Differential Equations for Engineers
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Lecture 25
Frobenius Method of Solutions

So in the last video we have seen seen the Frobenius method and we have explained we demonstrated with an example for the first case where you have the difference between roots of the indicial equation distinct and the difference is non integer. So we will see another example in which difference is a non zero non integer but they are complex roots, okay. So when they are complex roots if they are real they are to be distinct and the difference is non integer we have seen in the last example.

And if they are complex that we will see in the example obviously the difference will be imaginary which is non integer, okay. So we will try to solve the equation. So, x square y double dash minus x y dash plus 10 y equal to 0. If you have to solve this okay, so how do we solve this? So where is this? This is actually you have x is equal to 0 is a singular point so you have x positive or x negative. So if x is negative you have to work with solutions of the form y x equal to mod x power k sigma n is from 0 to infinity C n x power n.

So you have to look for solutions of this form when x is negative because when x is negative x power k does not make any sense, okay. So you have to write because k can be any real number, okay, k is not an integer so k can be any real number or can be complex number.

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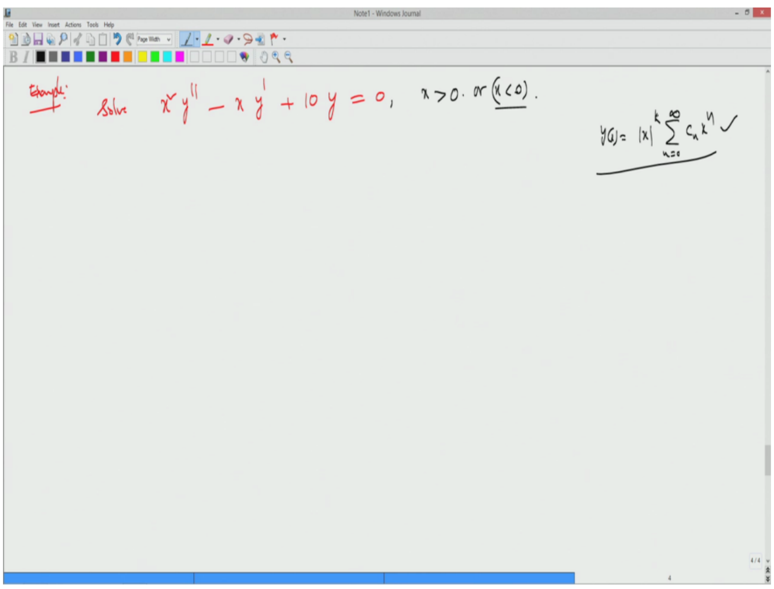

So you have to make sense this function. So you have to take x power. For example k is half, it has to be root of x and x is negative, root x does not make sense. You have to write root mod x. So that is why we look for solutions in the x negative case. But anyway we are working with x positive so we need not work with this mod x, okay. Otherwise so just for the completeness we are giving this, okay. So first see that x is equal to 0 is a singular point.

So you can see that is a singular point and you consider this limits since this limit x goes to 0 x times x minus x divided by x square. This is a 0, this is x square, a 1 is minus x. So this will go and this becomes minus 1 which is finite and other limit as x goes to 0 multiply x square with this a 2 divided by a 1 x square. So this is 10. This is also finite.

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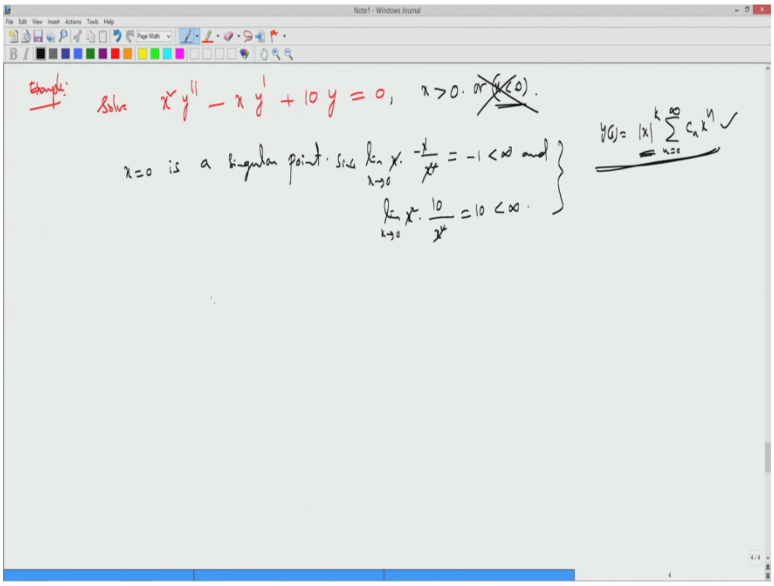

Because this is true 0 is a regular singular point. So given an equation how to check whether 0 is regular singular point or not, only when it is regular singular point it is of the form as of the generalisation of Euler Cauchy equation. So your equation is generalisation of Euler Cauchy equation. So you have to look for solution in the form of solutions of Euler Cauchy, generalisation for the solution of the Euler Cauchy equation. That is this Frobenius solutions, okay.

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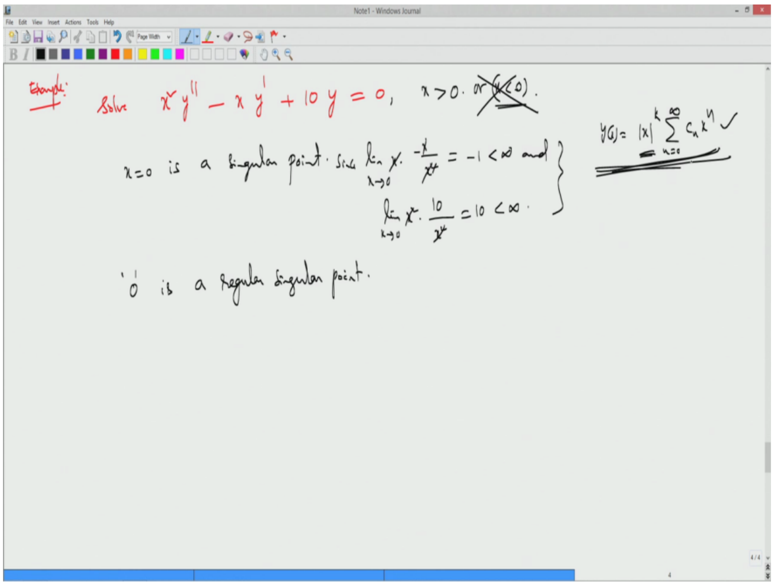

So you look for solutions in the form y of x equal to x power k sigma n is from 0 to infinity C n x power n. So if I am working with x negative so you have to take mod x power k. So, that you have to be careful. So this is the one.

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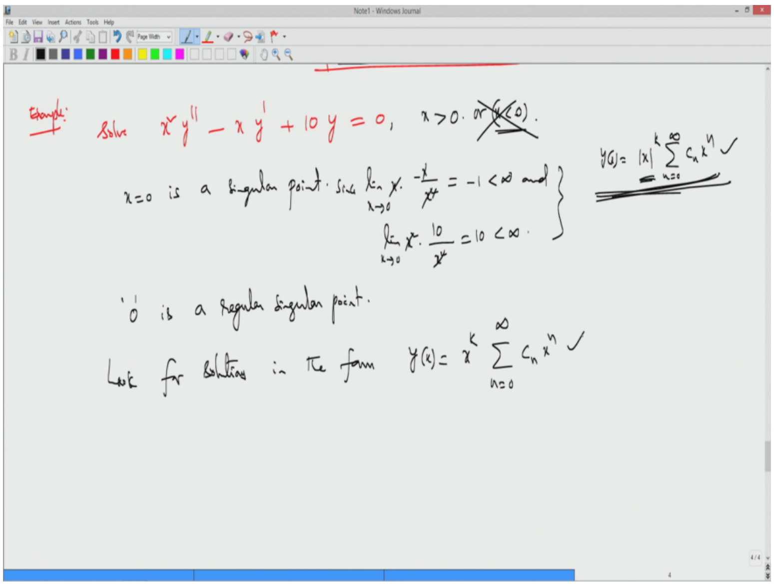

So you can now differentiate y dash of x on y double dash of x and if you substitute, I directly write this y double dash, okay, so you have x square times y double dash will be 0 to infinity n plus k and n plus k minus 1. So two times I am differentiating this.

