Differential Equations for Engineers
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Lecture 23
Properties of Legendre Polynomials

So in the last video we have derived orthogonal property. We are giving some properties of Legendre polynomials. First property is orthogonal property with most important property and in that we are saying that any two different Legendre polynomials will be orthogonal and we have seen that if you take the same polynomial and you integrate so you will have some value.

So while doing that so in this video so we will derive another expression directly from the recurrence relation of the Legendre equation from which you can directly derive this formula for the Legendre polynomial P n x, okay. So before I do that so in the last video we have done this integral value. So we have seen this integral value as this, okay.

So we can see that so 2 times when you do that d t by, so that 2 actually cancels, okay. When you are doing so 2 x d x equal to d t so when you are replacing with d x which is d t by 2 x so the 2 2 cancels, 1 by x is actually t power minus half. So you do not have this 2, okay.

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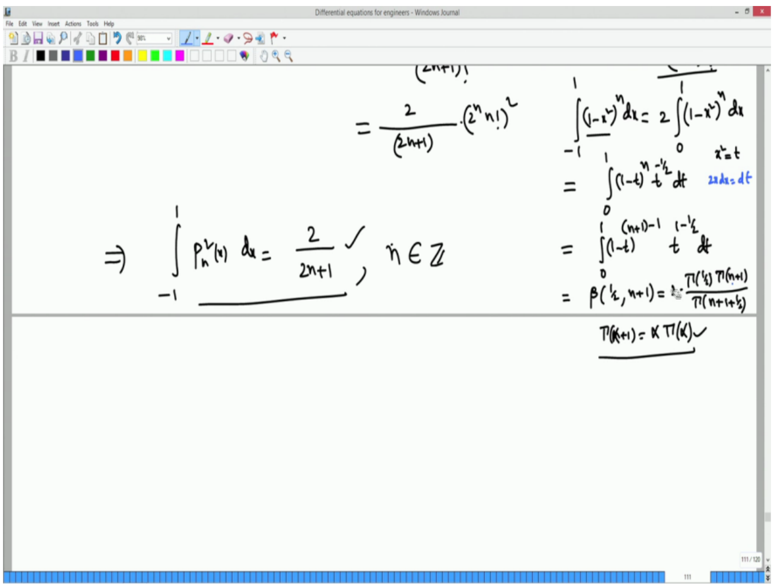

So this value you will see that is actually this one so 2 times finally you will see that this is what you will get, okay. So we will see that how it is done today. So what you need is this beta function at half n plus 1 which is Gamma half into Gamma n plus 1. So n is an actual number so you can find this n plus 1 plus half, okay.

So this is equal to so Gamma half is root pie and this is n factorial divided by, this you can write using this formula as n plus half Gamma, so Gamma n plus half, okay. So again I can rewrite like n plus half. So that is actually plus half, n plus half I write like minus 1 plus 1 plus half. So it is again like, right? Okay so half plus 1 rather n minus 1 plus half plus 1. So this is actually this, okay. And you actually see n plus half plus 1 is n plus half into Gamma n plus half, 1 1 goes.

Just add and subtract it 1 so again you can rewrite Gamma root pie into n factorial divided by n plus half times, again n minus 1 plus half that is n minus half. So this is going to be n minus half into Gamma n minus 2 plus half plus 1. So that is what you will see. What is this actually? N minus 1 plus n minus half. So n minus half into Gamma n minus half, okay.

