**Differential Equations for Engineers**

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**Lecture 24**

# Power Series Solutions around a Regular Singular Point

So in this video we will explain Frobenius method to solve second order linear homogeneous equation when 0 is a singular point that is regular singular point, okay. When 0 is a regular singular point the equation is much similar to Euler Cauchy equation. So we will just take the equation. So how it works, a 0 of x y double dash plus a 1 of x y dash plus a 2 of x y equal to 0. So this is the equation, 0 is a singular point.

So that means with a 0 of x equal to 0 at x is equal to 0, okay. We assume that 0 is a singular point. So once you divide it this is equal into y double dash plus a 1 x by a 0 y dash plus a 2 x by a 0 of x y equal to 0, okay. So because at x equal to 0 you divide it so you have problem. This has a singularity, this has a singularity. So this is defined either x is positive or x is negative but not at x equal to 0, okay. And this is your function p and this is your function q.

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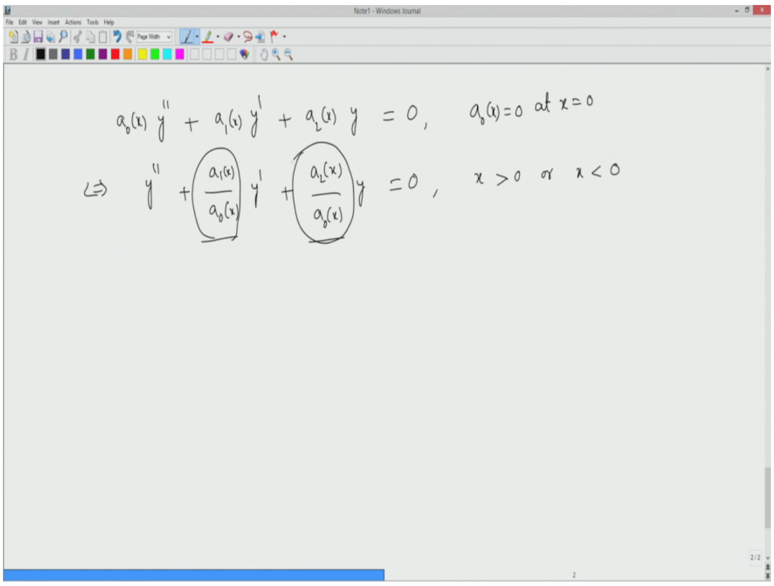


So you have y double dash plus p x y dash plus q x y equal to 0. So this is your q x and this is your p x. So p x is having singularity at x is equal to 0, q x is having singularity at x equal to 0. So both are not defined at x equal to 0. So the equal into equation y double dash plus p x y dash plus q x y equal to 0 which is defined as a positive or negative, 0 it is not defined but this is like p x is having a power series representation which is maximum allowed to have 1 by x singularity.

So this is maximum is like something like order of 1 by x. So this should be order of 1 by x that is same as saying so p x is having some power series some constant times C minus 1 by x plus C 0 plus C 1 x and so on. So that is the representation for p x. Similarly q x which is of order of 1 by x square is maximum. You are allowing it to of order 1 by x square.

So, that means if you write q x as a power series and this will become C minus 2 by x square. Some, okay, some it is not same C, you can say some d, d minus 2 by x square, d minus 1 by x plus d 0 plus d 1 x and so on. So, this is what is the representation for x positive or x negative.

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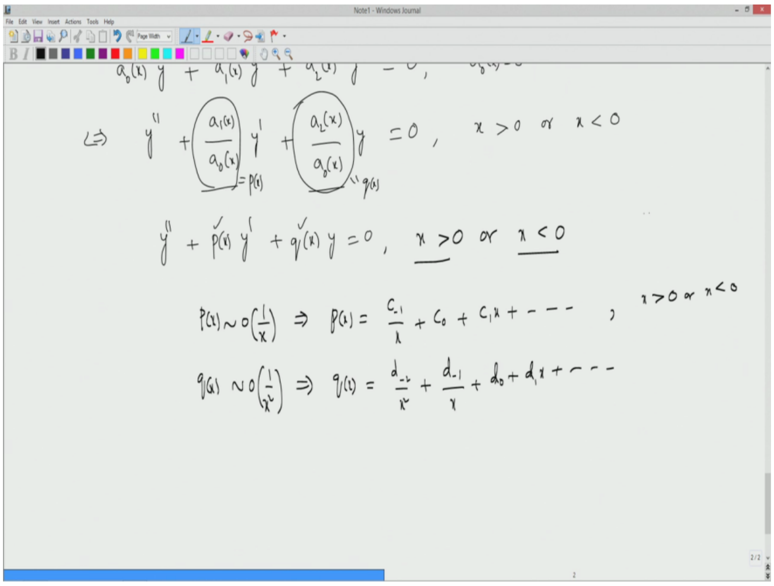


So assume that there is no other singularity. If you have a singularity so clearly 0 is the singular point. If you have a singular point somewhere here or somewhere here if you have and the minimum distance either from here 0 to this point or this point, whichever is the minimum, call this some distance d 1 and d 2, okay, some amount. So alpha 1 is the distance, alpha 2 is the distance here between this, okay.

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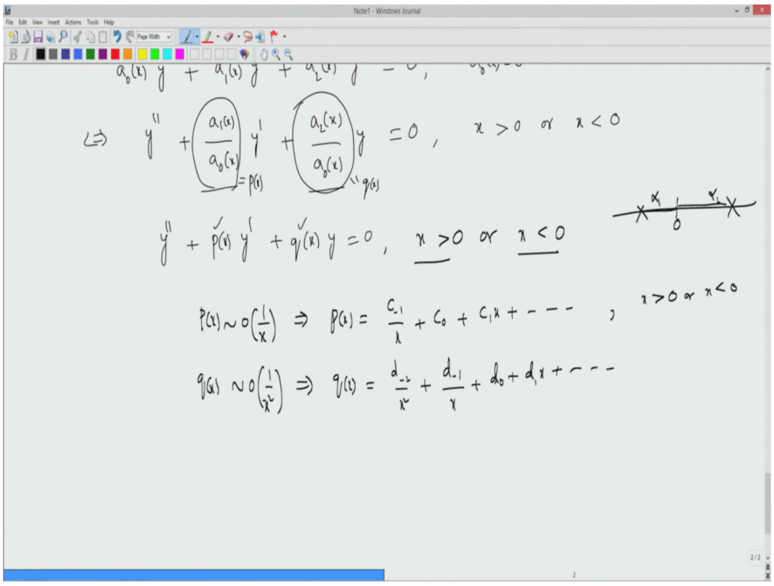


So whichever is the minimum, minimum of alpha 1 alpha 2 you call it alpha. So that is where it is defined. So alpha is 0 between some mod x less than alpha. So if you have a singular point at some point alpha 1 alpha 2 than the minimum of alpha 1 alpha 2 that is where this is defined. If there is no other singularity you simply can have mod x is, no you should write 0 to less than mod x less than alpha. So within this, okay.

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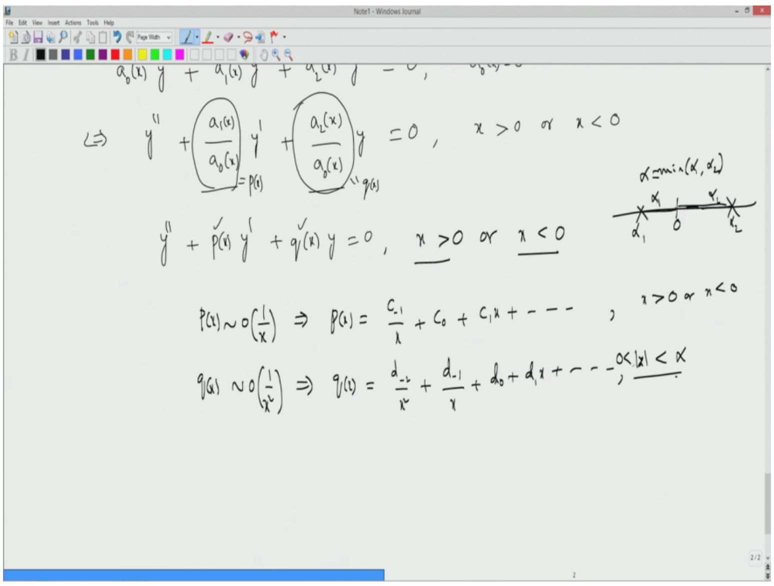


So we will say something like x is between it is minus alpha 1. So alpha 1 is minus alpha 1 so you have alpha 1 alpha 2 minimum. So this is positive side alpha 2 which is positive. So you have a minus alpha less than 0 or 0 less than x and less than alpha. So when alpha there is no other singularity. Alpha 1 alpha 2 are infinity so alpha is the minimum is infinity.

