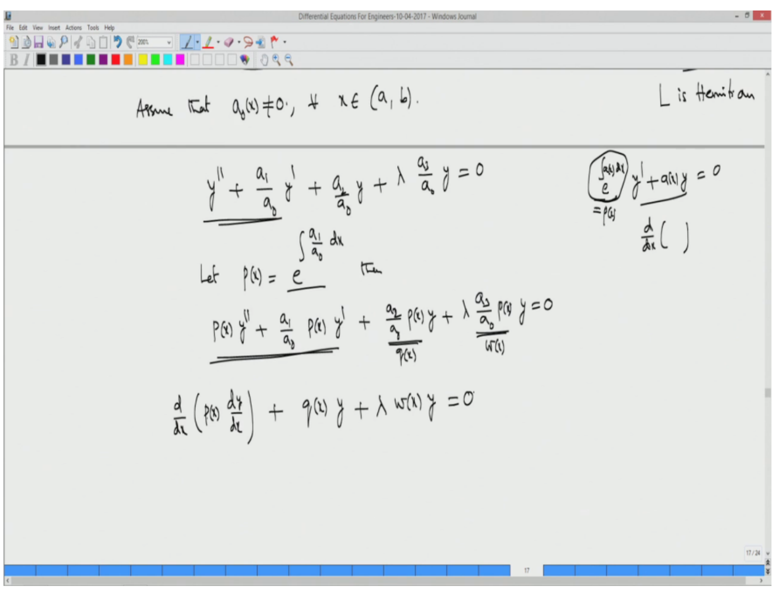

so how do first of all we will try to put in this form ly equal to lambda y so ax equal to lambda x for general matrix. A so first we will write this way ok so ly equal to lambda y so but this if you consider equation like this it's not proper since it is not in that form ly is not ax equal to lambda x you try to put it as ly equal to lambda y how do I do this you simply divide with assume that a not of x is non zero ok so assume that a not of x is not equal to zero why I assume this if it is zero it is not defined at that point so it's not in the domain part of the domain ok.

So you can always assume that it is non zero in that domain so it can be (even at A) it may be at A and b you may have a nit if x is zero so the domain always consider as open interval ok so I always consider as open interval so assume that this is true for every x in the open interval a to b then I can divide it a not if you do that y double dash plus a1 by a not y dash plus A to by a not into y plus lambda a3 by a not to y equal to zero ok.

So this implies now what I do is I try to put this together so if I try to how do you prove this together you have earlier you have seen y dash p of xy equal to zero if you want to write this as a ddx of something what you do you multiply some integrating factor so what your integrating factor is e power integral pxdx you do that so if you multiply this integrating factor so let's call this let p of x multiply p of x I call this p of x as e power integral integrating factor.

I am calling here its p ok so this is actually a, a of x you say then you have integral a of x so this whole thing I am calling px ok so let px as a1 by a not into dx ok then if I multiply this you have px y double dash plus a1 by a not into px ydash plus a1 by a2 by a not into px y plus lambda a3 by a not into px into y equal to zero that's what if I multiply now I can put this together as 1 so that is y dash into px for this you differentiate ddx of if I write like this ddx of dy by dx.

So what if you see this px into you differentiate this part px into y double dash first term second term px into y dash you have to differentiate p if you do because p is e power this e power itself that is p and you differentiate 1 by a not comes out so that is what is, so these two terms I can put it like in this form plus whatever here this you call it some q of xy equal to plus and then plus lambda some w of x this you call it w this is q this is w of x into y equal to zero ok.

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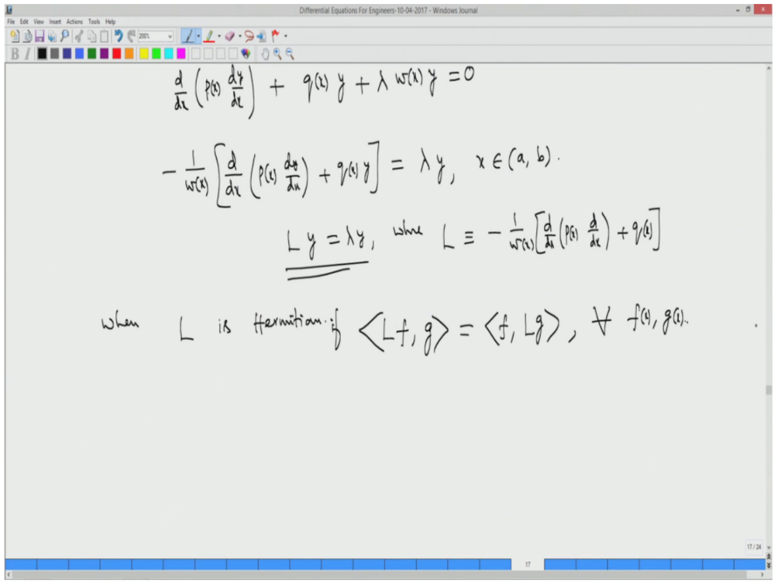

So this is what I have so now I can write this nice form so ly equal to lambda y ok this is like you can put it in that form so this means this you can write minus ddx of px py by dx ok plus qxy ok this is minus equal to lambda wxy I take this that's left hand side so you also divide this with wx so you have 1 by wx equal to lambda y this is x belongs to ab so the domain is this, so this is now nice form so this is ly equal to lambda y where l is minus 1 by wx.