Take this inside and then I differentiate term by term so you will get C n x power n plus k minus 2 that is your y double dash, okay, minus x times sigma n is from 0 to infinity n plus k times C n x power n plus k minus 1. You see because k is arbitrary, this index will not be changing, okay.

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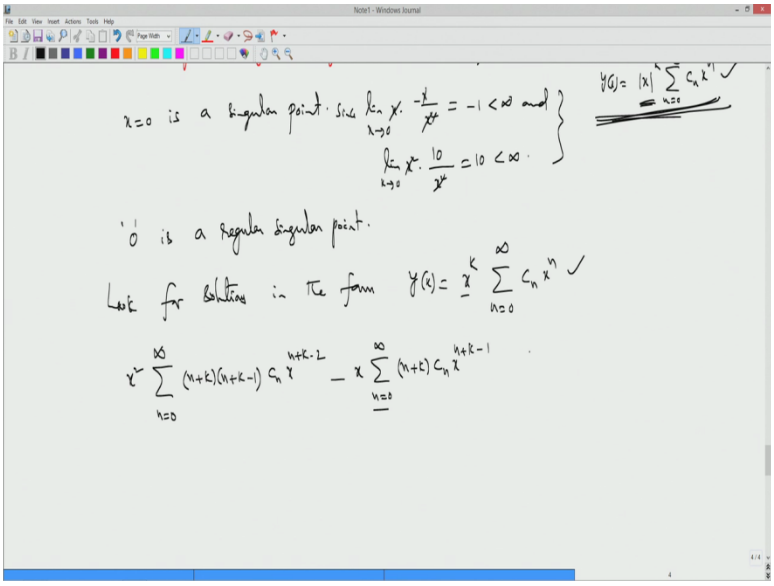

Then plus 10 sigma n is from 0 to infinity C n x power n plus k equal to 0. So you have a nice form, equation is so nice, n is from 0 to infinity, x square you can put it inside so that you have n plus k n plus k minus 1 into C n x power n plus k. Here also you have n is from 0 to infinity n plus k C n x power n plus k. So you take x inside minus 1 goes so plus 10 times, this is as it is.

So index is same, powers are also same in the equation. So you can simply you can simply write what is your indicial equation corresponding to always first one, n equal to 0, okay.

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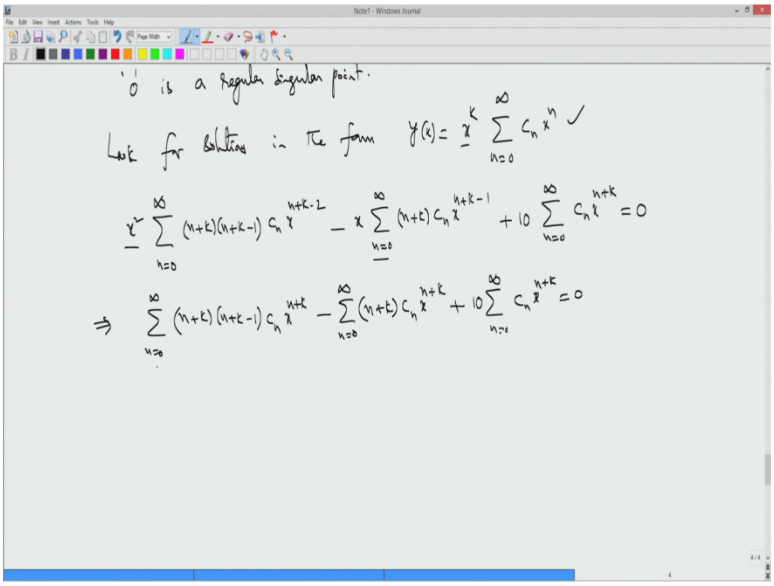

So actually because there is no isolated terms you can put it together as n equal to 0 to infinity, n plus k, n plus k minus 1 C n is common. So you have n plus k minus 2. So you have minus 2 here. So n plus k n plus k minus 1 minus n plus k, so n plus k is common, n plus k minus 1 minus 1. So that is what you have, okay. And then plus 10, this is the term for C n, right, into x power n plus k equals to 0. This is running from so this is what you have. So this we have power series which is equal to 0.

So put n equal to 0 that is the indicial equation. So whatever the starting one is indicial equation. If you have isolated terms that will also give indicial equation otherwise if no isolated terms so everywhere index is same, so you can have the first first when n equal to 0, first x coefficient that will be the indicial equation.

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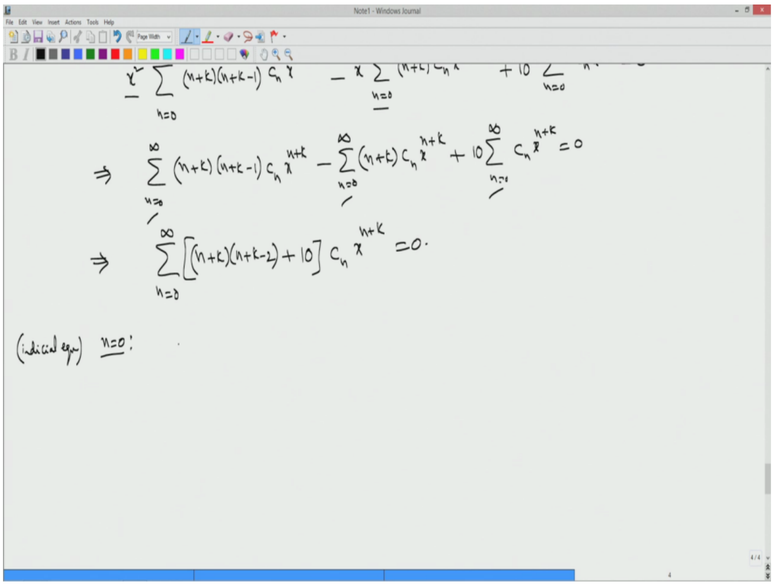

So you have indicial equation is n 0. So you have k into k minus 2 plus 10 equal to 0 into C 0 equal to 0, okay. But we have seen that C 0 cannot be 0. So you have to look for this that C 0 never be 0. So that is how you are looking for solution. So C 0 cannot be 0 implies k into k minus 2 plus 10 equal to 0. So this will give me k square minus 2 k plus 10 equal to 0. So you have k as minus b plus or minus square root of b square minus 4 a c, 4 a c 40 divided by 2.

So that is 1 plus or minus 36 you have, okay, it is 36 minus 36 so you have 3 6 that means 6 by 2 that is 3. So these are the roots, okay. So because they are complex you cannot say which one is bigger so k 1 is you can take anything you want, i 3 and k 2 as 1 minus 3 i, okay. So these are the two roots k 1 and k 2.