So you have x positive x negative. So that is how it is defined. So when you have this p q, when you have this order, when p and q are having this power series representation maximum they are allowed to have only 1 by x order, this is 1 by x square. Then we can apply this Frobenius method because assume that p x you can see that p x will become C 1.

Assume that C minus 1 is 1 by x or constant by x, all these terms are 0. Similarly all these terms are 0. Assume that such a p and q. Then what you have is y double dash plus C minus 1 by x y dash plus d minus 2 by x square y equal to 0. So this is exactly the Cauchy Euler equation.

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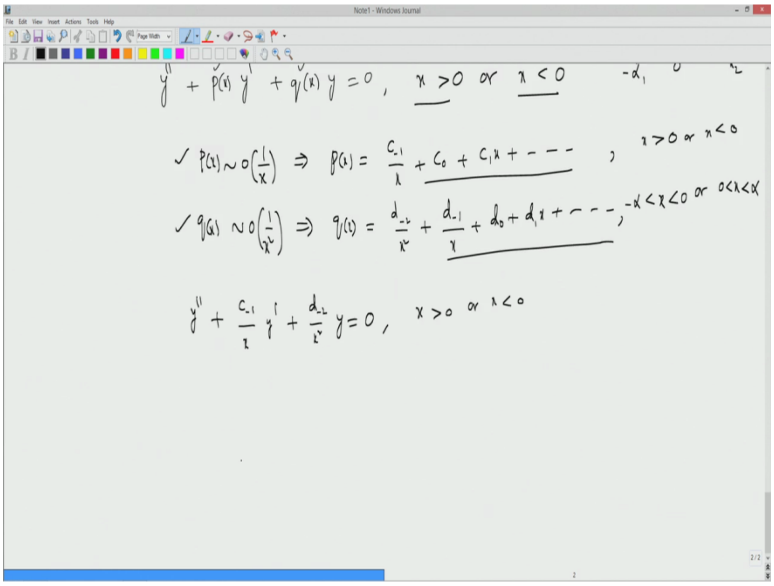


So what are the solutions? So you know that the solutions of Euler Cauchy, this is actually Euler Cauchy equation which we have studied already, okay. So solutions of the form we have already seen by converting x into z. So new independent variable z if you bring in by the transmission z equal to log x so what you have is e power some k z are the solutions, okay. If you use this transmission the equation becomes equation with constant coefficients.

Then you look for solutions of the form y z as in this form so that will give you y x as z equal to log x if you put this is simply x power k. So k is some constant so k can be even complex number. So you do not know exactly what. So k is a scalar. So you have solutions of this form x power k for this Euler Cauchy equation, okay. K is a scalar. Now this is all you had seen.

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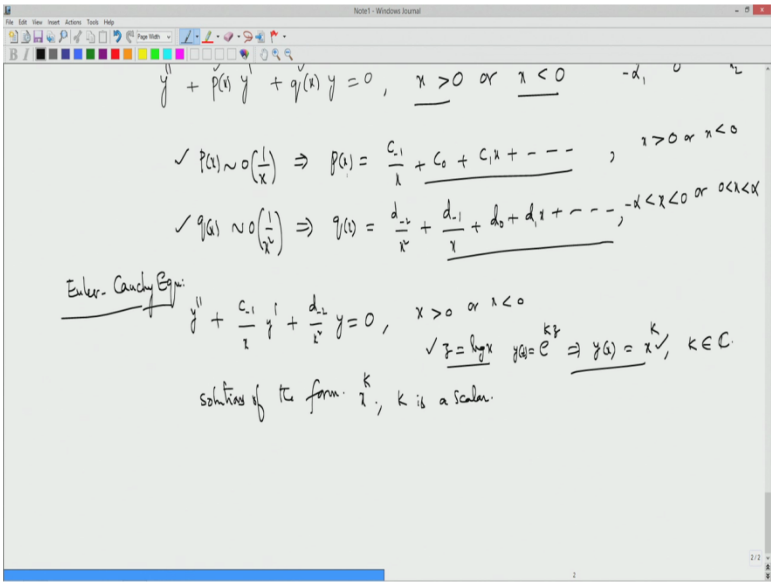


But if you have p x and q x is also having this representation remaining. So it is not really 1 by x, 1 by x square then it is like you are bringing out the singularity. So you have x power k, k can be even negative, k can be anything, okay. So you bring this kind of solutions you bring out. So you have a singular point but it is regular that means you maximum allow p x is of this type, q x is of this type.

So what you do is instead of looking for the solutions of the given equation as a power series which is n is from 0 to infinity C n x power n, okay. Instead of this you can look for this fashion but you multiply with x power some or let us say some k. So same k I can use, some k, okay.

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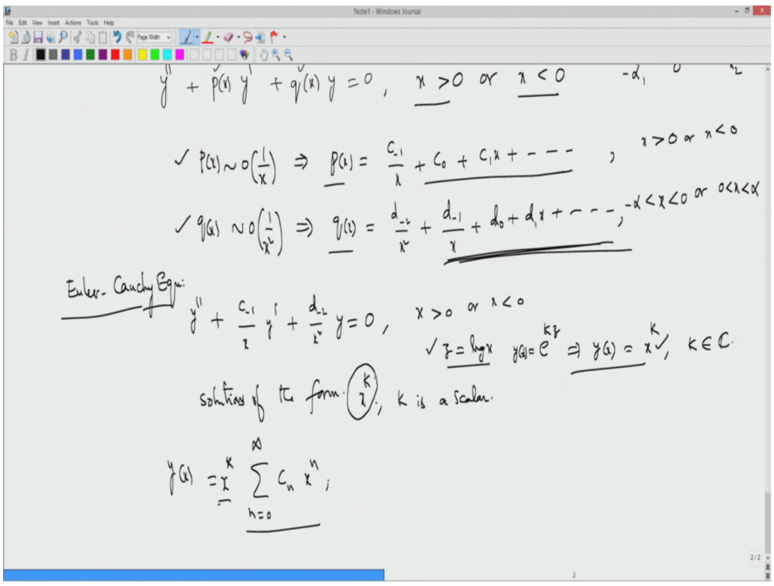

So I brought this singularity part outside x power k. So you can see that this if I look for solutions of this form, this power series is actually finite at every point, even at x equal to 0. Only part is x power k. If k is negative and you have a singular point k will always be some kind of negative. So you do not know sometimes. Even if you have a singular point you may have some k power positive k, you may have solutions, okay.

So this equation when you look for x power k sigma C n x power n, what you look for ? So in the Frobenius method you look for solutions of this form, okay. Why? Because there is a singularity that singular part as k is unknown, okay, where k is unknown scalar and C ns are unknowns as usual, okay, unknown scalars.

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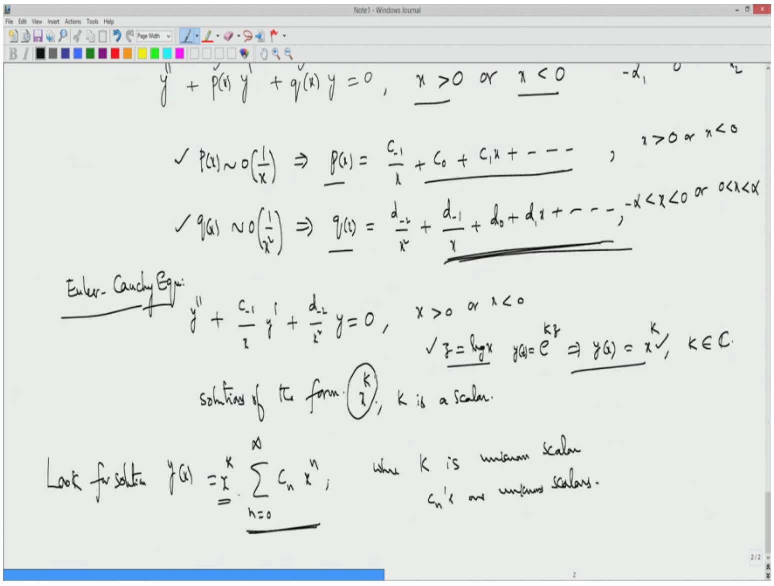


So what we did is so we brought out this singular part as x power k and then you multiply this as a power series solution. So this is on par with the Euler Cauchy equation. There x power k some constant is a that is how you look for a solution, okay, for the homogeneous equation. So because this works. Actually if you take this form it works. That is called Frobenius method. We will demonstrate it with an example.