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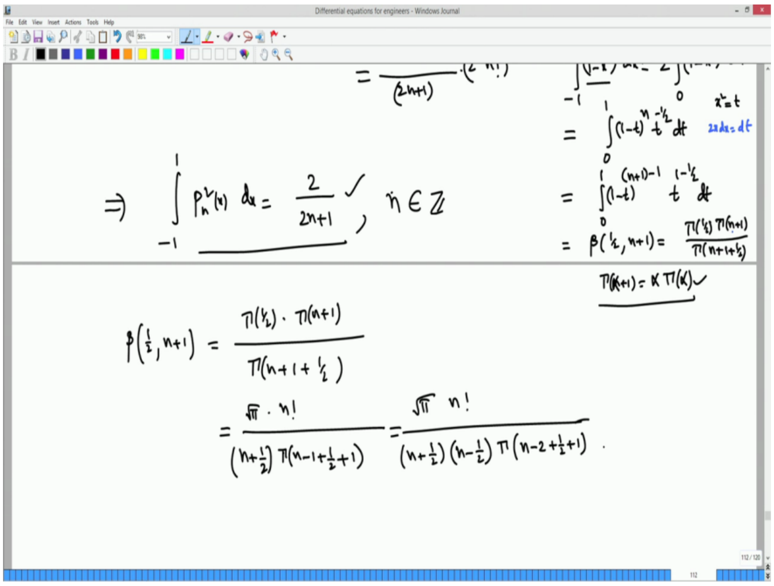

So like that you go on every time you do this. This is actually n minus half, n minus half I am rewriting n minus 2 plus half plus 1, okay. So next time when you write n minus 3 plus half plus 1 will be this is actually this is this into Gamma of this into n minus whatever is here. So this is going to be n minus 2 plus half. N minus 2 plus half is n minus 3 by 2. So this into this is actually equal to this.

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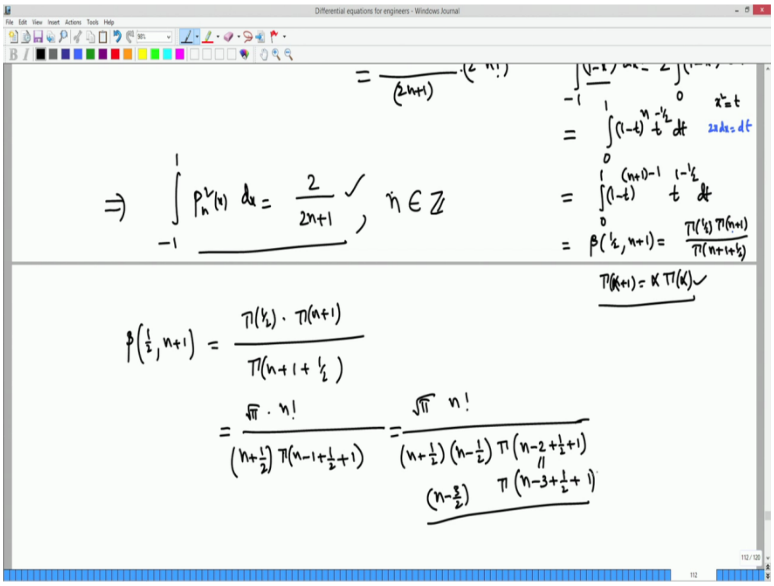

So you can replace like this, you can go on writing like this, what you see is you will end up n minus 3 by 2 and so on. Finally you will end up just 5 by 2. So if you keep on adding so n is a fixed number, fixed natural number so we have 3 by 2. So you see that 5 by 2, 3 by 2 and finally we get 1 by 2. So when you get 1 by 2, Gamma will be n equal to 1 in this case for example, okay, n equal to 1 this is going n half plus 1 that is half into Gamma half. The Gamma half is again root pie. So this root pie this root pie goes, okay.