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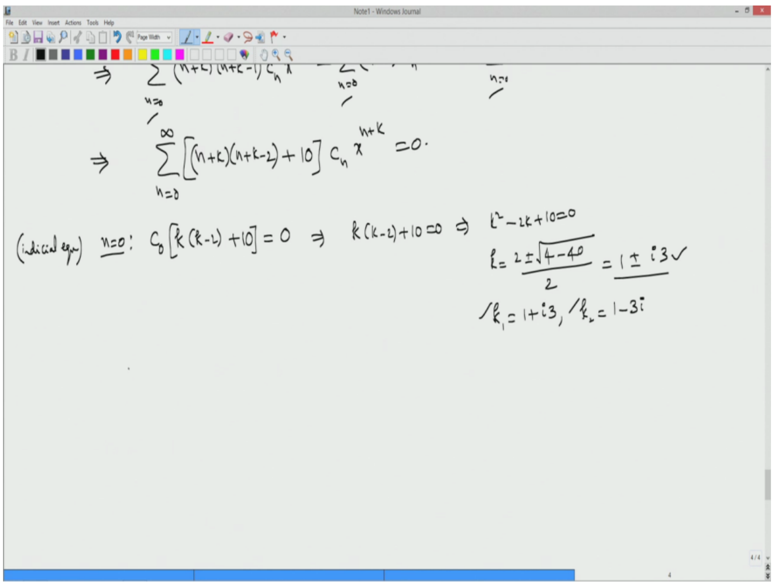

That is what you got when n equal to 0. N equal to 1 you get k plus 1 and k minus 1, n equal to 1. So you have this plus 10 into C 1 equal to 0 because now what are your k values? This is this. K is either 1 plus 3 i or 1 minus 3 i. If you put it here 1 plus 3 i is 2 plus 3 i into 3 i plus 10 that can never be 0. This cannot be 0, okay, for those k 1 and k 2 values, okay. That means C 1 has to be 0.

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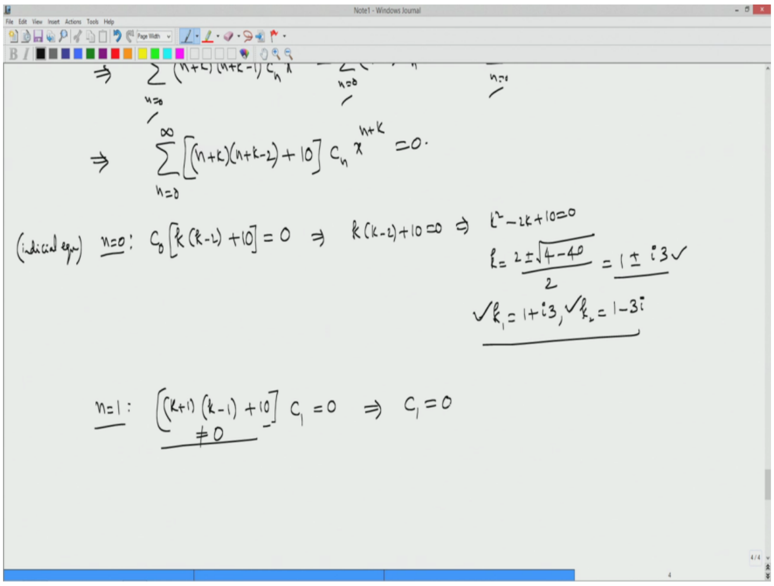

N equal to 2 or in general k equal to 2 and so on, you always see that k plus 2, right? So n equal to 2 is k plus 2 into k plus 10 into C 2 equal to 0. Again for these two k 1 k 2 values this shall never be 0. So implies C 2 equal to 0. Similarly like this you can go on and you see that all C ns are 0 for any n and n equal k, C k will be 0, okay, for any k 3, 4 and so on. This is what happens.

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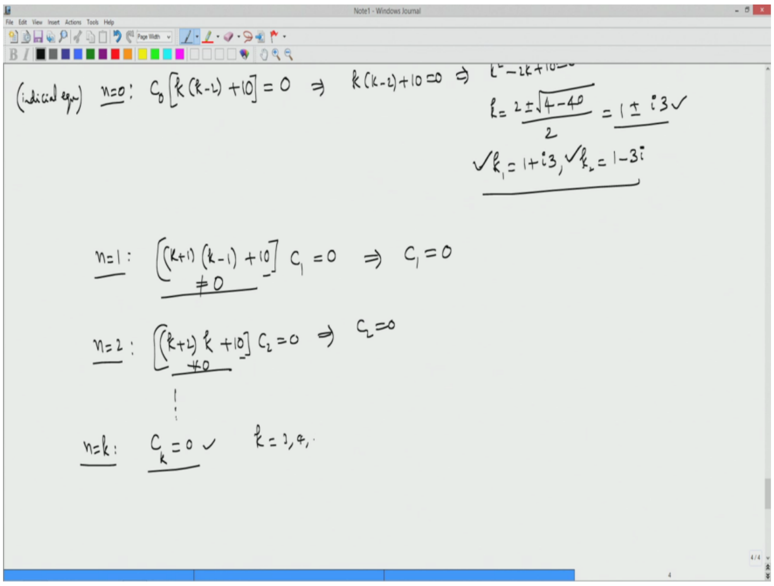

So what happens you know put your k 1 you will get k 1 as 1 plus 3 i, choose then your y 1 x is you have the form of y x at k equal to 1 plus 3 i. So what is this 1 plus 3 i? X power k, k is 1 plus 3 i . C 0 is non zero. C 1, C 2, C 3 are 0. So you have a constant C 0. So that is one solution.

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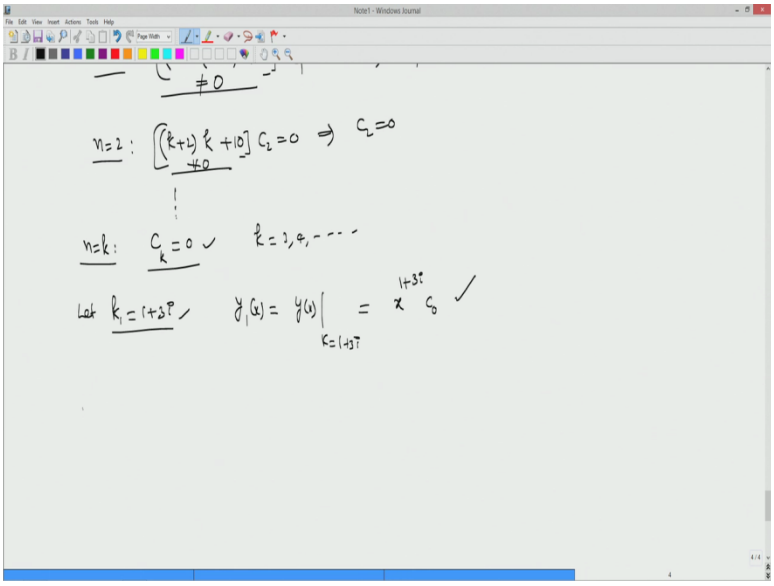

What is other solution? So k 1, 1 minus 3 i if you choose this y 2 of x is y x and you put k equal to 1 minus 3 i. So again x power 1 minus 3 i into C 0. Both the cases C 1, C 2 they are constants. Whatever maybe it does not depend on k. C 1 C 2 all C n are 0. So you have these are the two linear independent solutions. What are these x power complex numbers? It is not really clear. So what you have what you do is you have seen already x power k, any x power k we can write let us say this is some L.

L is this. We want L, okay. So your log L thus will be k log x. Assume that x positive so when you do log x makes sense, okay. So what happens? L is actually now take the e power, this is actually log x power k. So when you take both sides exponential L will be e power log x power k. So that is actually x power k, okay. So you can write like this in this fashion, okay.