So what we do is we always look for solutions of this form whenever you have an equation this one with p and q of this form. So maximum when you have a singular part. So when you have x equal to singular point you have seen already when it is regular x into p x as limit x goes to 0, this limit will be finite. As you can see if I multiply x into p x and what you have is C minus 1 and C 0 x plus C 1 x and so on, as limit x goes to 0 this will be simply C minus 1, okay, it is actually C minus 1.

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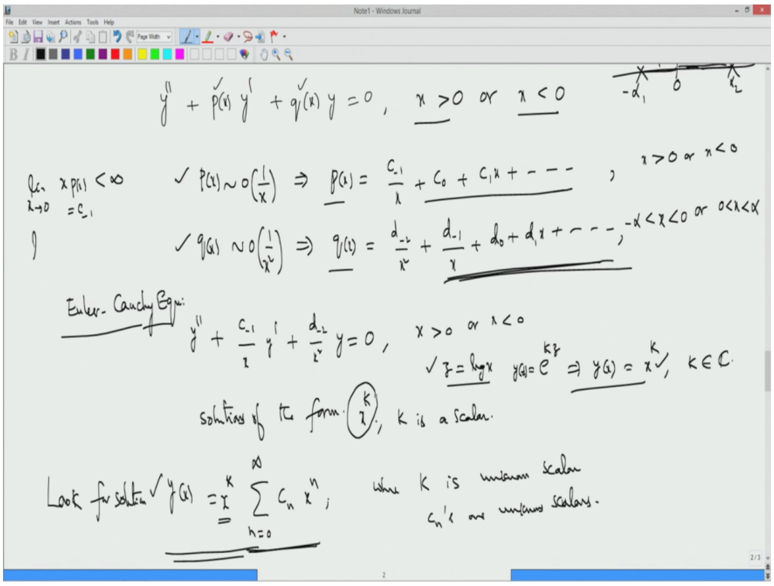


Similarly when you have this one x square q x this is actually d minus 2 which is also finite, okay. So you have x square, you multiply to q x what is left is simply d minus 2 plus x d minus 1 plus d 0 x square and so on. So as x goes to 0 this is simply d minus 2. So this is actually regular singular point. When you have a regular singular point as 0 so this is what is the definition I gave that is actually equal into saying p and q are having this one, okay.

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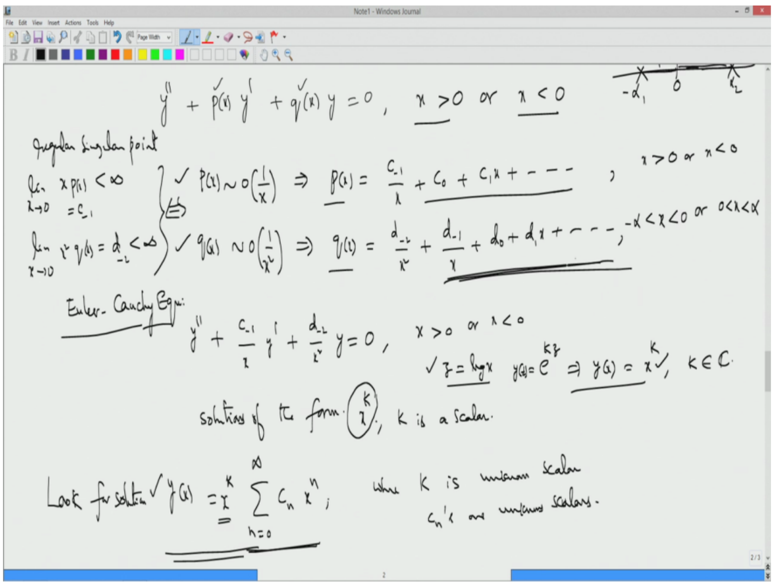


In such a case you can look for all this solutions like this. This is a Frobenius method. I will give you both the solutions, two linearly independent solutions you can get. So I will demonstrate you will have three cases, in each case first of all you will find what are the values of k for which you have the solutions.

You will immediately find, you look for solutions in this power series and you substitute the equation you will find what is your k and then you will be able to find C ns, remaining constants in a power series based on this k. C n is based on depending on this k value you will have different C ns, okay. That will determine two linearly independent solutions. So based on the roots of this k 1 and k 2 so you will get two values for k. If they are repeated that means they are same.

If they are not repeated, if they are distinct roots but the difference is integer or the difference is non integer, these are the three cases you will see, okay. So we will have three cases. So let us first substitute, once you find what is that k we will see those three cases. Substitute into this equation. So we will start with. So what we will do is that we substitute. So I start with an example, okay. So we will understand those three cases in each example. Each case you will see in each of these examples, okay.

I will start with an example in which case you will able to find k values. You will find two k values for which those two k values a different, they are distinct and their difference is non integer, okay. so actually k satisfy some polynomial equation of second order. So that is actually will give you two roots. Those two roots if they are real, if they are distinct, the difference is non integer. That is the case you are looking at, okay.

And when they are complex obviously the difference is non integer because it is when the two complex conjugate roots when you take the difference it is actually becomes imaginary part which is non integer, okay. So these two cases we can cover in this example, in this type first of all. So I start with this first example, the difference is non integer, roots are real, okay. So start with an example, I will give you an example.

So solve 2 x y dash plus x plus 1 y dash plus y equal to 0. If you want to solve this you can easily see at x equal to 0, 2 x is 0. So this is defined there is no other singular point. This is only one. This is a polynomial.

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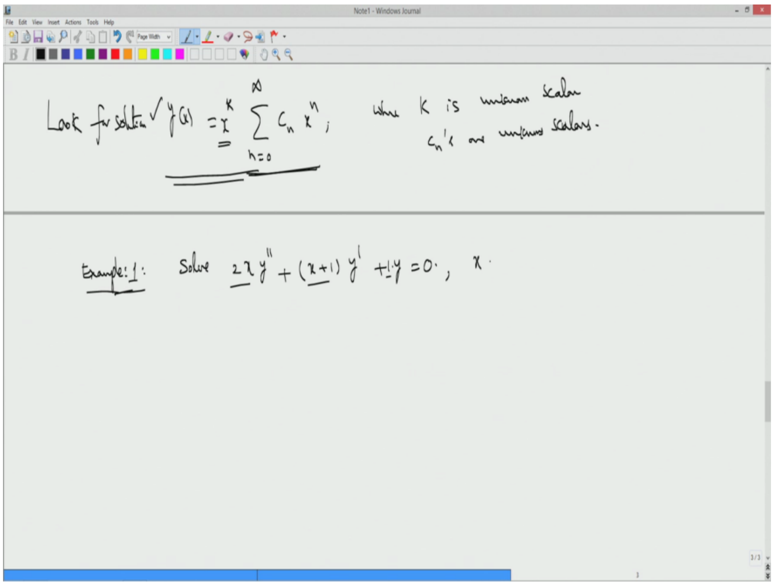


So x is equal to 0 is the only singular point. So you have either x positive or x negative, you can look for solutions, okay. So in each case if you take this one that is one equation in this domain. If you want to work here you have to work differently. So take the same equation but x is less than 0, okay. So either this or this. So let us work with the positive, okay. So you look for because 0 is a singular point.

So 0 is a singular point first of all, 0 is a singular point and because x plus 1 divided by a 2 divided by a 1, take this limit x goes to 0 and multiply with x. What happens? X x goes, you will have half which is finite, okay. And now you look at a 2 divided by a 0, 1 by 2 x multiply with x square, take this limit, this becomes simply 0 which is finite. So this implies since these two limits are finite 0 is a regular singular point. Immediately if it is a regular I can nicely extract the singular part outside okay in the method.

So I look for a solution in the form y x equal to x power k, k is unknown and into some power series, n is from 0 to infinity C n x power n, okay.