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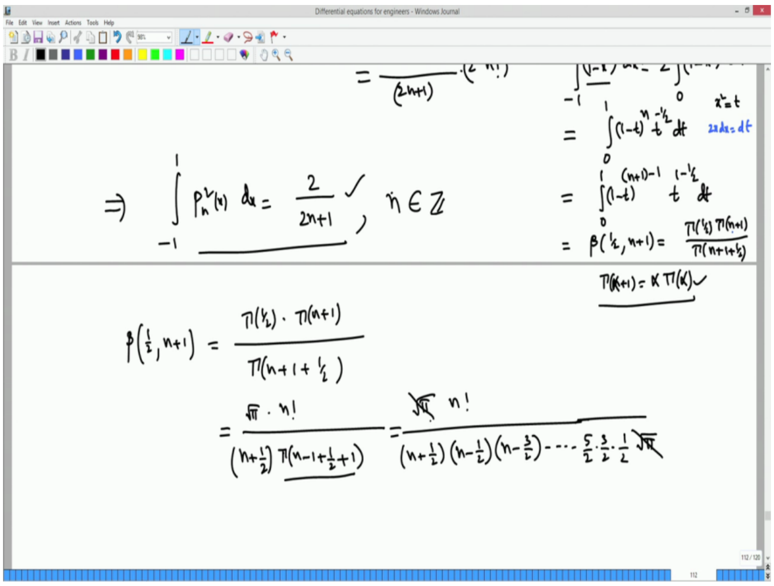

So we can easily see that this is going to be n factorial divided by, if you write n plus 1 divided by 2, this is going to be 2 n minus 1 divided by 2, this is going to be 2 n minus 3 divided by 2 and so on, 5 by 2, 3 by 2 and 1 by 2. This is what you have. So in between you add 2, 4, 6, okay, and so on. You have here 2 n minus 2. You have 2 n. So what you added 1, 2, 4 actually 2, 4, 6 8 up to 2 n that is actually 2 power n into n factorial, okay.

And you see this how many are there? I have in the division one, two, three, four, five, up to I have n plus 1 you have in between, okay. These 2s in the denominator how many are there? If n equal to 2 this is going to be 2 into 2 plus 1, 5 by 2 up to here. So if n equal to I have three 2s. If n equal to 3, 7 by 2 that means I will have 4. So if n equal to 3 I am getting 3. So like that if you have n you will have n plus 1 2s, okay. These you can take it up, bring it up.

So what you have is n plus 1 into 2 power n into n factorial divided by, now this is 1, 2, 3, 4, 5, up to 2 n plus 1 factorial, divided by 2 power n plus 1 that will go here, okay.

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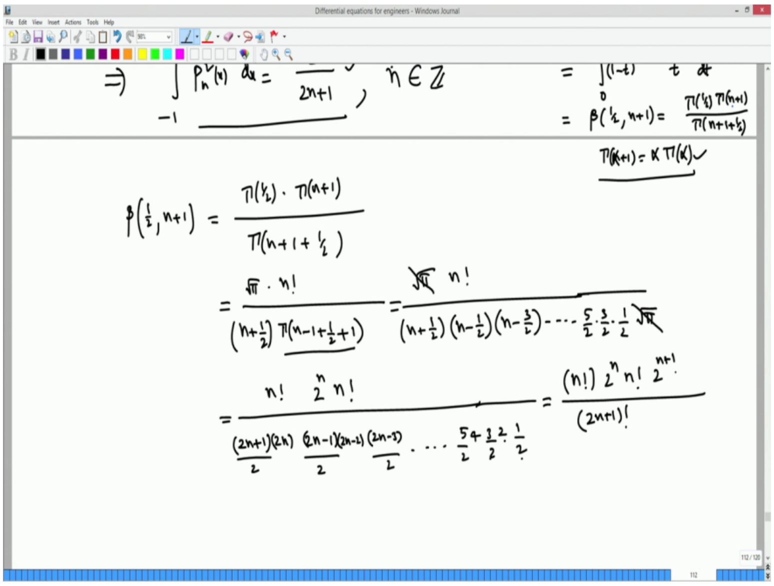

So this is equal to n factorial square 2 power n, n factorial rather we write 2 power n, n factorial whole square into 2 divided by 2 n plus 1 factorial. This is exactly what you have here. This is what you get, okay. So let us see how we get a different formula for the P n of x? A new formula, okay, new formula for P n of x. So this is also one kind of property, okay. And so we have seen one property that is the P ns are orthogonal, okay. So P n P m are actually satisfying orthogonal property.