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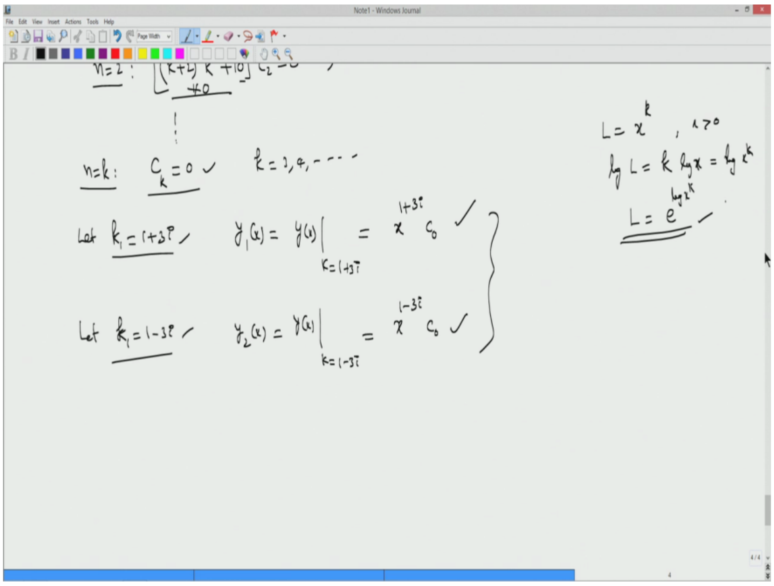

So if you do that k equal to 1 plus 3 i. So when L equal to x power 1 plus 3 i so log L is 1 plus 3 i into log x, right? So this is equal to log x plus 3 i. So log x plus log x power 3 i. Now simply you can have L. L is e power log x into e power 3 i log x. So what is this one e power log x into e power i log x cube, right, log x cube. So you have seen that when L is like this, this is L as this form. So you can see that e power log x into, so this is one solution, okay. When I take minus, you work with minus, then we have minus.

So you have plus or minus, okay, in both the cases. So this is a two linearly independent solutions. That is what we have, right? And C 0 equal to 1 both cases both time you ignore C 0. So these are the two linearly independent solutions. So if you see that e power log x into e power i log x cube, both the cases, e minus log x cube, e power plus log x cube, both are two linearly independent solutions. So it is like e power i x and e power minus i x are two linearly independent solutions.

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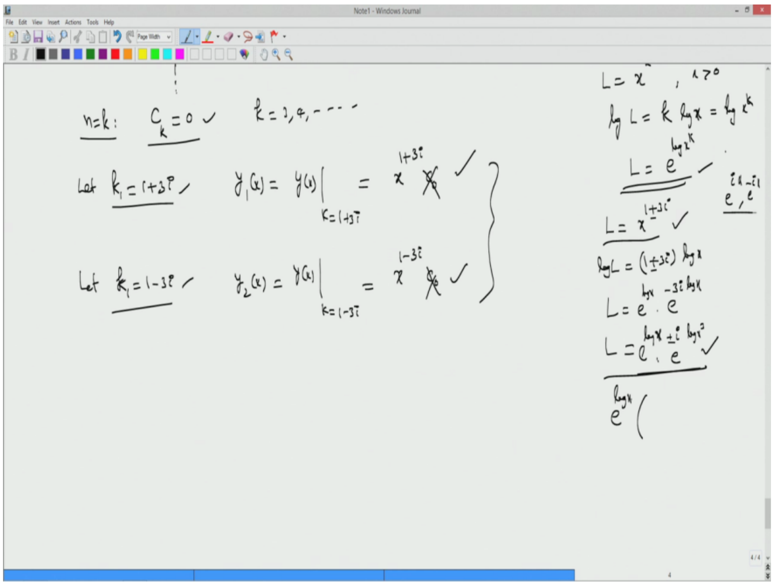

What equal into? You can get summon difference. What is the summon difference? Cos x and sin x are two linearly independent solution, okay. That is also you can rewrite as e power log x, this is anyway common, cos log x cube is one solution, e power log x sin log x cube is another linearly independent solution. So either plus or minus you take as a complex solution.

As a real solution if you write these are two linearly independent real solutions, okay. So if you write like this the general solution is as a complex form C 1 times x power 1 plus 3 i plus C 2 times x power 1 minus 3 i. These are complex solutions. It does not make any sense in practice, okay.

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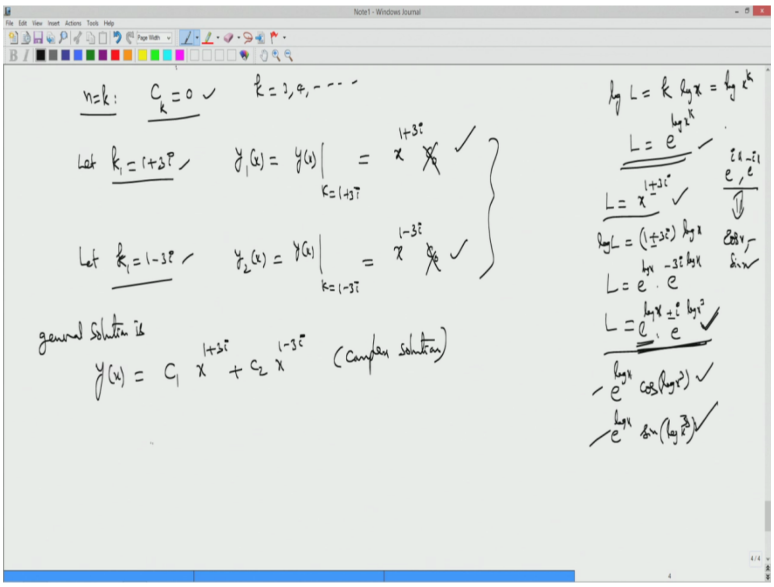

So you have to write it is a real solution or y x real solution if you write C 1 log e power x. So (co) log x is common, okay, e power log x is actually x, right? So this is actually x. So this is x. This is simply x. So you have x times, x is common anyway, so x times C 1, x power 3 i is cos log x cube. X power 3 i and x power minus 3 i linear combination you can rewrite as some new linear combination d 1, okay. Summon difference if you can write so they are also linearly independent just like here, okay.

So d 1 plus d 2 sin log x cube. So these are two linearly independent solutions x cos log x cube and x sin log x cube. So their linear combination will give you the general solution. So this is your general solution of real solution. So these are your real solutions, okay. So this is your general solution what you are looking for.

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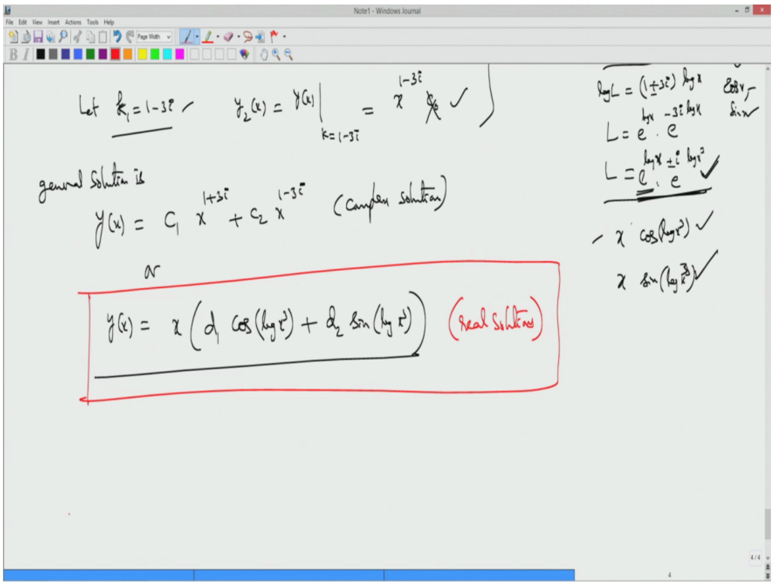

You can also write like this, only problem is this is a complex solution. It does not make any sense in reality. So we will demonstrate with another example where the indicial equation give you two different roots but the difference is integer value. If the positive integer value what happens? So you will be able to get only one solution.