L is this operator this is like a matrix ok so you have ddx of px ddx plus qx it's just an operator it can act on any function y ok so this is what is your matrix, it is like in this form so I want to see when L is Hermitian our question is when L is Hermitian, what is the Hermitian meaning I want to make the dot product (so with any) so is Hermitian if I work with some functions so you have a functions here earlier you have vectors now y of x is a function.

So I have a function lfg fg are two functions if this is a definition of Hermitian if this is what is true so flg if I bring this here ok for every this is true for every fx gx ok

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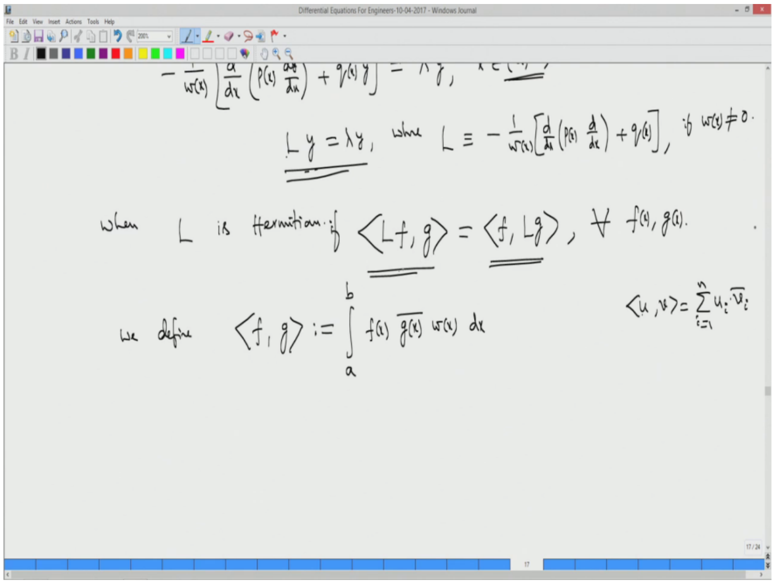

So earlier you have vectors now we have functions for every functions this is true so you just what is this dot product so you define this dot product now, so this dot product in this case for the functions is we can see we can define the dot product so I define this dot product, dot product of fg ok is.

So is depends on this L this actually this dot product is not usual one as a summation because you are involving with functions you integrate earlier dot product with v1 and v2 is simply summation i is promound to n v1 i so let us say uv so have a ui into vi bar now what you do is you do the same thing here so what is the domain so we were doing earlier from because only finitely many elements u1 u2 un v1 v2 vn each component you multiply.

Here also you do fx into gx gx bar and x is between x is continuously running from a to b so you integrate your edition is like integration a to b fx gx bar dx ok you see that and see when so when you have in the L when you have this w (the senior product the annual product you diff) sorry when w is non zero here so you can see that in the operator L w what is this w, w is actually a3 by a not into wx right a3 by a not into px.

Which is you can assume always that it is positive ok so we will see this is always assume that is non zero ok so assume that we can assume that this w is, see because you divide it right divided means you assume that if w of x is non zero whenever you divide it you should that should never be zero ok so you have such wf if you have what we do is I have this wx into f into g so you have a weight function this is called the weight function ok.

So in the dot product you have a weight function if it is 1 it's the usual dot product in for the functions ok, so this you this is the definition you define a dot product whenever you have when you differential equation is having this form with L or 1 divided by the w the w whatever w you divide it that should come here ok with this definition ok we define the dot product we define this dot product so with this dot product we apply here.