So we have seen this orthogonal property that is as a property 1 and we see that second if you want to write as this property, second property is property 2 you can write, P n of x, n is from 0 1 2 onwards or n belongs to Z in fact, any Z, okay. Any integer n you have is a bounded solution for your Legendre equation. What is that equation? Y double dash 1 minus x square minus 2 x y dash plus n into n plus 1 y equal to 0. So this is what you have, okay.

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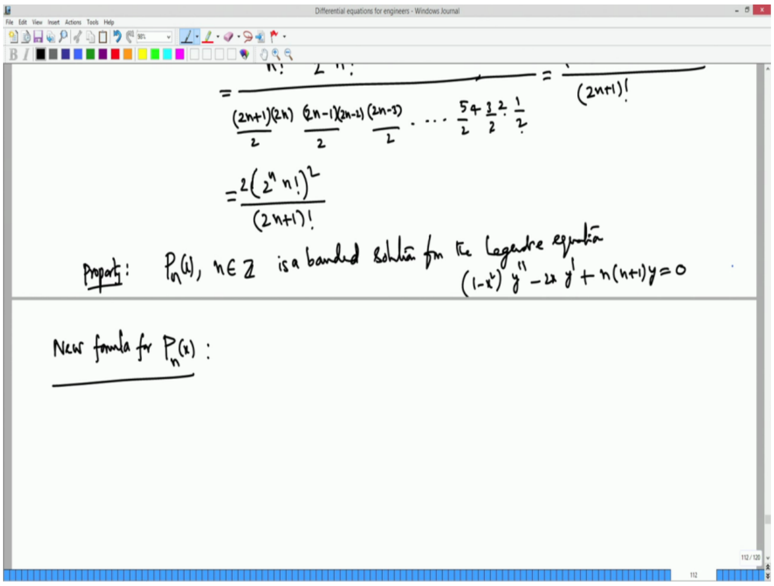

So you see that how minus 1 and 1 are singular points but this function is defined because these are polynomials, we have a solutions which are as a function they are defined everywhere, but as a solution they are actually valid between minus 1 to 1. Even at those points minus 1 to 1 they are actually bounded. So its value is defined, okay. So you see that this is the bounded solution. So you have a bounded solutions for this equation. So that is I can take it as property.

We will give now a new formula for this. So to give this let us go back to the equation. So what we have done when we started solving this equation, what we derived is you put it as series solutions. When you put the series solution you get a recurrence relation like this. When you equate this x power 0, n equal to 2, n equal to 3, okay. What you have is this and now what you have is this one. So where is the recurrence formula. You have a recurrence formula here. So what you have is this is equal to 0. This formula is called recurrence relation.

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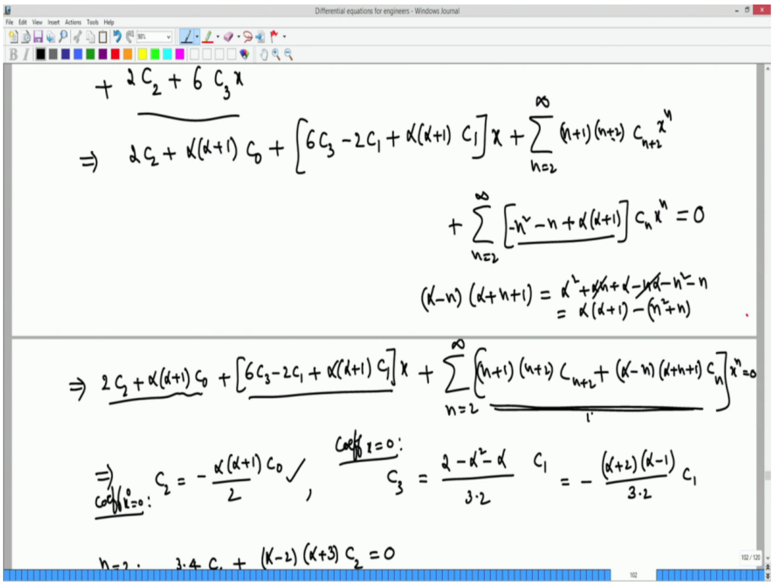

