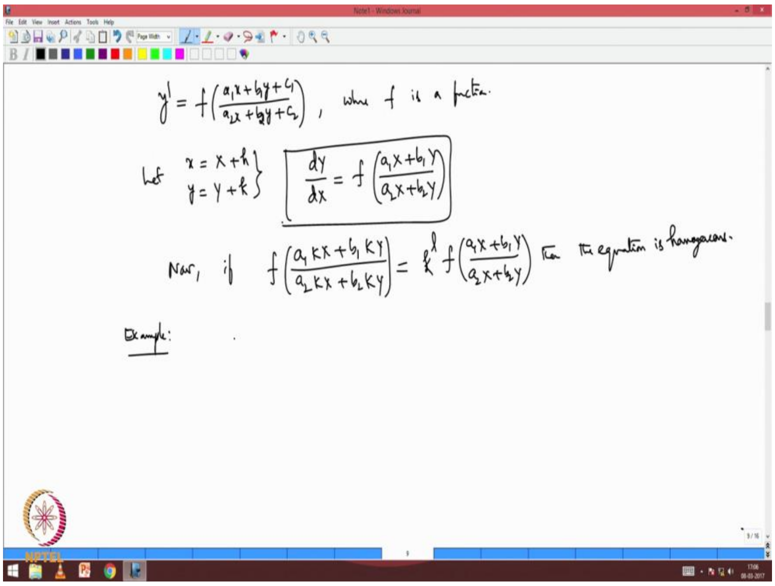
Differential Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
Department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-3
Methods for First-order ODE's- Exact Equations.

So in this video we will see what other equations that can be reduced to homogeneous equations. you know how to solve homogeneous equations, we will see what are the other equations and also you have seen some simple equations that can be reduced to homogeneous equations. So in this video we will see what are the other kinds of the questions that can be reduced to homogeneous equations. Also further we define what is exact differential equation and we will learn how to solve, how to integrate them.

(Refer Slide Time: 0:59)

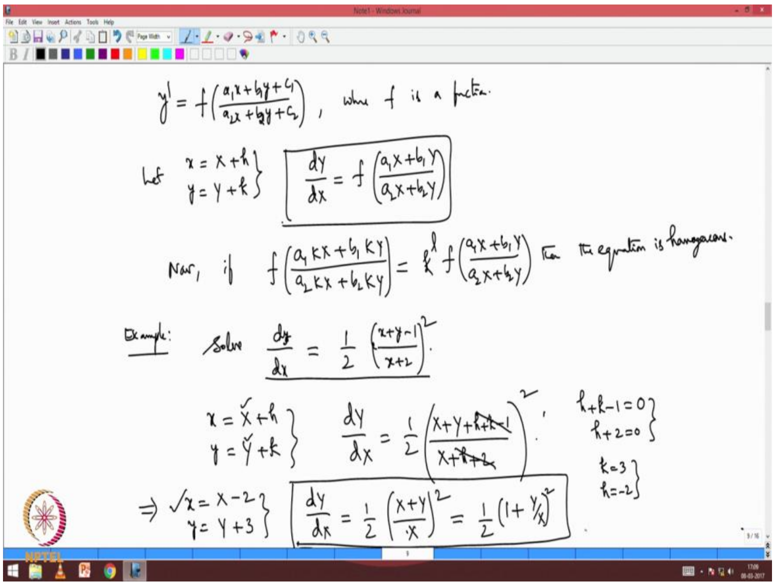


That means how to, how to solve these exact equations sometimes directly by integrating or by a method, okay, so we will see that. So we had reducible to homogeneous equations. What we have chosen is simple form earlier. So we can now, we can have some function of that form A1 x plus B1 y plus C1 divided by A2 x plus B1, B2 y plus C2. Okay. So how do we solve this, where F is, F is a function, certain function. So you follow the same procedure, you look for x, y as new variables for some H, y as some new variable y plus K. Okay.

So you have this transformation. Then you reduce left-hand side as d, available, dy by dx is equal to some F of A1 big X plus B1 big Y by A2 big X plus B2 big Y. If you repeat the same thing, so what earlier we had, only without F is one. F of x equal to1. Now we have general sum, f of something, the whole thing as variable, so this is general F, it can be square of this, it can be anything. Only thing is if F of this, so inside is homogeneous.