With this you want to see whether it is Hermitian or not ok

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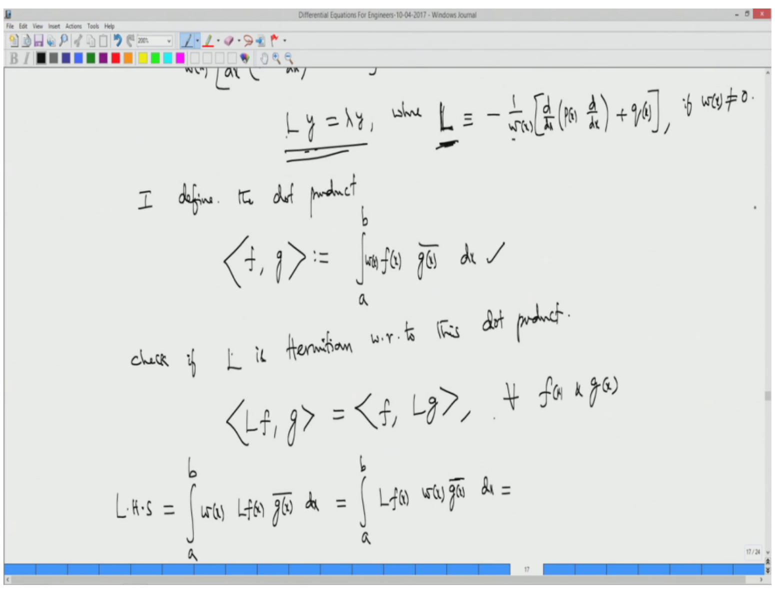

So (you have) actually you have to define this first so let us write let's do this first in a logical way so once you have this operator ly equal to lambda y I define the dot product what is a dot product here, dot product as for the functions f(x) and g(x) as a definition so earlier dot product means dot product of sum of the products of each components of a vector so now you have you have fx into x you phase fx gx bar ok.

This is a component and you doing it over all x which is in range in the interval a to b, so it is continuous so you have to integrate from a to b and you have because for this L once you see that this operator is having this w which is non zero that w is called the weight function that should be there in our dot product so that is how with this dot product I define this dot product this is the with this dot product.

I want to see whether L is Hermitian ok check if L is Hermitian with respect to this dot product the above dot product ok with this dot product you want to verify so what is a meaning of L is Hermitian you have L f g you want to verify this ok flg for every f and g fx and gx ok this is what you want to check ok so then only this operator L is differential operator L is self weight joint just like in our matrices, A is Hermitian means a star equal to a you can directly check.

With your matrix ok so that is a same as saying earlier we defined ax equal to y equal to xay that is definition we have given so both are same equal length so you want to verify this you start with the left hand side, what is the left hand side so write a to b wx lfx and g of x bar dx ok so this is equal to what is this one so a to b lf you write like lf x wx g bar of x dx

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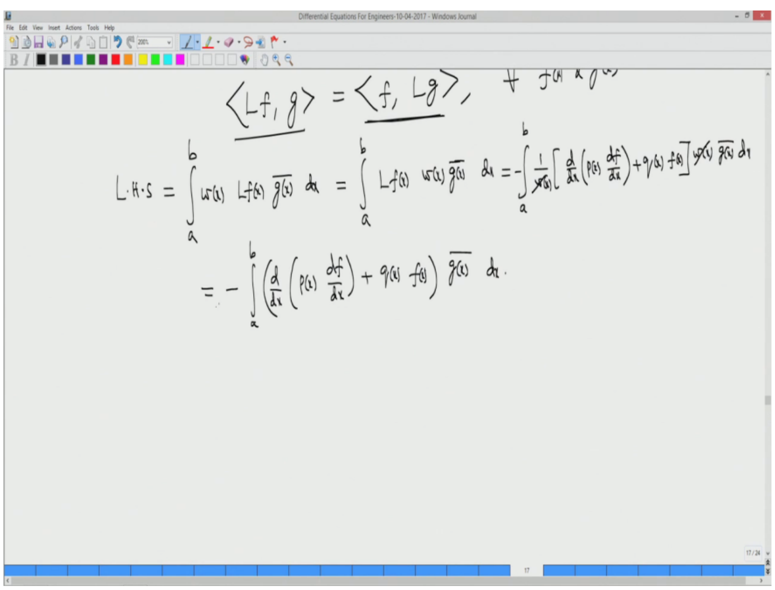

You can now define what is your Lf, lf is you can use this one so this you write it. So that we give you a to b minus comes out lf I am replacing one by wx px there is px f dash of x for this whole dash ok so if you want to nicely can write ddx of px df by dxplus qx into f of x you acting so acting L acting on f this is what is your Lf into wx gx bar dx so this is equal to minus a to b this ww goes now so what your left is simply ddx of px df by dx plus qx fx into gx bar dx ok so we will see the left hand side is actually right hand side.

You want to see that ok so we will see exactly what is a left hand side and when this left hand side is (you just show) so we will see this we started with the left hand side that we defined here now we just use the integration by parts in the next video and see at right hand side place certain terms we will get we will make that when you so you have to define you want to see that that is actually left hand side is equal to right hand side.

When the left hand side is equal to right hand side so you need to give impose certain conditions ok further differential equations so you need to give some boundary conditions so once you give the boundary conditions you can see that left hand side will be equal to right hand side so then it will be Hermitian operator so once you have the Hermitian operator all the properties which you have seen for the matrices will be valid here so we will just use directly analogally ok.

And then we see this in the next video.