If this is equal to 0 when you substitute the power series solution into the equation which is equal to 0 all the powers of x power n coefficient should be 0. So this is a recurrence relation, okay, recurrence relation. So we will pick up this one. So this is the recurrence relation and actually n is from 2 to infinity. So if I put n equal to 1, n equal to 1 is actually 2 into 3 C 3 minus, okay, so that is what you have 6 C 3 and then you have here, so alpha into alpha plus 1 and n equal to 1 alpha into alpha plus 2, okay.

Alpha minus 1 into alpha plus 2 into C 1. So that is exactly you have alpha square plus alpha minus 2, okay. So this is what if you actually sum it together this is what you have, okay, into C 1, right? Alpha square plus alpha minus 2 into C 1. So that is what you have.

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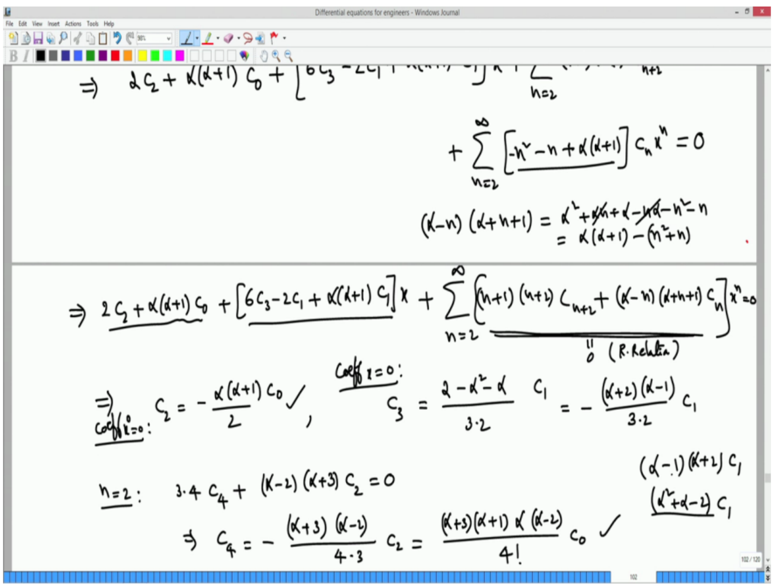

So this is exactly even when n equal to 1 you get this expression. So n equal to 0 you get this expression. So this recurrence relation is actually valid from n equal to 0 to infinity, okay. So we can put this together inside. By just writing n equal to 0 to infinity this is what we have. So let us use this n plus 1, n plus 2. So we use this recurrence relation. So n plus 1, n plus 2, so let me write into C n plus 2 equal to minus alpha minus n into alpha plus n plus 1 into C n, okay.

For every n greater than or equal to 0. So this I included both the first two terms. What we derived earlier, okay. So this is true. So this implies what we did is if you fix n as a natural number, okay, if you fix your n and when n is odd or n is even what you see is that n is even. So let us say some 2 k. Even let us say some 2 k. So if it is 2 k, 2 k you get C 2 k, okay. So C 0 you are not able to use. So you get C 0, C 2, C 4, okay. C 2 in terms of C 0, C 4 in terms of C 0, like that you go up to C 2 k in terms of C 0. That is what we did, alright?