Second solution if you can get it, if you actually work out with the general equation eventually you will see at the end that it will take certain form. That form I will directly give and workout, okay. Just have to remember the form in the case 2 and case 3, okay. Let us work out with the case 2. The difference is non zero but integer. So this is another example, example 3.

So example 3 is solve this x square y double dash plus x into 2 plus x y dash minus 2 y equal to 0, 0 is a singular point, either 0 or x is less than 0, okay, whichever is the domain. So if you do this so 0 is a singular point, okay. Solution is 0 is a singular point, 0 is a regular singular point. Regular singular point because x square is 0 at x is equal to 0. So 0 is regular singular point. Regular singular because these two limits are finite, x goes to 0 x times a 1 divided by a 0.

So that is 2 plus x divided by x. That is going to be 2 which is finite and limit x goes to 0, x square minus 2 divided by x square. So that will be minus 2 which is also finite.

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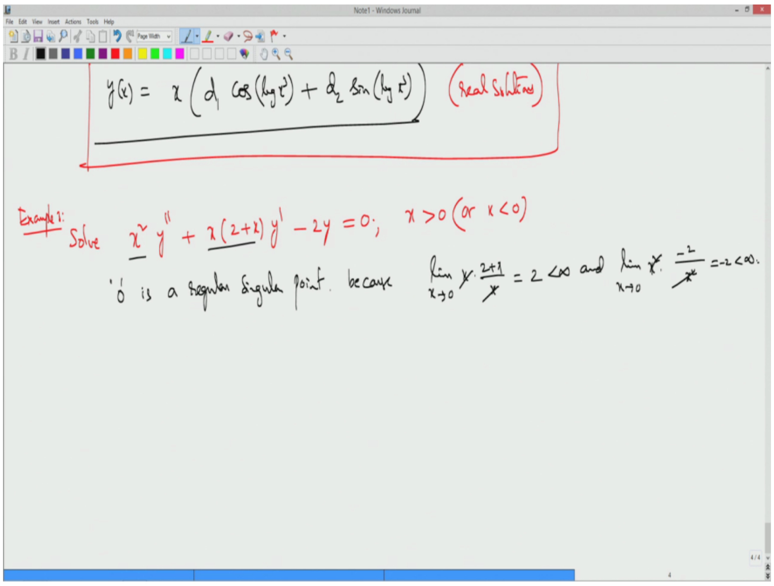

And then always you choose these two limits, okay. So you can directly summon at this step here itself. At this step itself you can find indicial equation by writing k into k minus 1 plus this first 2 k, this is your limit, first limit 2 k minus q 0 that is other limit. So minus 2 equal to 0. So you can see that k square plus k minus 2 equal to 0. So what are the roots? So now minus 1 plus or minus square root of b square plus 8 divided by 2. So you have minus half plus or minus 3 by 2.

So you will get 1, other one is minus 2. So these are the two roots, okay. So always see k into k minus 1, this is common and this first limit value put it here into k and second limit value you put plus second limit value. That is equal to 0 that will be the polynomial equation that will have roots. That is one way.

That is another way of getting directly indicial equation. Even after substituting you get the same thing. You will see that, okay. So 0 is a regular singular point so you look for solutions in the form y of x equal to x power k as usual, okay, n is from 0 to infinity C n x power n with always C 0 is without loss of generality. C 0 is never be 0.

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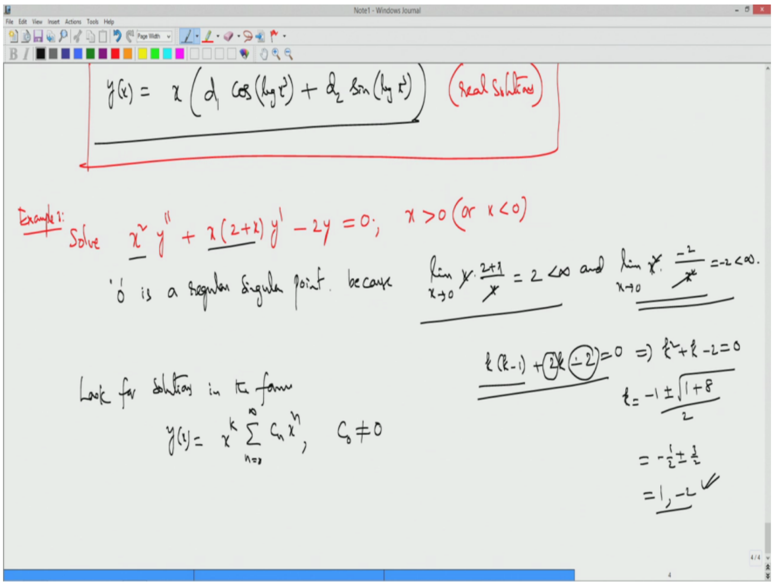

So substitute the equation so you will get x square y double dash. X square y double dash will be I am directly writing n is from 0 to infinity k plus n, k plus n plus 1 n minus 1 C n x power n plus k minus 2 and you are multiplying with x square so that minus 2 will go. Plus 2 x, okay, 2 x plus x square. So this you have to be careful. So you have 2 x that means 2 x times y dash I write n is from 0 to infinity, n plus k times C n x power n plus k minus 1, I will multiply with 2 x. So you have 2 x minus 1 will go.

So this is what your first term. 2 x y dash I wrote then plus x square y dash. That is n is from 0 to infinity n plus k C n x power n plus k minus 1. So if I multiply with x square this becomes plus 2, right? Plus 1, okay, plus one. So minus 1 into x square. So 2 minus 1 is plus 1. So this 1 minus 2 times n is from 0 to infinity C n x power n plus k into 2. So 2 is there so you have this is 0. So this is what is the equation and so substitute all the solution y, y dash, y double dash into the equations.

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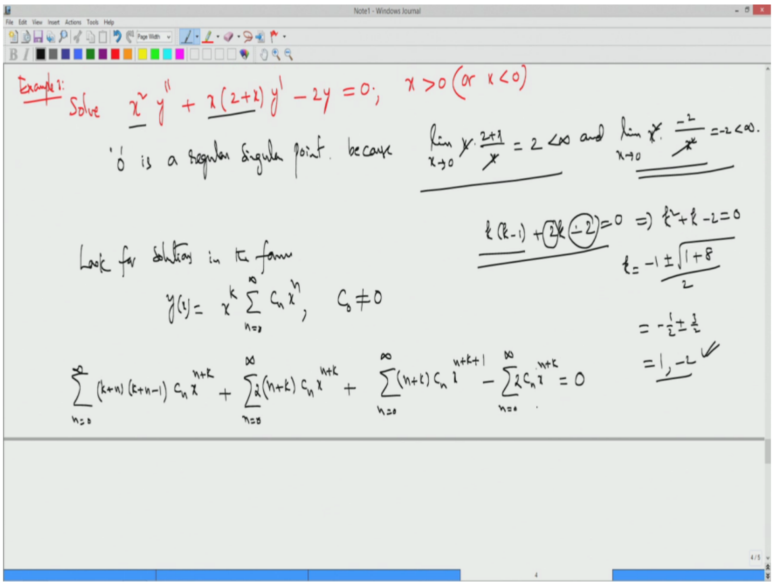

Now what happens so this n is running from 0 to infinity all the times except this. This you rewrite, so if you rewrite, change this sum. You convert this into x power n plus k by replacing n equal to n minus 1. N minus 1 equal to 0 means n is from 1 to infinity, n minus 1 plus k C n minus 1. That is what it is. Other sums you write as it is 2 C n x power n plus k equal to 0.