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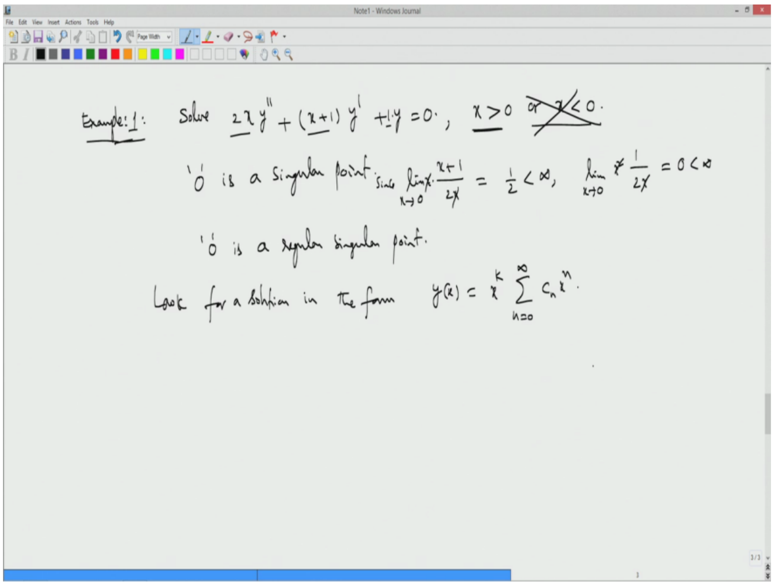


So your job is just to substitute this y, y dash, y double dash. You calculate and put it here, okay. So earlier we have seen that power series properties we have seen. Power series is actually when it is a power series and you have seen that it is actually converging uniformly so you can differentiate term by term, okay, you can differentiate term by term so you can calculate y dash as term by term differentiation and substitute into the equation here.

Here also you can actually show the way if you look for solution in this form, you can actually substitute, finally you see that this power series solution once you find is constant C n and this k, once you see that this power series is actually converging uniformly and absolutely.

So 0 is a singular point and if there is no singular point, x is positive, and actually this power series is actually converging uniformly and absolutely in this and any closed interval within this positive. So that means this power series is actually converging at any closed interval here, okay.

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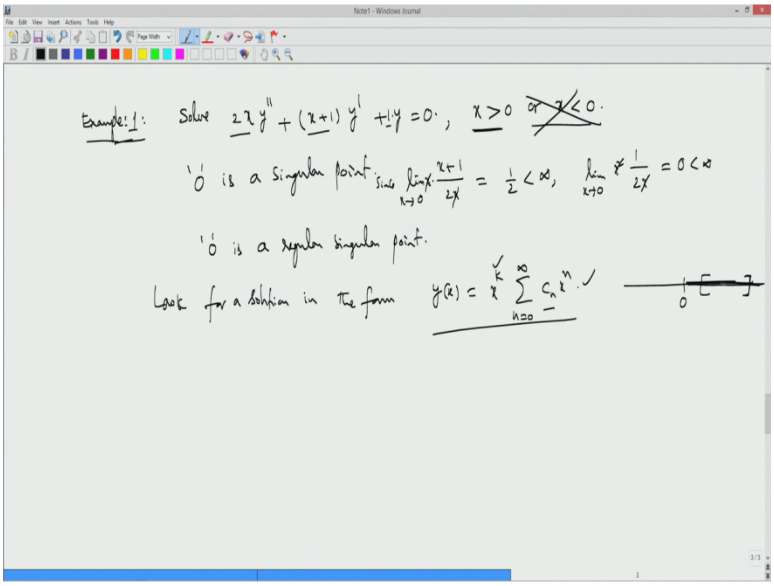


That means you take any point you can cover with this closed interval, okay. So and x positive actually this series is convergent so because it is uniformly convergent on any closed interval on the x positive part, right side of 0, so you can actually differentiate term by term in this interval, okay. If your point is here, you include that in a closed interval by including in a bigger closer interval, okay.

So that way this series one can prove that is actually converging uniformly and absolutely up to the next singular point, okay. If I do not have any other singular point x 0, any wherever is so. Otherwise 0 is a singular point. Suppose alpha 1 is a singular point so here within this any closed interval it is absolutely and uniformly converging. So this is open so 0 to alpha 1 is open interval. So in any closed interval in this series one can show that this converging uniformly and absolutely. So you can do term by term differentiation, okay.

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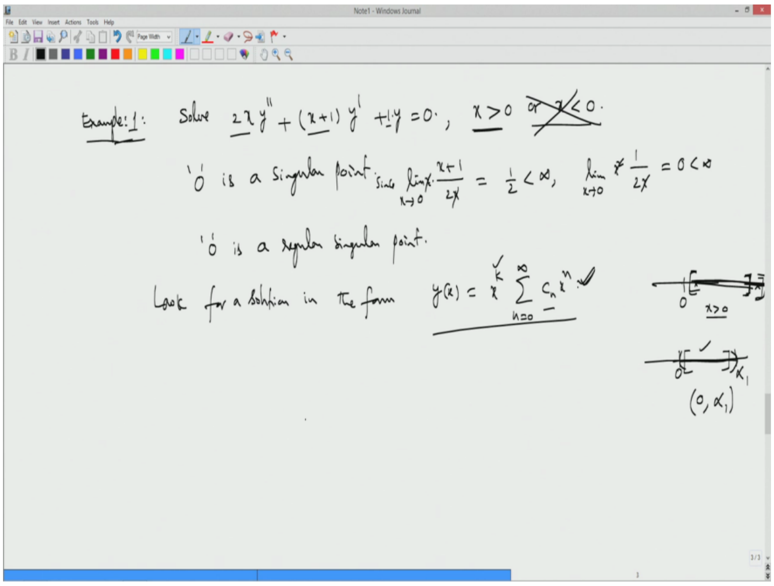


So having assumed that part you can differentiate this y dash of x as a term by term. So this you can write it like x k you can put it and you can write n plus k. So if you do that k is not an integer, okay. K is not an integer or an actual number. So when you differentiate everything will remain. So this index will never be 0 unless, it is not like when k equal to 0 when usual power series when you differentiate the power series that constant part will go, n is running from only from 1 to infinity after differentiation but here it is not so.

So you have n plus k C n x power n plus k minus 1. Simply differentiate (do) y x, y double dash will be n is from 0 to infinity. On more differentiation of this will give you n plus k, n plus k minus 1 into C n x power n plus k minus 1. So this is what you have, okay.

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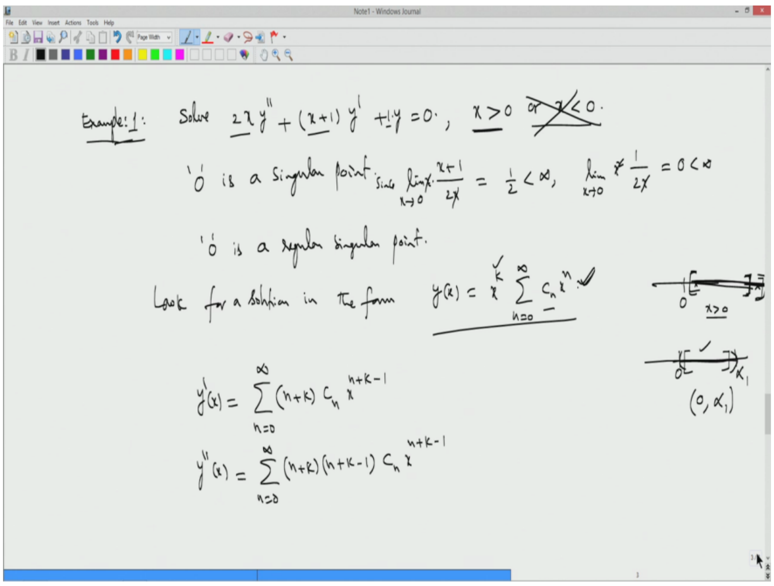

So you substitute this three into the equation. So e if you substitute this 2 x, y double dash is this, n is from 0 to infinity n plus k, n plus k minus 1 C n x power n plus k minus 2 is actually when you differentiate it twice it will become 2. So minus 2 plus x plus 1 times y dash that is n is from 0 to infinity n plus k C n x power n plus k minus 1. That is what is the second part. So the remaining part is simply y. That is n is from 0 to infinity C n x power n plus k equal to 0. So this is what it is after substituting.