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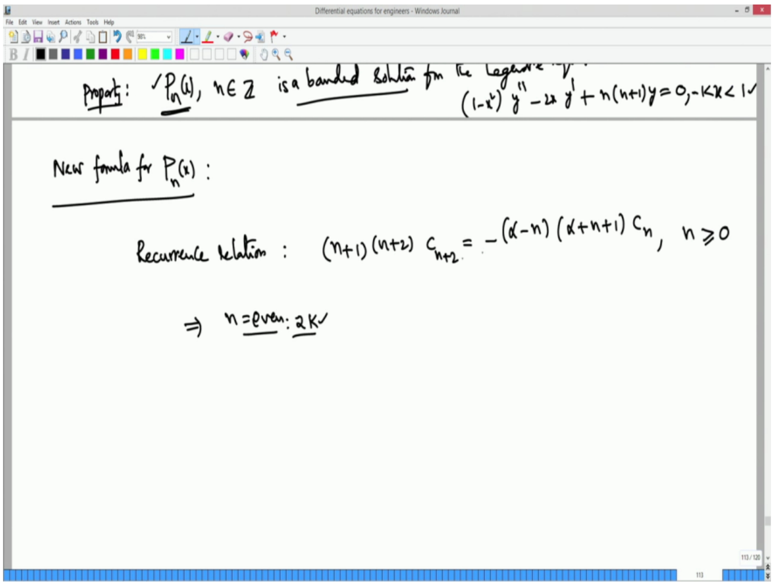

So you put n equal to 0 I get C 2 in terms of C 0. Like that C 4 in terms of C 2 which we already know that this is again C 2. C 2 is also in terms of C 0 so we have C 4. Like that you can go up to 2 k is in terms of C 0. This is what we did. This is what normally we can get and substitute into the equation, right? When n is even, y 1 first solution in terms of C 0 you will have all beyond C 2 k plus 2 will be 0, okay. So this one is the polynomial that will stay up to 2 k terms, okay.

So you basically you get a polynomial. If n is odd, okay, this is even. If n is odd say 2 k plus 1. In this case what you have is C 1. You start with C 1 so you take only odd coefficients C 1, C 3, C 5 up to C 2 k plus 1. So in this case here C 0, C 2, C 2 k.

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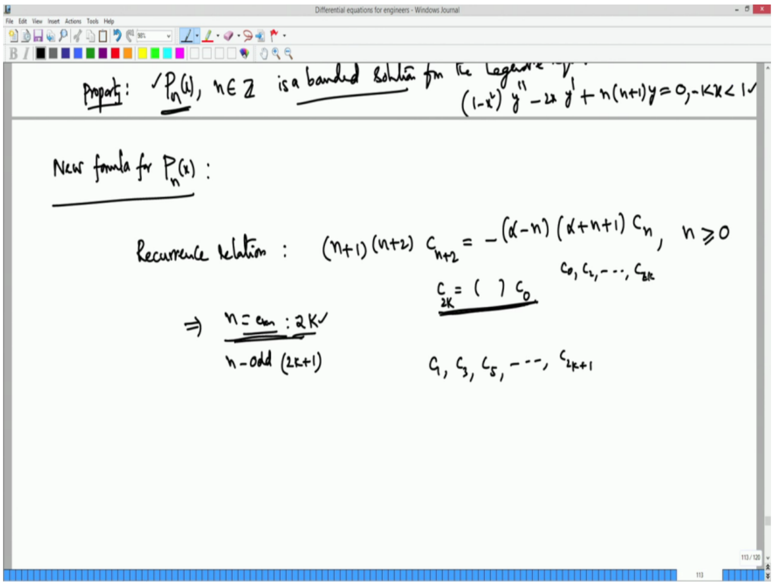

So when it is even, even case I can write all these things C 2 k up to C 2 in terms of C 0. So you have C 0 coefficient and a polynomial of degree 2 k, some polynomial, okay. Polynomial of degree 2 k. So here in this case here also from the recurrence formula, so if you have n is odd, C 1 you will not be able to evaluate, get it. C 3 you can get in terms of C 1, C 5 in terms of C 1 up to C 2 k plus 1 in terms of 1.

And once you substitute you get C 1 times a polynomial of degree 2 k plus 1. This is what you get. So what do we do is to derive the formula for P n, instead of writing C 2 in terms of C 0 or here C 1 in terms of or C 3 in terms of C 1. We do the reverse way. So we consider C 2 k or C 2 k plus 1. I do not find this one, okay. What we do is I try to keep this as arbitrary constant. C 2 k minus 2 I write in terms of C 2 k.