Now here this side also n is from 0 to infinity, n plus k, n plus k minus 1 C n x power n plus k plus sigma n is from 0 to infinity 2 times n plus k C n x power n plus k. This is what you have.

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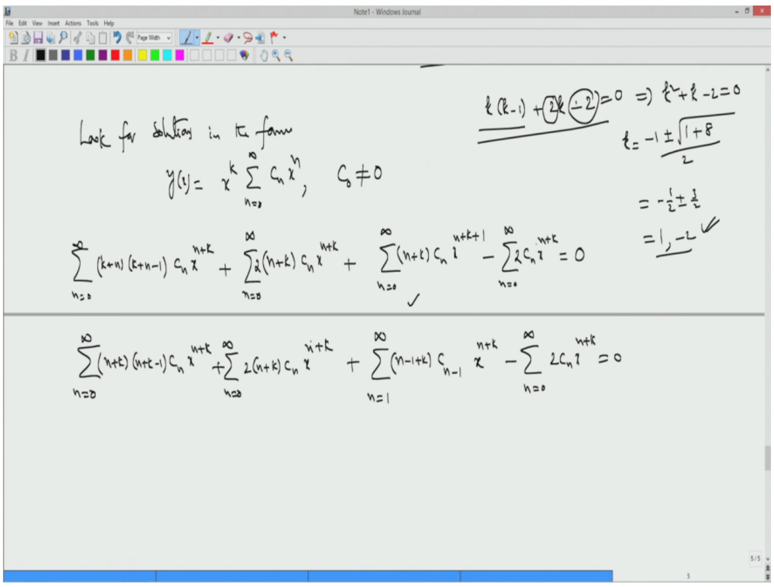

So this now everything all ports are same. You can sum it together so you get n plus k, m plus k is common here in both the cases, so you have n plus k into, n plus k into n plus k minus 1 plus 2 into n plus k, n plus k is common. So you have n plus k minus 1 plus 2 so that means plus 1 into C n x power n plus k, okay. And then you add this one here.

So this one plus n minus 1 or n plus k minus 1 into and then okay so you have C n, C n you put it together so you have minus 2 here. Minus 2 so this C n okay this C n. Now what is left is C n minus 1. So that is n plus k minus 1 C n minus 1. This is with x power n plus k equal to 0, sigma n is from 0 to infinity. So this is what you have.

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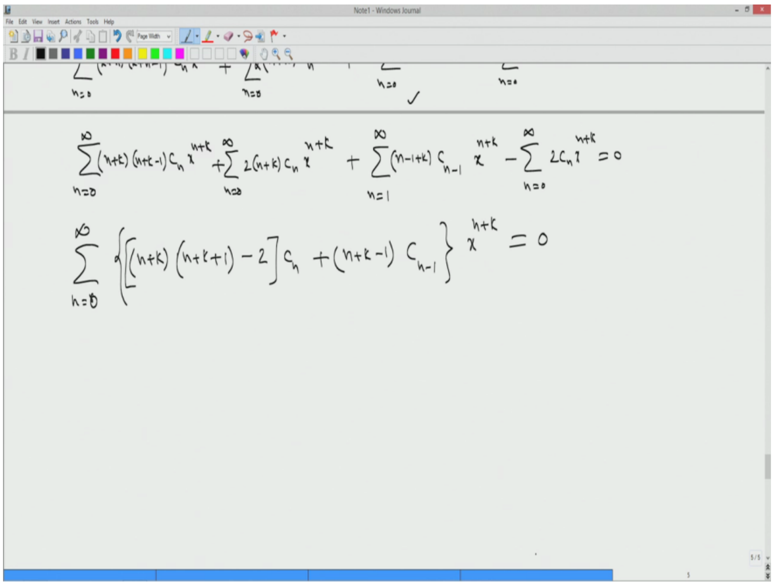

If you actually put C n together so this is what you get with the power. So now put n equal to 0. So now this is a power which is 0. Now n equal to 0 coefficient of x power k, 0. Coefficient of x power k is 0. That is what gives so n equal to 0 means coefficient of x power 0 that you have to make it 0. So this is a recurrence relation. So obviously you have a recurrence relation. First you write recurrence relations.

So recurrence relation is n plus k times n plus k plus 1 minus 2 C n plus n plus k minus 1 C n minus 1 equal to 0, n is running from 0, 1, 2, 3 onwards because this is power series which is 0 power series. All the coefficients should be 0. That is what is the recurrence relation. So you have n equal to 0. Now look at the n equal to 0 is nothing but x k into k plus 1 minus 2, okay, sorry I made a mistake so this you cannot combine. So this is this is actually n equal to 0 or isolated here, n equal to 1 only common, okay.

And then n equal to 0 parts you write separately, n equal to 0 is k into k minus 1, okay, into C 0 plus 2 times k. And here minus 2 k C 0 and you put okay, here minus 2 C 0 into x power k equal to 0. So this part is 0 separately and this part is 0 separate.

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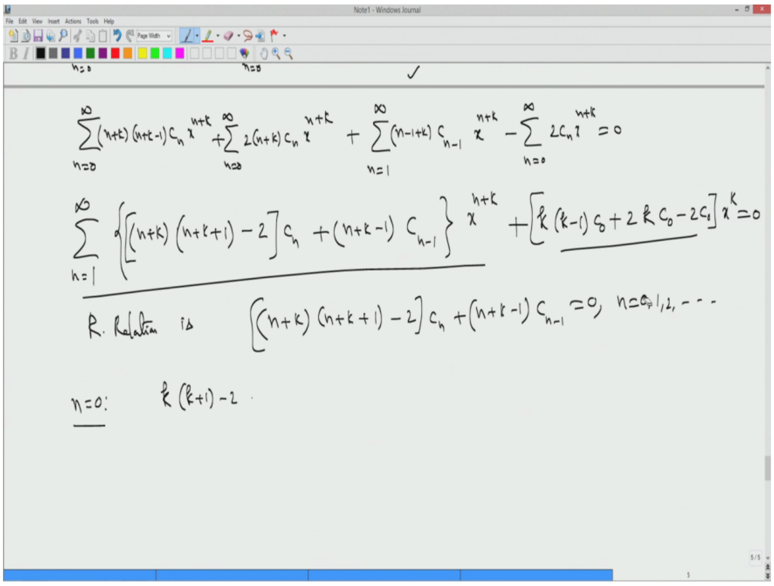

So n is running from 1, 2 onwards is the recurrence relation. This is indicial equation so indicial equation is simply k into, because C 0 cannot be 0, so you have k into k minus 1 plus 2 k minus 2 equal to 0. this is what exactly you seen earlier. So k square plus k minus 2, okay. So you have roots. What are the roots? (())(25:47) minus 2. So the bigger root is 1, k 1 equal to 1, k 2 is minus 2, okay.

So you are with the case 2. So by looking at the indicial equation you have k 1 minus k 2 which is bigger root is 1 minus of minus 2, this is 3 which is non zero and a positive integer. This is case 2 okay. This is with case 2. So we are with the case 2.