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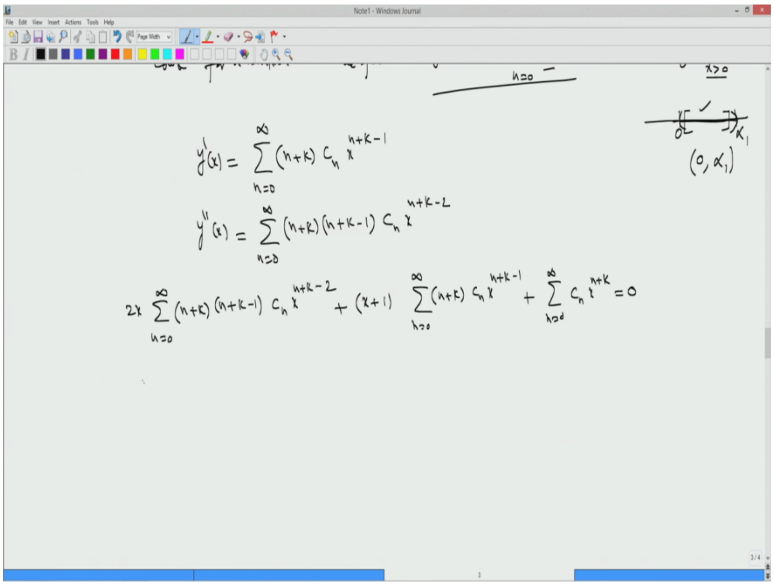


So what is the next step? So what you do, you just take the inside so you will get n is from 0 to infinity. So 2 times n plus k, n plus k minus 1 C n x power n plus k minus 1 plus this part n is from 0 to infinity, n plus k C n x power n plus k plus sigma, n is from 0 to infinity, n plus k C n x power n plus k minus 1. So I expand it x into this and 1 into this, plus sigma, n is from 0 to infinity C n x power n plus k equal to 0, okay.

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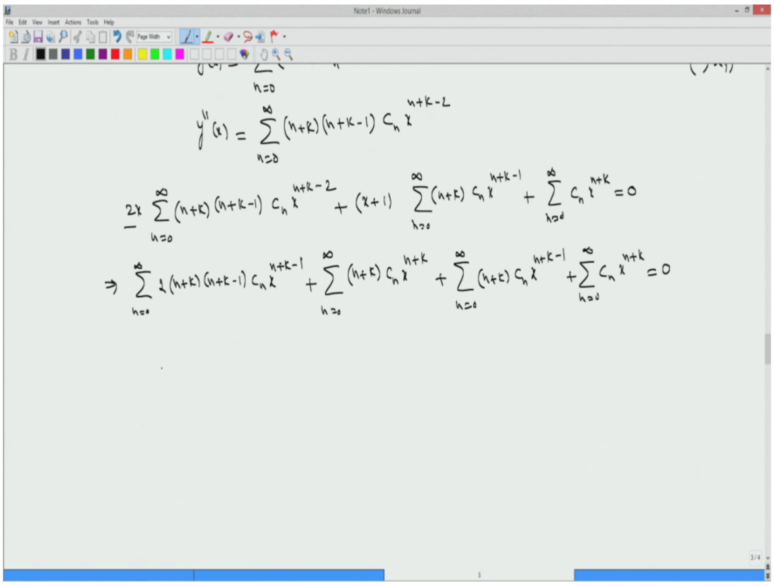


So what is the common term here? all the series are starting from 0 to infinity but the common part is x power n plus k minus 1, powers are x power n plus k minus 1 here. Here x power n plus k, here n plus k minus 1, here x power n plus k. So you convert this you reduce this x power n plus k minus 1 into n plus k by changing this index. If I want n plus k minus 1 into n plus k so that means n minus 1 I have to replace with n, okay. So wherever n is there you replace with n minus 1.

So in this for example here, okay, you can change the index. You know how to do it now. So you do the same thing here, n equal to n minus, so this is running from 1 to infinity. So, 2 times n minus 1 plus k, n minus 1. So that means n minus 2 plus k C n minus 1 x power n minus 1, sorry, n plus 1 you have to replace, right? N plus 1 if you replace this is going to be n is from n plus 1 equal to 0. That is n equal to minus 1 to infinity, 2 n plus 1, okay. Here 2 n plus 1, 2 n plus 1 will go.

So we have n plus k, C n plus 1 x power, in the place of n, n plus 1 so it will become n plus k. So this is already in the powers of n plus k. So you write as it is. So we have n plus k C n x power n plus k, similarly here. So this you change the index n equal to n plus 1 if you put this will become x power n plus k. So n equal to n plus 1. Sp n plus 1 equal to 0 so that means n equal to minus 1 to infinity, n plus 1 plus k C n plus 1 x power n plus k plus this is as it is because it is already in the powers of x power n plus k, equal to 0, okay.

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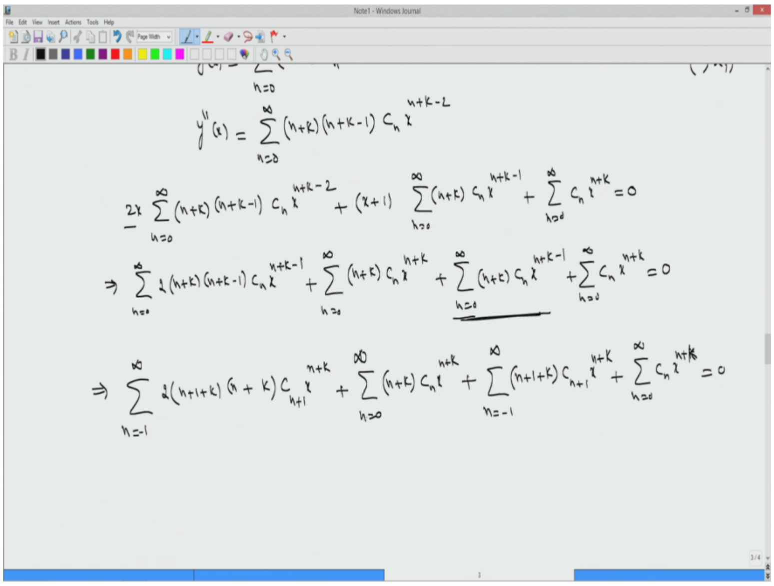

So 0 to infinity it is common and n equal to minus 1 here, n equal to minus 1 here, those are the only isolated terms. If you write separately them you have n equal to minus 1 means 2 times n equal to minus 1. This is going to be k into k minus 1 into C 0 that is your n equal to minus 1 here, plus n equal to minus 1 you have k times C 0 equal to 0. So you just write this as separately plus sigma n is from 0 to infinity C. Now you can combine all the four series you can combine that is C n.

So C n plus 1, C n plus 1 here. So you have n plus 1 k, n plus k, okay. This is common. When you take common out so you have 2 times n plus k plus 1 times C n plus 1 into x power n plus k. So that is also common. So plus what else you have, n plus k plus 1. So n plus k plus 1 times C n into x power n plus k. So that is common in both places. So you have x power n plus k which is equal to 0, okay. So this is what you have.

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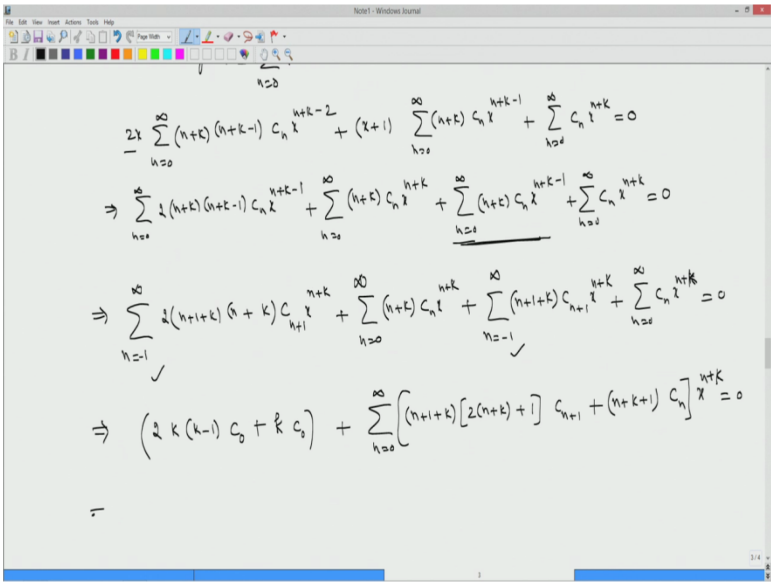

So now you have this is 1 power series into n equal to minus 1. What you have is x power, I missed x power, k minus 1. So when you have this power series like this which is equal to 0 power series you can actually make the coefficient 0 so that will give you C 0. You do not want C 0 to be 0. If C 0 is 0 what happens? If C 0 is 0 without loss of generality you can always assume that C 0 is non zero here. If C 0 is 0 I can take one x out and anyway k is unknown.

So once C 1 is 0 so you can write it will start from C 1 x plus C 2 x square and so on. So you can take one x outside, you can write x power k plus 1 with the power series now starting with n equal to 0 to infinity or you have C 1 plus C 2 x plus C 3 x square and so on. So that is like same power series here.