If F is also homogeneous, if you can take it out, when you replace K x, okay, so if you reduce this into this form, now if F of K1 K x plus B1 KY divided by A2 K x plus B2 K y, if you can write this some K Power L times F of A1 x plus B1 y, A2 x plus B2 y, then it is a homogeneous, then the equation is homogeneous. So we can solve, we can proceed. Okay. So I will give an example how to solve, how to go and come out to apply this method. Example.

(Refer Slide Time: 3:37)



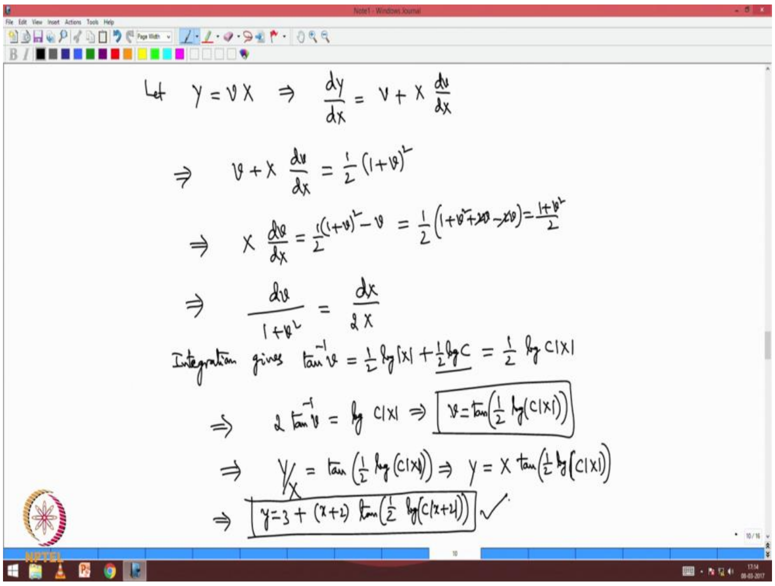
So let us do this example. Solve dy by dx equal to 1 by 2 x plus y - 1 divided by x + 2. So this whole square. It is like this. You can see inside, inside is of this form, F is 1 by2 x square. Okay, if you can go on to recognise like this, in terms of x, F of x equal to this whole thing is x, here I have one by 2 into x square, whole thing is one variable, view it as one variable. So, so you the transformation as usual, you look for x equal to x plus H, y equals to new variable y plus K. These are new variables x, capital X, Y. you want to some H, K some numbers so that the right-hand side of this equation is homogeneous function. Okay.

Can we do that? So if you want to do this is what happens, with this transformation, we, the equation becomes, what we have seen is dy by dx equal to 1 by 2 this square. What do you have, capital X plus capital Y plus H plus K minus1. What did I do … So I replaced x by capital X plus H, y by capital Y plus H, y plus K. This is what I have, so bottom also, the denominator you have capital X plus H + 2. So now I choose my H, K such a way that H plus K minus1 equal to 0, okay and then H +2 is equal to 0.

So what are the transformation, what are the numbers I get that solves this? H equals to -2, so that makes K equal to 3, H equal to -2. So this is the solution of this that makes it 0. So if I choose my H is equal to -2 and K is equal to 3, okay so that implies my transformation is capital X -2, y is capital Y +3. So this is my transformation, the question becomes dy by dx equals to 1 by 2 x plus y by x whole square. Clearly when I replaced KX, KY, KX, K square comes out, so it is homogeneous function. So this equation is now homogeneous with this transformation. Okay.

With this transformation, the given equation becomes homogeneous equation now. So we will solve this homogeneous equation now. So what is the step, so we have to write this right-hand side is equal to, so we remove this, equal to, so 1 by 2, x you take it out, it becomes 1+ y by x whole square. Okay. So x square, x square goes, this is what is your differential equation. Okay.

(Refer Slide Time: 7:29)



So now you choose y by x is equal to v, new variable as usual, how do we solve the new procedure, the procedure to solve the homogeneous equation, we have chosen capital Y is equal to, independent, dependent variable is equal to some new dependent variable v and independent variable x. That gives me dy by dx, big variables is equal to v plus x into dv by dx, okay, which we know is equal to dy by dx is equal to half plus 1+ y by x is v, 1 plus v square whole square, equal to, okay, so.