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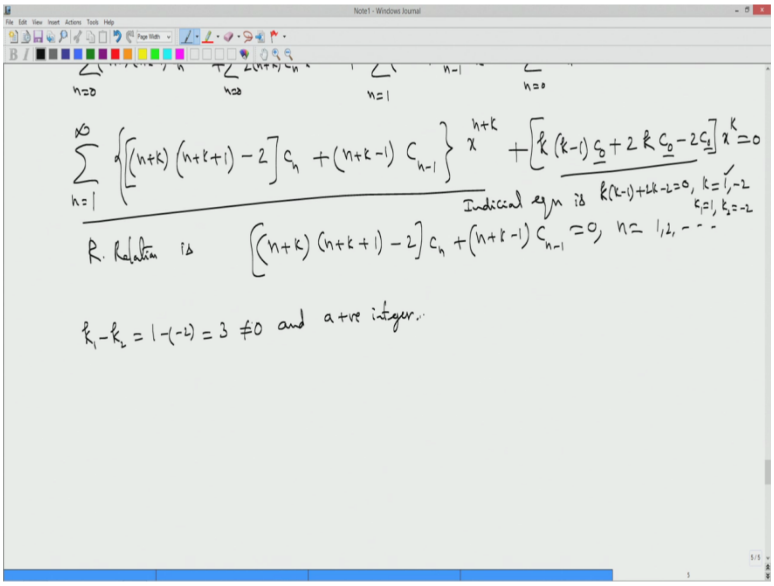

So in this case what happens? The bigger root will always give you a solution, okay. So you do not have a problem. You will be able to get all your C n from this recurrence relation. You will be get all your C n if you put k equal to k 1, okay. Put k equal to k 1 bigger root get all unknown constants, unknown C n. It will give you y 1 of x. So this is the algorithm. So you get solution y 1 by putting bigger root.

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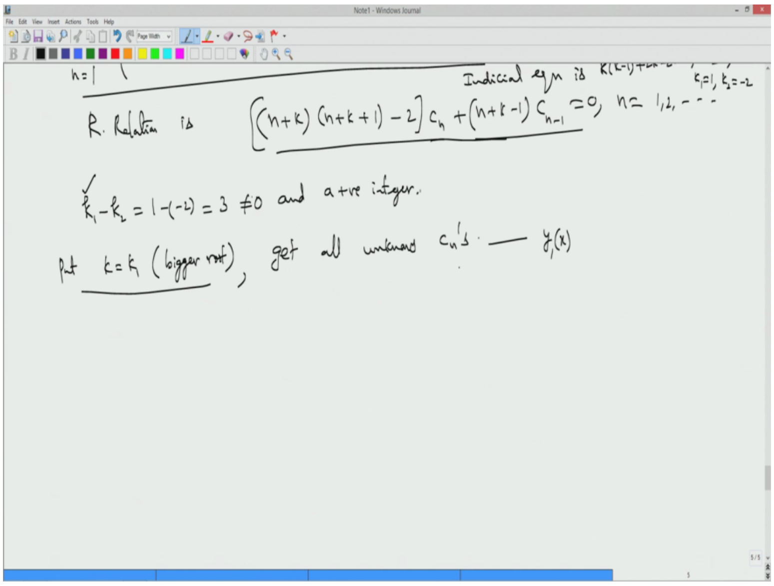

But for smaller root especially in this case you should look for in a different way. So let us first get this first solution. What happens when put n equal to 0 and k equal to k 1? So what is k equal to? K 1 is 1 so you have n equal to 0 is 1 plus, n equal to 0 means k plus 1. So that is 2 minus 2 times C 0 plus n equal to 0. So n equal to 0 is, sorry n equal to 1. So you have to put n equal to 1. So n equal to 1 because it is a running from n equal to 1 onwards. So n equal to 1 means k equal to 1.

So you have 2 times n equal to 1, k equal to 1, okay. Put k equal to 1 that is k 1. So and then into this recurrence relation. If you put this recurrence relation becomes n plus 1, n plus 2 minus 2 . So this is what you have into C n plus n because k equal to 1, n C n minus 1 equal to 0. So this is what is running from n 1 2 3 onwards.

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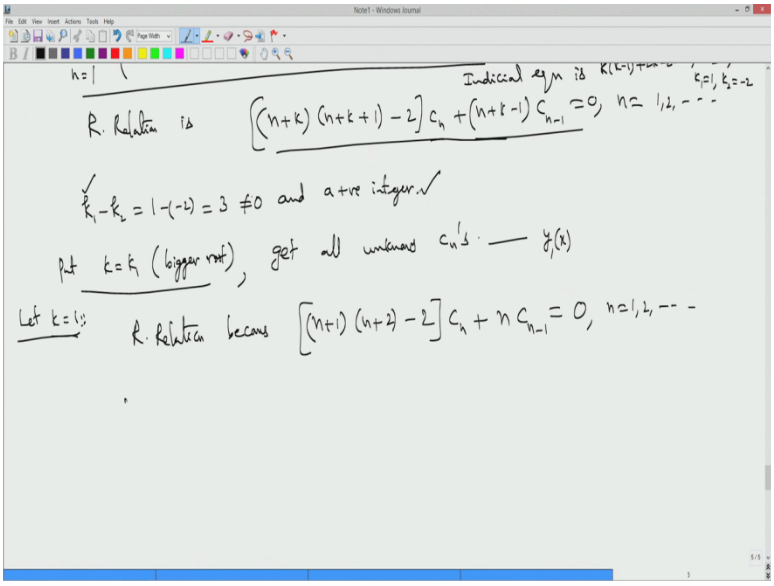

So now put n equal to 0 here into this recurrence relation so you will get n equal to 0, you get 2 minus 2. So you have sorry n equal to 1 here, n equal to 1 you get 2 3 6 minus 2, okay. 4 C 1 plus n is 1 C 0 so C 0 equal to 0. So it will give me C 1 is minus C 0 by 4. So n equal to 2. So in this case n equal to 2 will give me 3 4 12 minus 2 that is 10 C 2 plus n equal to 2, 2 C 1 equal to 0.

So that will give me C 2 equal to minus C 1 divided by 5 that is plus C 0 divided by 4 5. N equal to 3 give what is this 4 into 5 20, 20 minus 2 so that is 18. So 18 C 3 plus 3 C 2 equal to 0. So this will give me C 3 is minus C 2 divided by 9, okay, 6 sorry 6, 3 6, okay. So C 3 is minus C 2 by 6 that is equal to minus C 0 by 4 5 6 and so on you can go on getting it for different n values.

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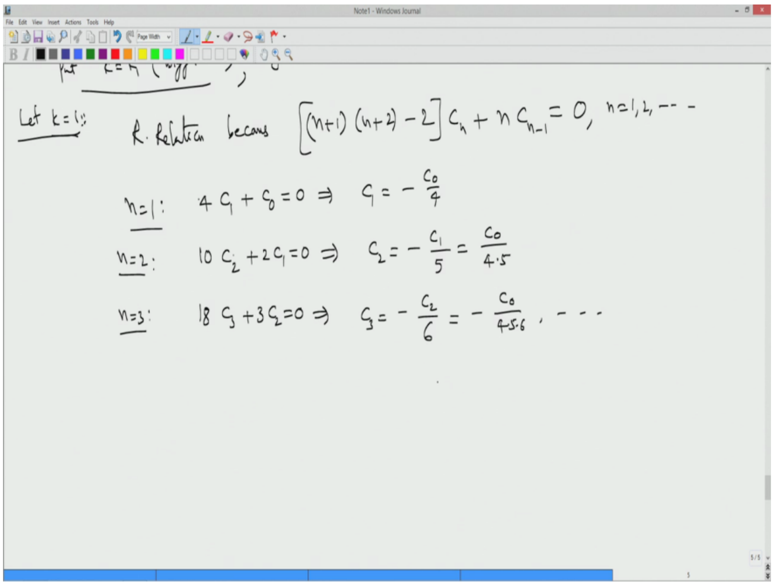

Now we know what is k, what is your C n so go and substitute your y 1 solution. So first solution is y at x when you put k equal to bigger root which is 1 which is x power k equal to bigger root 1 and the summation n is from 0 to infinity. So I will write this as an expression. So you have C 0 which I could not find, C 1 which is C 0 by 4 x plus C 2 that is C 0 by 4 5 x square minus C 0 by that is C 3, C 3 is minus C 0 4 5 6 x cube and so on. So this is what you get.