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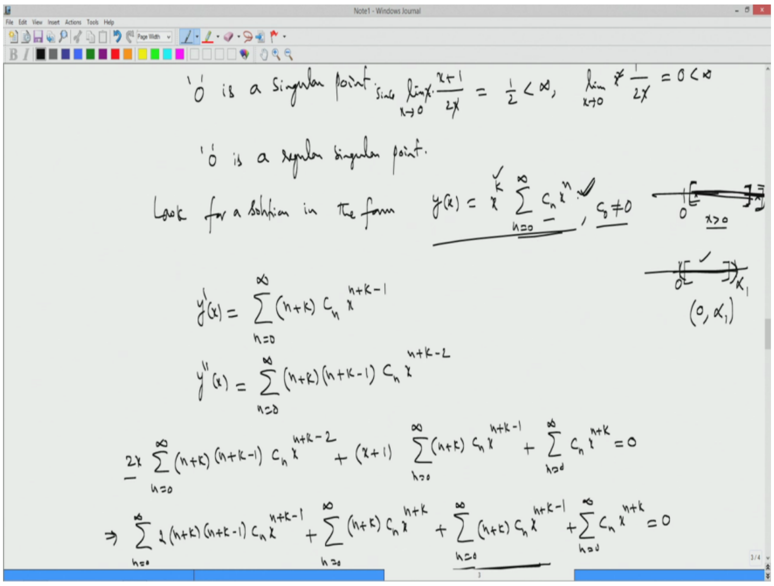


Instead if C 0 is 0 you will have x power k plus 1, k is unknown, k plus 1 is also unknown so it does not matter. So without loss of generality you can assume that C 1 is non zero. So because C 1 is non zero you have this 2 k into k minus 1 plus k has to be 0 by equating k power minus 1 coefficient, okay, 0.

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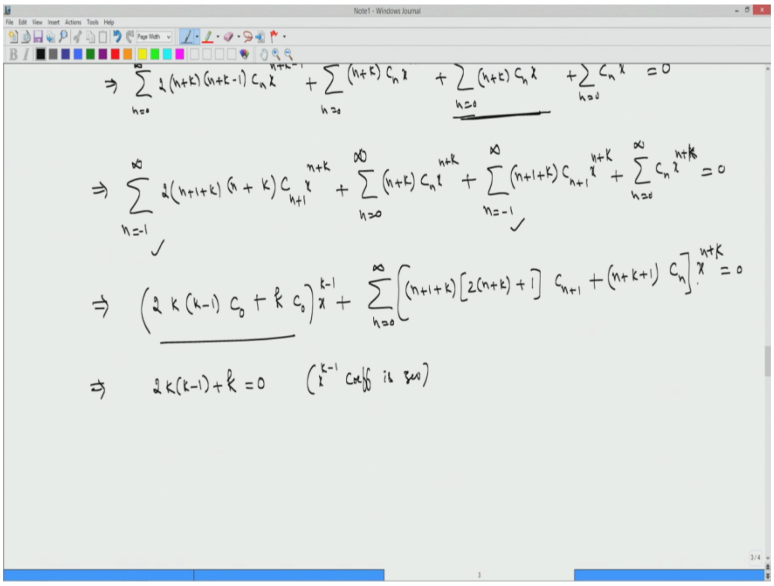


Next thing is k power n plus k, n is running from 0 to infinity, you can equate. That will give you the recurrence relation. So what happens this is equal to 2 k square minus 2 k plus 2 k. So that is going to be minus k equal to 0. So that will give me k equal to 0 or 1 by 2. So you always choose k 1 which is always big. What is big here? So bigger is root is this, smaller root is 0. So always assume k 1 is bigger than k 2. So when they are same I choose like this.

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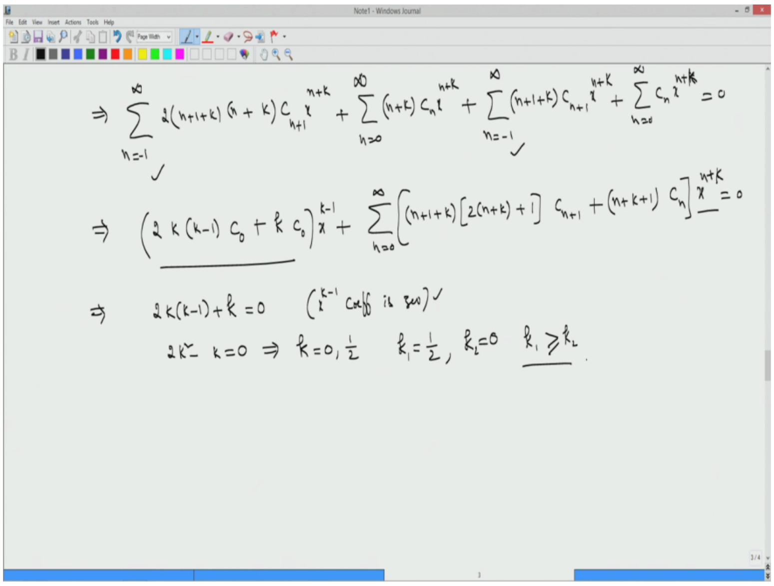


If they are complex it does not matter whichever you can choose as k 1 k 2. So this is called indicial equation, okay. So this is called indicial equation. So it has these roots, okay. So when their roots different r 1 minus r 2 is non zero and non integer. This is the case 1. Case 1 we are dealing with. In this case, okay, so this is the case we are dealing with. So if this is 0, okay, obviously the difference is r 1 is non zero and it is a non integer, okay. If it is 0 it is one case. So you have a three more cases, okay.

So you have r 1 minus r 2 is non zero but integer. And case 3 because I have chosen r 1 is always bigger, when it is non zero it is a positive integer. So it is a positive integer. This is non zero or rather non integer, it is not a positive integer. Here also you can write not a positive integer because r 1 is always chosen as big, okay, not a positive integer. So this is one case, this is another case. Now you have last case when r 1 minus r 2 is 0. That means roots are repeated.

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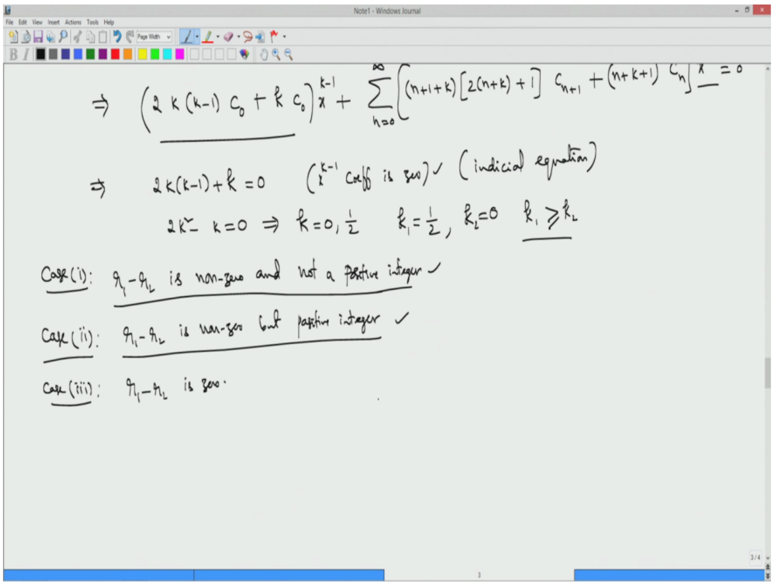


So this is the first case we are dealing. That is what we have seen, k 1 minus k 2 is 1 by 2. So this example belongs to the case 1. In this case it is relatively simple. So you will see that finding the solution is relatively simple because just by equating we will be able to get two linearly independent solutions very easily. So what we did is we equate the coefficient of x power k minus 1, you got this indicial equation.

If you do now all powers of x power n plus k, so powers of n plus k, coefficient of x power n plus k, n is from 0, 1, 2, 3 up to infinity, if you do, okay, is 0, okay coefficient of 0, this will give me recurrence relation, this is called recurrence relation, okay. So you will get a recurrence relation.