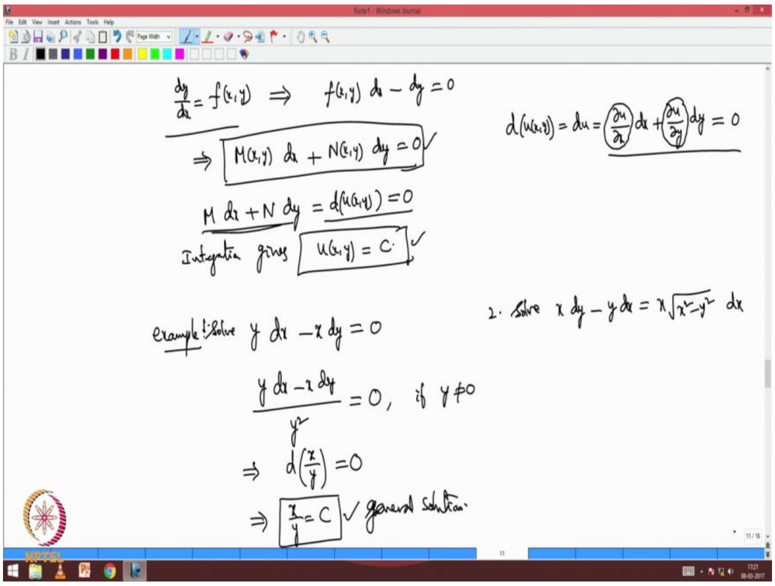
So this implies dy by dx which is v plus x into dv by dx, which is equal to, just replacing here. So left-hand side dy by dx I replace with this and this I replace half 1 plus y by x is v, y by x is v. So I have half 1 plus v whole square. So this will give me x dv by dx equal to 1 by 2, 1+ v square. So 1+ v whole square minus v. So this is equal to 1 by 2 common, so one, 1 + 2V plus v square plus 2V and minus 2V. So this, okay. This is closed, you have 1+ v square by 2.

So this implies, you can see that now the equation is variable separation, you can easily, variables are separated, so you can easily, dv, v variables you can put this side equal to x variables you can put at this side by x, 2 x is there. And then so you can integrate. Integration gives again tan inverse v equal to 1 by 2 log mod x plus arbitrary constant, I call it half log C. C is arbitrary, log C is also arbitrary, half log C is also arbitrary constant.

Why I did this, so that I can easily simplify this as 1 by 2 log C mod x. Okay, so this implies 2 tan inverse v is equal to log C Mod x. This gives me tan inverse v, v equal to 1 by 2 log C mod x, then take tan both sides, this is what is your general solution of the equation, this. Okay. So this is dy by dx square, so still you have to put in the new variables, this is in the new variable. So v is, v is given by y by x, y by x, big X is equal to tan of, of log C mod x. Mod x, this implies y of big X is equal to big X times Tan function of half log C mod x.

Okay. Now going back to original variables, capital Y, capital X is small x +2, capital Y is y - 3, you put it. Small y is y - 3, capital x is x + 2. So tan of log CX is capital, small x +2 with mod. So this is your general solution. If you really want equal to 3, you can bring it on this side and write it like this, this is your general solution of given equation in the variables small x, small y. So this is, this is the equation, we reduce it into homogeneous equation and then use the method to solve the homogeneous equation, homogeneous equation, finally used the transformation, brought the new, old variables to get the general solution of the given equation as this. Okay.

(Refer Slide Time: 12:43)



So this is how we solve some equations which are, which can be separated when the variables are separated, when you have differential equation dy by dx equals to F of x, y. This is the form you have general differential equation of first-order. When F of x, y is, you can separate these variables x and y, you know how to solve. When F of x, y is a homogeneous equation, you know how to solve it, you know how to find the general solution.

And some form, when if F of x,y is having certain form, like I showed earlier, earlier method where you have this form, you know how to solve, you know how to reduce them into a more generous equation and then solve it. Next what we do is, this is actually we can write it like F of x,y into dx minus dy equal to 0. Right. So in general we can write M of x, y the x plus N of x, y dy equal to 0. So this way also we can represent some first-order ODE.