Similarly going backwards you see that C 2 will be in terms of C 2 k. C 0 will be in terms of C 2 k. So finally instead of C 0 you will get C 2 k. Again we will get a polynomial that is also one way of doing it, okay.

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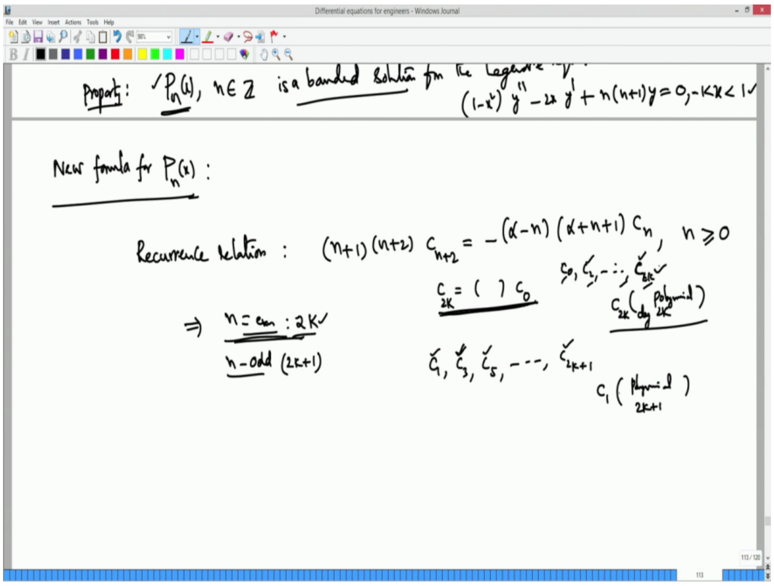

We do the same thing. So you write instead of finding this C 2 k plus 1 from this recurrence relation you try to do it reverse way. You keep this 2 k plus 1 as it is and then C 2 k minus 1 you write in terms of C 2 k plus 1. C 5 also in terms of this, C 3 will be in terms of 2 k plus 1.

Similarly C 1 in terms of 2 k plus 1. Then once you substitute back the polynomial solution will be 2 k plus 1 into this polynomial degree, something like similar you will get it, okay. Now you fix this constant which depends only on this 2 k or 2 k plus 1 which is n, okay.

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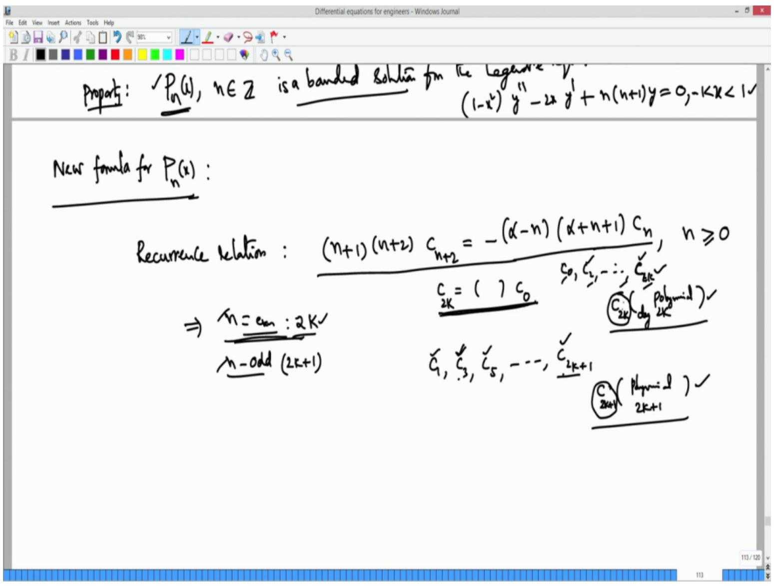

If I can give you this polynomial that is this is the polynomial of degree 2 k that means equal to n. This is your C n. If I give my C n which is the coefficient of x power n in this polynomial, okay, so that I hope I have to give you exactly what is P n. I will tell you because you know that P n is a polynomial of degree n with some coefficient. Same coefficient if you give here because from the Rodrigues' formula you know what is the P n of x.