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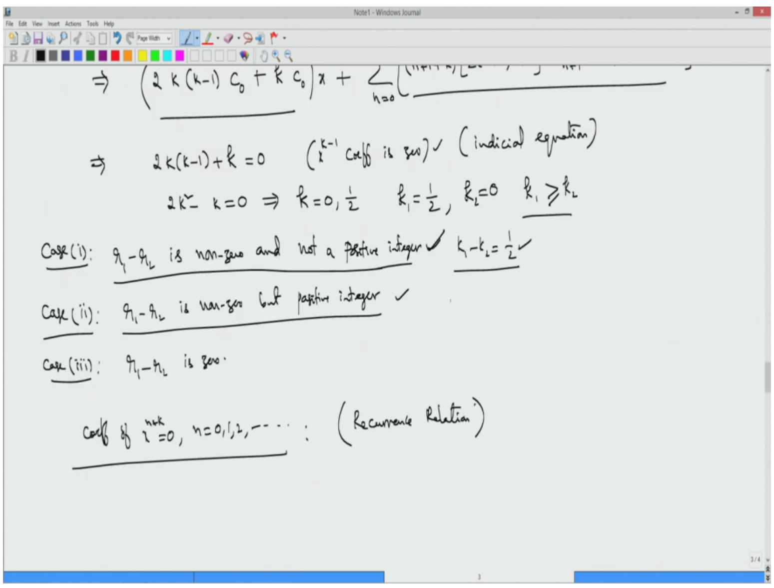

What is that recurrence relation? This one. So n plus k plus 1 that is common. So what you have is 2 n plus k plus 1 into C n plus 1 plus C n which has to be equal to 0. Where is it running from? Running from n is from 0, 1, 2 and so on. This is what you have because n equal to k equal, now you fix k equal to 0 or half plus 1, 3 by 2 plus n is 0. So this will never be 0, okay.

This will always be since n plus k plus 1 never be 0 for every n 0, 1, 2, 3 and k is 0 or half, okay. So this has to be your recurrence relation. You have 2 n plus k plus 1 into C n plus 1 plus C n equal to 0 for n is from 0, 1, 2, 3 and so on.

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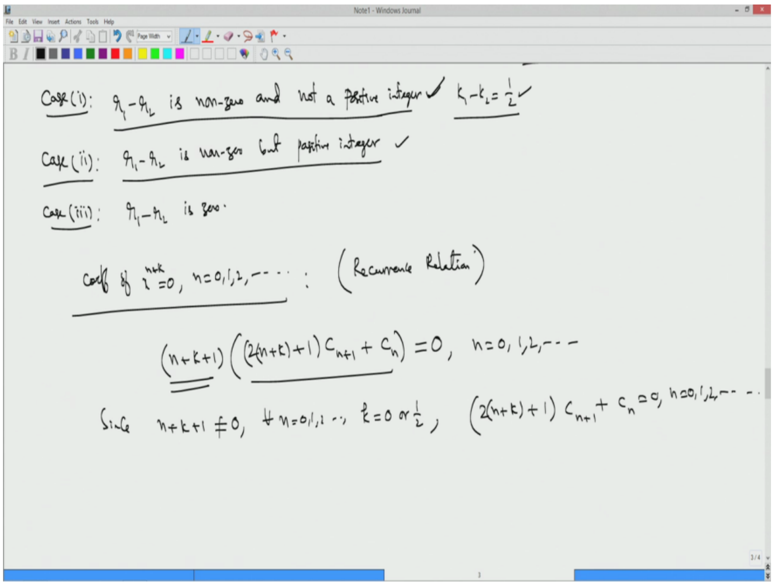

So this will determine your C n plus 1. So put n equal to 0 in the recurrence relation here so you will get C 1 equal to minus C 0 divided by put 2 times k plus 1, okay. And put n equal to 1. If you put n equal to 1 into this you get C 2. C 2 is n equal to 1. So you have C 2 that is minus C 1 divided by 2 times n is 1. So you have k plus 1, 2 k plus 1 plus 1. And I already know what is this is going to be plus C 0 divided by 2 k plus 1 into 2 k plus 1 plus 1, okay.

So this is actually C 0 divided by 2 k plus 1 into 2 k plus 3 and so on. So n equal to 2 you will get C 3 as minus C 0 divided by 2 k plus 1, 2 k plus 3. And what is the next step? 2 k plus 5 and so on, okay.

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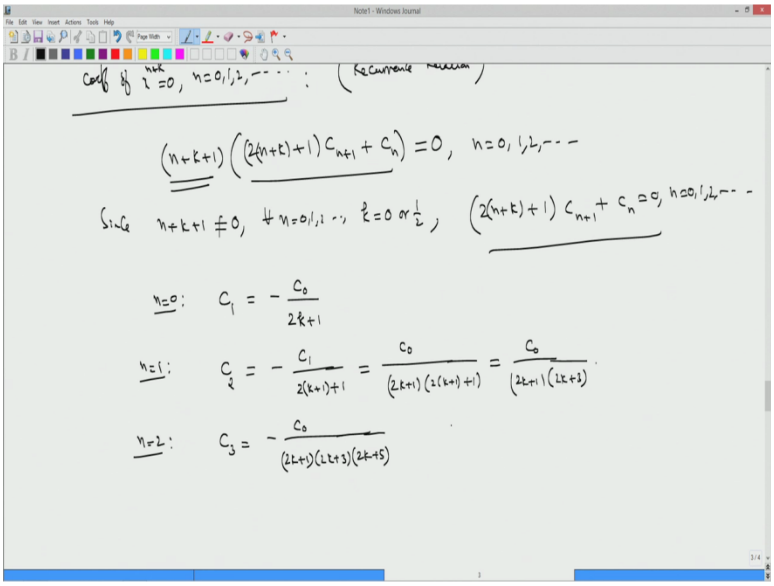


So in general you will get you can write the nth term if you like, okay. So we can go on getting like this. Now what is k value? K is either 0 or half. So start with this case 1 when the difference is non integer you put k equal to bigger root k 1 which is half. If you do this what we gain is that C 1. What happens to C 1? C 1 is minus C 0 divided by 2 by 2, so you have 2. So C 2 will be C 0 divided by 2 k plus 1. So k equal to k 1 which is half so it is going to be 2, it is going to be 4, okay.

So similarly C 3 will be minus C 0 divided by, so 2 4 6 and so on you will get like this. So this implies you go back and (subs) now you have got most of the root Cs. C 1, C 2, C 3, C 0, okay. Except C 0 you got everything and you found one k. One k is so you have that solution y 1 as y x with k equal to half and corresponding C ns, okay.

That is actually equal to x power k which is half so when you say x power half, x has to be positive because this function when you are working with the equation with x negative side. If you are working for the equation that x negative side, when you get this x power k, when k is something. So when your write like this you have to write mod x, okay. Mod x means it works both for x positive and x negative, okay.

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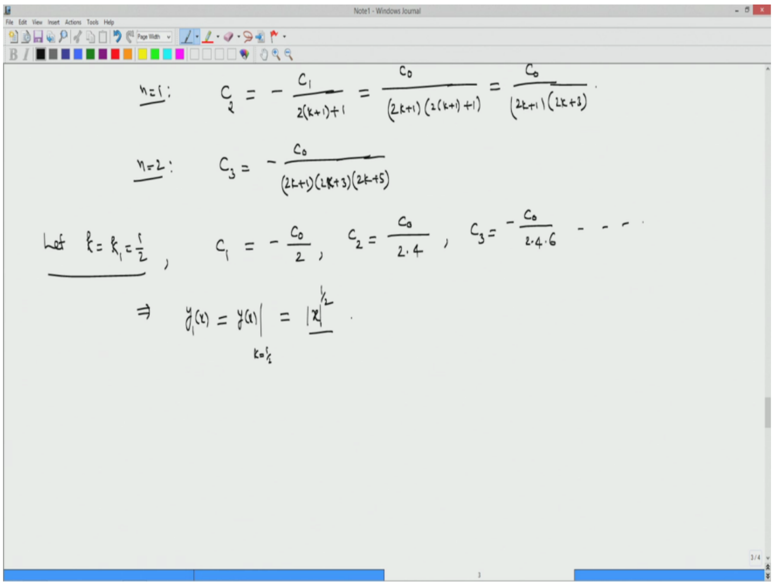

That is how your solution should be. So if you are looking for x negative side you should look for your solution in this mod x form, mod x power k. So because you are working with x positive your solution looks like this, okay. So anyway we are only working for x positive so you can say x power, this is simply x is positive because x power half you can write. And then what is your power series? Power series starting from C 0 plus C 1. What is C 1? You found here.

So minus C 0 by 2 x plus C 0 by 2 4 x square minus C 0 divided by 2 4 6 x cube and so on. So this is your first solution. This is equal to C 0 is common, root x times 1 minus x by 2 plus x square by 2 4 minus x cube by 2 4 6 plus and so on. So minimum three terms are there so you have this is one solution.