Finally differential equation, we can also see that, we can also see in this form. When I have like this form, if I can write this M dx, I do not repeat M of x, y, so I write simply, for simplification I write M dx plus N dy equal to, say some total derivative of, this is a function of x, y, dx, N is a function of x, y, dy. So I want this to be the derivative of, total derivative of some function U of x, y. If I can write like this, okay and what is the differential equation, M dx plus N dy is equal to 0. So M dx plus N dy is equal to total derivative of U of x, y which is equal to 0.

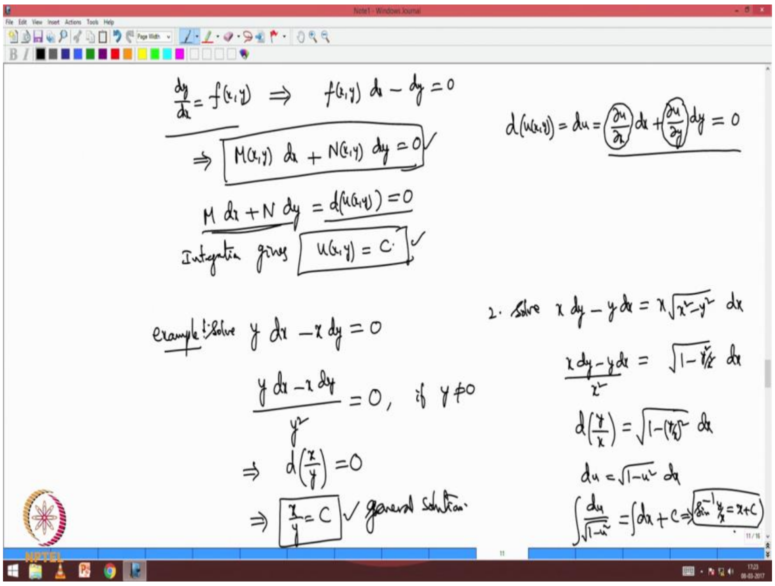
So that means that is easy, right, I can simply, now I can easily, I can simply integrate this equation. Now the equation is this. If I integrate, integration gives U of x, y is equal to simply constant. So this is your general solution. So what is that it did? you have a differential equation, put it in this form M dx plus N dy, this if I can write like this, DU, total derivative, so what is the total derivative of a function?

D of U of x, y, so there is, so you write du, which is, it is a function of symbol to variables, so it involves dow u by dow x, partial derivative into dx plus dow u by dow y dy, so this is the form you have, this is the form is equal to 0 for some function u of x, y. If this is your differential equation, that means this is your M, this is your N. M, M and N, if you can write like this form with some function u, then you can integrate, directly integrate. For example, y dx minus xy equal to 0. If you want to solve this yx dy, if you want to solve this equation, there is nothing you can do.

It is already separated, the variables are separated, this I can easily put in that form. So how do I write this? I can simply write y dx minus xdy divided by y square, I can write, if this is 0, this is also 0, provided y is not equal to 0, those domains, okay. So I have to consider the domain of the differential equation where y is equal to 0 is avoided. So what is this one? This, this now the equation is this, this I can write like this, this is actually the derivative of x by y equal to 0.

It is like in this form, total derivative of function of x, y square u is x by y is equal to 0. So you integrate, so you will get x by y is equal to constant is the general solution, right, general solution, directly. So some equations you can directly integrate, that is what it means. you can write, you can take the differential equation, see whether you can write it as a total derivative of some function, so you integrate, so such an equations are called exact equations. Okay. So this is one example, I can give one more example, so I can give one, one more example if you want to solve.

(Refer Slide Time: 18:24)



Little complicated but you can do. x dy minus y dx is equal to x into x square minus y square dx, let us say we have this. We want to solve this, again you can do this same way. You can nicely put it in the form, so how do we do, this is, this is with commonsense you have to do. So you 1st take this right-hand side, x you take it out inside this root, so you will have x square, this x coming out, with 1 coming out, x is coming out, this is multiplied by x square, square root of 1 minus y square by x square is equal to x dy minus y dx.