So this is nothing but C 1 is common and you have x minus x square by 4 plus x cube by 4 5 minus x cube by 4 5 6 and so on, x power 4, okay. So this is what is the solution. So you can consider this as 1 that will be taken as 1. So if you take it as 1 this is your first solution. So you can get one non zero solution always in the case of root 2 difference is non zero but it is an integer.

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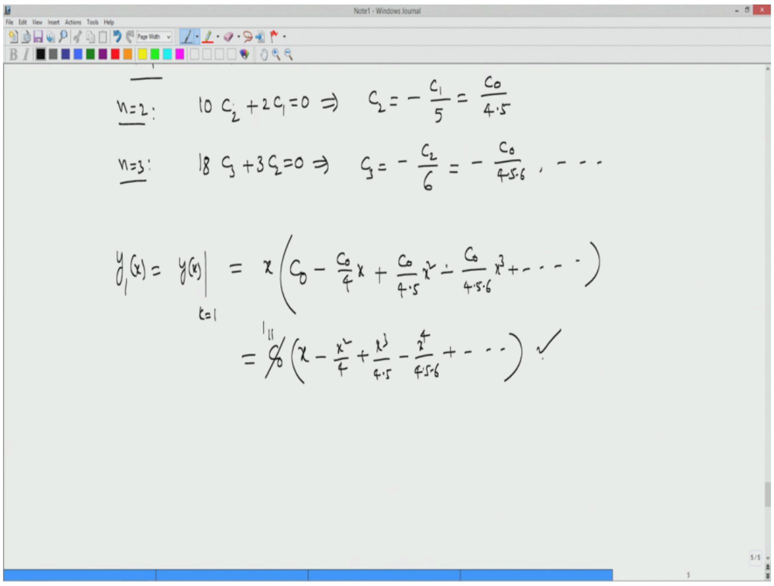

So to get the other solution when it is an integer case you may have trouble. If you simply substitute k equal to minus 2 into the recurrence relation and try to get your C values you may end up getting somewhere you will get something like 0 by 0 form. You will not be able to find, okay. If you actually work with the general equation you deal with the 0 by 0 form and finally you end up, you will see that 0 by 0 form wherever you get a 0 by 0 form that is like kind of, 0 by 0 form means indefinite form, okay.

That means it can be any number. That is like arbitrary constant. So if you work out finally you can see that the second solution, okay, let k equal to k 2 which is minus 2. The second solution is of the form y 2 of x which is equal to, I will give you the second form when the root difference is a positive integer. So that is some arbitrary constant A log x because x is positive. If x is negative A log mod x you take okay into y 1 x. So you write y 1 you already calculated y 1 x into log x, x is positive.

If x is negative, is a domain, log mod x you take plus x power or mod x power, if x is negative, x power smaller value k 2 that is minus 2. So it is always of the form so k 2. You have is n from 0 to infinity, some d n x power n. So this is the form you should remember. So this is what you get. The second solution is always of this form, okay, that is A y 1 x log x plus x power minus 2 sigma n is from 0 to infinity d n x power n.

So this is the form you should look. If you look for in this form because I worked with the general equation and I see that second solution final will be in this form. I will be able to get these constants, okay, that is why I know (())(33:53) this is the form that works.

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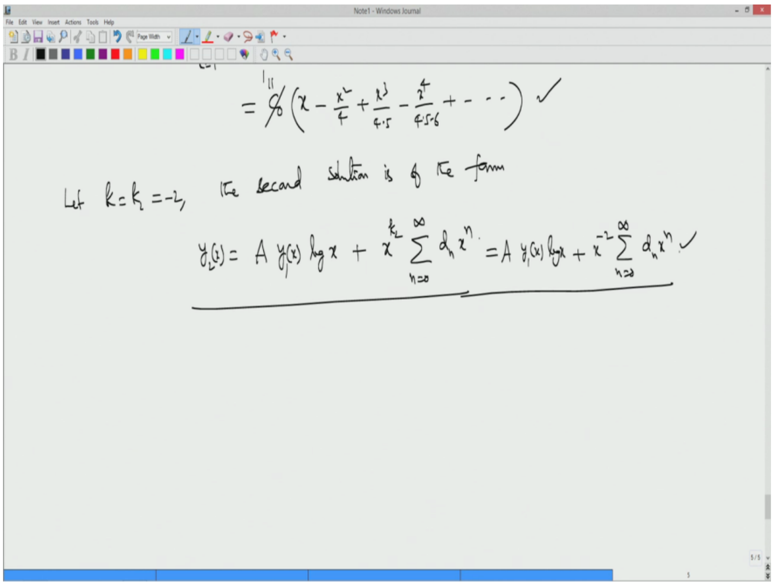

So if you can remember just work out problems, okay, and then so if you look at the indicial equation roots and the difference is non zero but it is a positive integer. So this is a case 2 you can always get one solution for bigger root. For smaller solution you can always look for the second solution but if you put it in this form rather than the other earlier solution form, if you look for in this form you will be able to get your constants and you see that it will be different from your y 1 and it is linearly independent from y 1.

So that you have two linearly independent solutions, okay. So you write your solution as y 2, y 2 in this form. So you look for solutions in this form, okay, with your y 1 is involved, some arbitrary constant. You have to calculate now A and d n. These are arbitrary constants. A and d n are constants to be found, okay.

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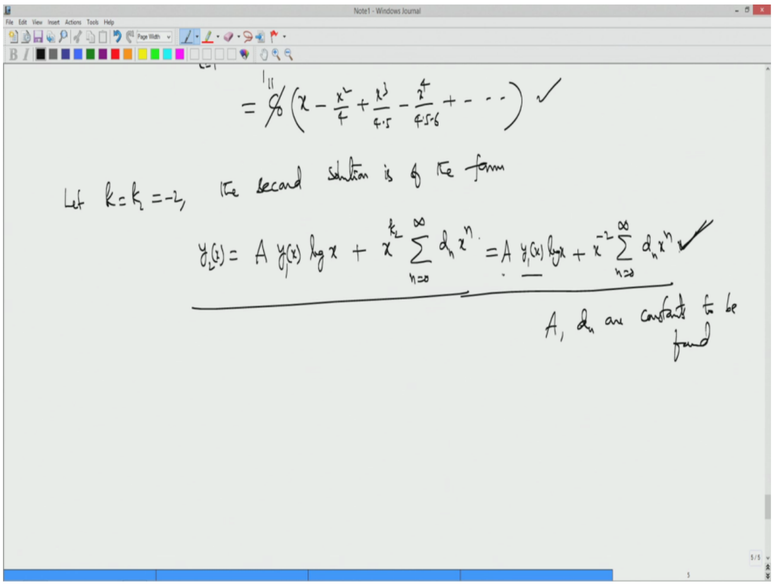

So when the roots of indicial equation when you construct k 1 and k 2, when k 1 is bigger root and k 2 is smaller root and the difference is non zero but it is an integer we can always find one solution. A second solution we should look for special form that you can remember so that we will do in the next video. So we will try to find the second linearly independent solution in the next video and write the general solution in terms of y 1 and y 2, okay. So we will see that in the next video.