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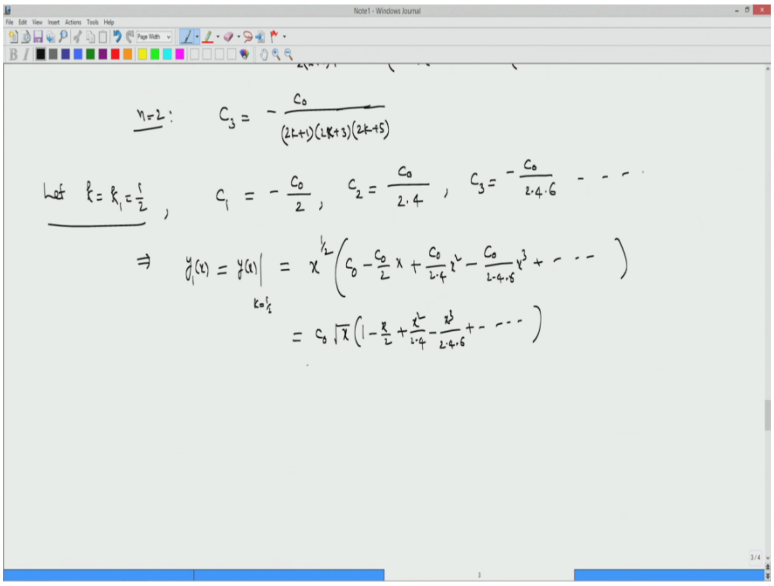


You can take C 0 as 1 that is your first linearly independent solutions, non zero clear, right? Now you allow k equal to k 2 which is 0. If you do this what happens to your C 1 from your recurrence relation? C 1 is if I put k equal to 0 so you have minus C 0. C 2 will be minus C 0 minus C 1 divided by 2 plus 1 so 3, minus C 1 divided by 3. So this is going to be C 0 divided by 1 3. Similarly you can get C 3 as minus C 0 1 3 5 and so on, okay. Put k equal to 0. K equal to 0 if you put 1 3 5, okay. So that is what you will get and so on. So like this you can get all C ns.

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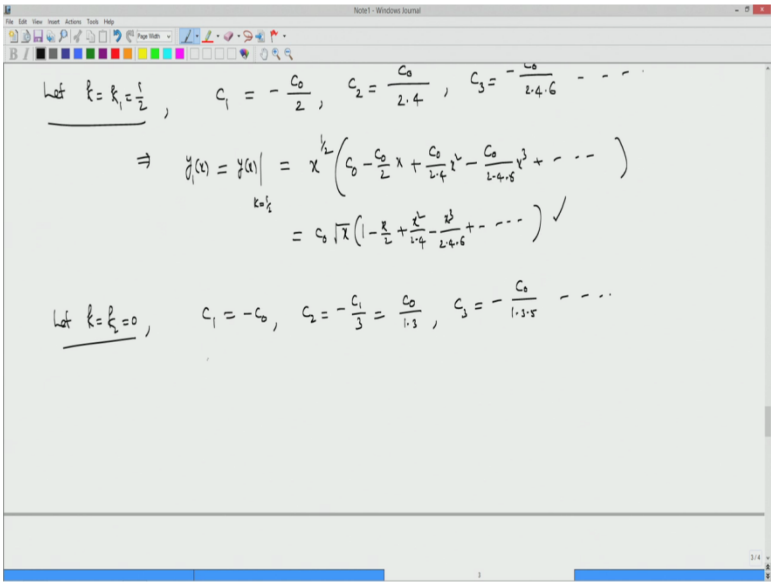


So if you do this you can get now you know what is your k and you know what is your C 1, C 2, C 3. So if you substitute y 2 of x, y x at k equal to 0, okay. Chose (an) this function at k equal to 0 which is x power k 0, C 0 which we could not find, C 1 minus C 0 plus C 2, C 2 is plus C 0 by 1 3, C 1 into x, C 2 into x square, C 3 into 1 3 5 x cube and so on. Again here x plus 0 is 1, C 0 is common, we have 1 minus x plus x square by 1 3 minus x cube by 1 3 5 and so on. So this is your second solution.

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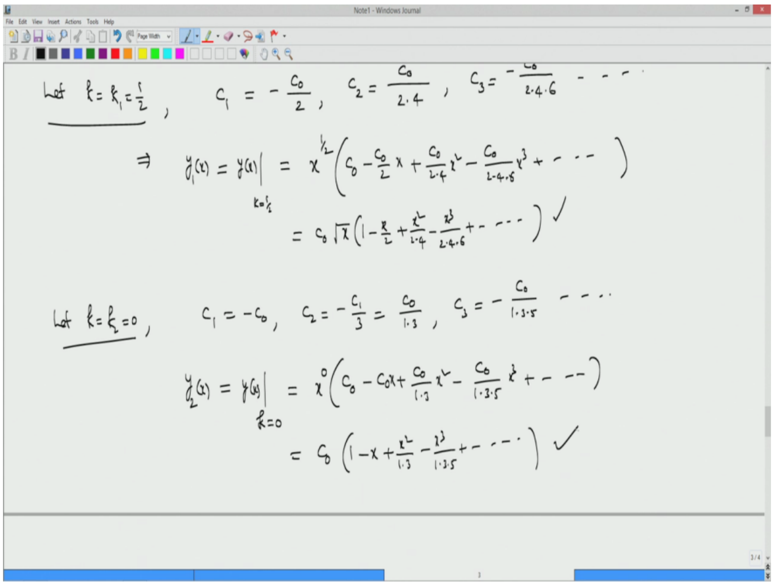


If you choose C 0 equal to 1 you will get a second linearly independent. So you can see that y 1 by y 2 is actually two different things. This is not constant. So that implies there are two linearly independent solutions that implies y 1 x and y 2 x are linearly independent solutions since y 1 by y 2 is not a constant, okay. You can see that they are not same. They are not constant multiple of other. So that implies the general solution of the given equation is y x equal to C 1, some arbitrary constant y 1 of x plus C 2 y 2 of x.

What is y 1 and y 2? By considering your C 1 as 1, okay. Similarly here you take it as 1. We chose that is your C 1 and C 2 so you have C 1 and C 2 are arbitrary constants. So this is your general solution, okay. So this is your final general solution you got. So this is your general solution you got, okay, with y 1 is this and y 2 is this one, okay.

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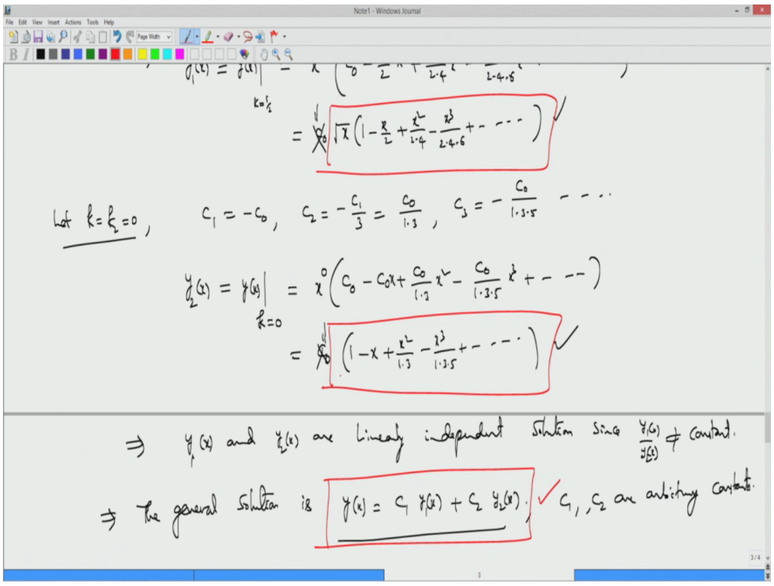


So you got this general solution of this equation in the case 1. So what is the case 1? You find the indicial equation, you look for solution in this form and when you find the indicial equation the difference of the roots r 1 and r 2 is non zero and not a positive integer. So we will see another example in the next video where I will consider the equation. You get the same case, they are distinct but they are complex roots so that the difference is always imaginary.

Two complex conjugate roots you take the difference it will be imaginary numbers. That is not a positive integer. So we will look into that example and then we will see the other examples, these case 2 case 3, where the difference between the roots will be one is, they are distinct but integer and they are same. So that means the difference is 0. So these two cases we will see in the next few videos.