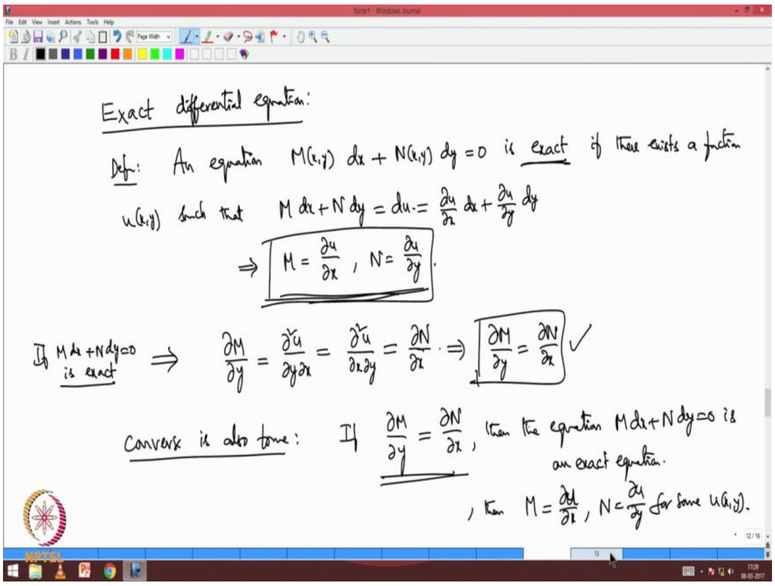
If I divide with x square, I remove this and put it here, so this is what is the equation now. Now what is this point? This is total derivative of y by x is equal to 1 minus y by x square. This also I can integrate, what you have is something like du equal to square root of 1 minus u square. Do not you know how to solve? So u equal to integral 1 minus u square, right. Rather, sorry du by square root of 1 minus u square is equal to 1, so you integrate both sides, okay, dx.

u is a function of x, y, so this simply your, what we have, so this is the total derivative, right. du is this, you simply integrate both sides with respect to u, okay. So what do you get? u is a function of x, y, so total derivative of u, so this will give me, if you integrate u is equal to integral of square root of 1 minus u square. Sorry, so du by square root of 1 minus u equal to, I miss dx here. So you have a dx, you have a dx, so you have a dx. So this is equal to dx.

Now you integrate both sides, you get what is this integration, integration of this plus C. This is, this integration we know, this is, this means sin inverse of u, u is y by x is equal to x plus C. So I got the general solution directly by integrating. So this way, so some equations you can make use of this, total derivative. So you can write, part of it you can write it as a total derivative, something or full equation you can do it so that you can integrate. So when you integrate, you can easily, direct, direct integration will give this.

Okay. So I will give you what is called exact equation, so we will define what exact is. So if you can find such a u function so that you are given, given ODE, you can put it in this form, du is equal to 0, then you say that the given differential equation is exact.

(Refer Slide Time: 21:46)



So let me define 1st what is exact equation, exact differential equation. So definition, an equation is said to be exact M dx, M x, y dx, I always write, now onwards any exact equation I will write the equation dy by dx is equal to F of x, y, I will put it in this form. M of x, y into dx plus N of x, y into dy is equal to 0 is exact, if there exists, if there exists, there exists a function u of x, y satisfying your x, y such that, I can write this left hand side as total derivative of that, M dx plus N dy is equal to du.

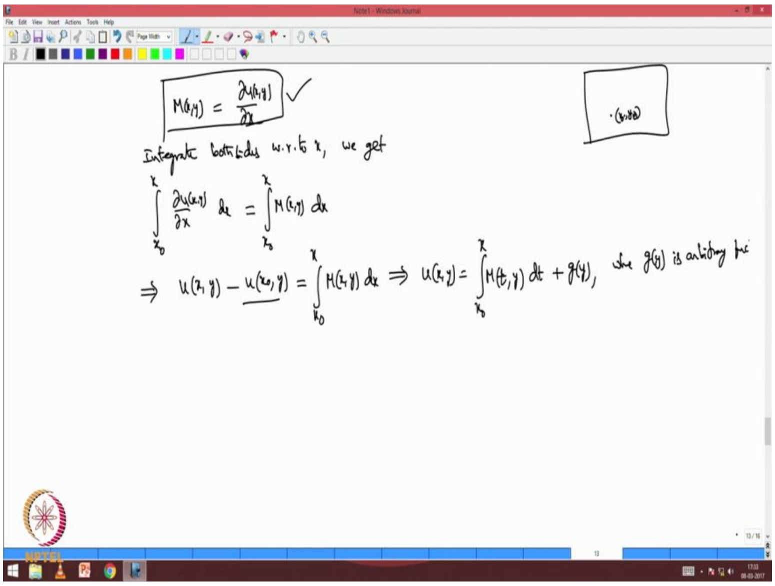
Okay. That means what is du, du is dow u by dow x dx plus dow u by dow y dy. You can easily compare both sides, you can get M as dow u by dow x and N as dow u by dow y. So this is what it is, the equation is said to be exact if these M and N, you can write like a partial derivatives of some function, function means x, 2 variables x and y. This is the definition to start with. Once you have this, we have result, this is the solution procedure, so, now we consider the equations, exact equation.

Suppose you have given equation is exact, that means this is satisfied, immediately what you can say? Dow M by dow y which is dow square u by dow y dow x which is equal to dow square U is equal to dow x dow y, assume that is u is satisfying this, u is this, these mix derivatives are continuous. This is equal to, what is this, dow, dow x of dow u by dow x, dow u by dow y is N, dow N by dow x. Okay.

If M dx plus N dy equal to 0 is exact implies, this is what you get. It implies dow M by dow y equal to dow M by dow x. Okay. If an equation is exact, you have this result. It so happens that if this is, this is, if this is true for the given differential equation, the equation will be exact. So simply converse is also true. Converse is also true. What is the converse, if dow M by dow y is equal to dow M by dow x, then the equation M dx plus N dy equal to 0 is an exact equation.

That means if this is true, then there exists some U, I should be able to get, right this is the definition. So you have a definition here. If it is an exact, so you have this is true. So I will be able to, so if something is exact, then from the definition I will be, then I can write M equal to dow U by dow x, N equal to dow U by dow y for some U of x, y. If this is true, if this is true, this should be true, that is what is the meaning of converse. So we have to show such a function exists, okay.

(Refer Slide Time: 26:53)



So assume that such a function exists, if it exists, it has to be M equal to dow U by dow x, you start with that. Okay. M equal to dow U by dow y, dow x, dow x. So what is M, so you can integrate both sides. If you integrate, so U is a function of x, y, M is a function of x, y, you integrate both sides with respect to x. With respect to x we get integral dow U by dow x of x, y into dx equal to integral M of x, y dx plus some arbitrary function of T, G of y.

Where G of y is an arbitrary function. Okay. A small digression here, so whenever you have, you are integrating both sides, when the domain is not specified, you are simply using indefinite integration, so that you are writing as a constant, if it is the only variable of x, you simply integrate plus integration constant C, because we have another variable, G of y is also constant in that sense.

You can also do, you can see you have, so that means once you have this equation, this is an equation, differential equation you are integrating. There is an underlying domain, so in that domain you pick up, you pick up a point. So new domain x, y, x0, y0, you can always pick up. Pick up x0, y0, normally you choose it as 0 to, in your problems if 0, 0 0 is involved, you can choose your point x0, y0 as 0, 0. Otherwise general x0, y0 which is in your domain, where the equation is defined.

So I use definite integration from x0 to x, if you do that, I do not have to write this arbitrary constant, right. If I do the integration both sides, definite integral, I did not write this arbitrary function, it will become naturally. So this implies U of, this is only dummy variable x, you can also write t dt so U of x, y minus U of x0, y is equal to x0 to x, M of x, y dx. You can see this part, this is function of y. x0 is fixed, so this is the arbitrary function, because U is the solution, U is the solution of this differential equation, U is, U is an arbitrary function. So when you have arbitrary function, this is the constant which you have, this you are calling as GY. Okay.

So this means U of x, y equal to integral x0 to x M of, I can choose this x as dummy variable t, y dt plus U of x0, y, x0 is fixed, so you call this arbitrary constant, so GY, where G is arbitrary function. Okay. So we have seen what is the exact equation and we have seen the converse part of it. So we have seen what is the exact equation and we have given the necessary condition when it is exact. And then to see the converse, if you check that condition, the same condition whether the equation is exact or not.

That means once you check that condition, dow M by dow y equal to dow N by dow x you want to, you want to see whether you can get a function U of x, y such that that makes derivative of that should be your differential equation du is equal to 0, du equal to 0 is your given differential equation. If we can do that, we can find such a U, it is called, then it is, that means the converse part is done. We will continue this in the next video.