The coefficient of x power n is something which you know from the Rodrigues' formula. The same coefficient you also assign here as a C n then what you get is a polynomial solution with the same x power n coefficient, okay. Polynomial solution of the Legendre equation with same coefficient as in the P n. So if you know that P n is the solution and this new polynomial is also a solution with the same x power n coefficient.

And then if they are the solutions of the Legendre homogeneous equation which is Legendre equation second order homogeneous equation. So their difference if you take these two polynomial differences what happens? With the same x power n coefficient when you take the difference it will become n minus 1th degree polynomial solution, okay, for the same Legendre equation which is having alpha equal to n.

But you know that you do not have n minus 1th degree polynomial solution for Legendre equation alpha equal to n. And then it has to be 0. It has to be series but it is not series here. And because it is a homogeneous equations there is no other solution except the trivial solution, 0 is a solution. That means the P n and this new polynomial should be same, okay, either this or this. This is n, okay, and n is odd this is this.

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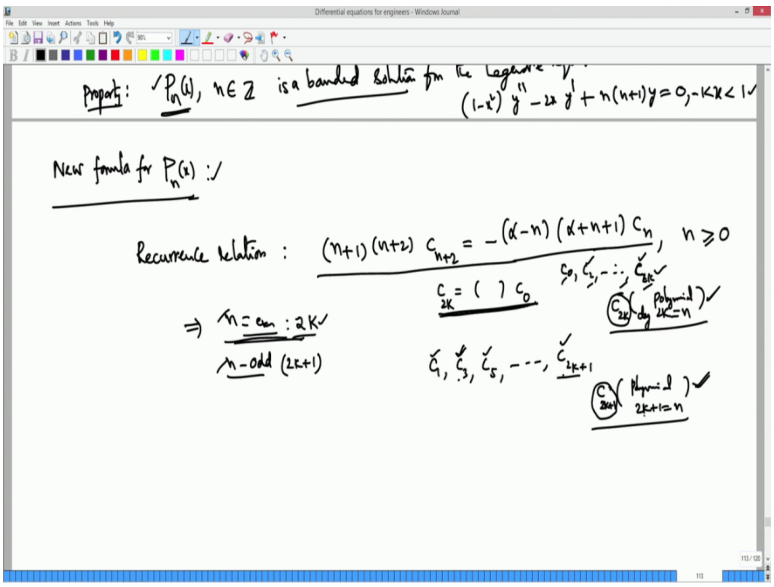

So we will see that will give you a formula for P n, okay. That is the idea. So let us work out. So you take this one. So instead of writing C n plus 2 so we normally do like this, alpha minus n, alpha plus n plus 1 divided by n plus 1, n plus 2 C n. This is what we normally do, okay, n is from 0, 1, 2 up to. This is what we do but instead we reverse this thing. So if you want reverse, this you want C n. you want this to be rather you write reverse, it is reverse way.

C n equal to n plus 1, n plus 2 divided by, sorry how minus alpha minus n, alpha plus n plus 1 into C n plus 2, n is from 0, 1, 2 onwards, okay. N is fixed either even or odd so what we see is wherever you start, okay, so you will not be able to get C 0 and C 1, right? So it should start from either 2, 3 onwards, okay. So that means this I can write n equal to n minus 2, if you put it, okay, and this will become minus n equal to n minus 2, okay.

So if I do this, this side will be C n. Everything in terms of lesser coefficients will be in terms of C n. So n equal to n minus 2, so this is going to be n minus 1, n equal to n minus 2 n divided by alpha minus n minus 2 that is plus 2 into alpha plus n minus 1, n minus 2 plus 1. So this is what you have, into C n, okay. Into C n, n is from 0, 1, 2, 3 onwards.

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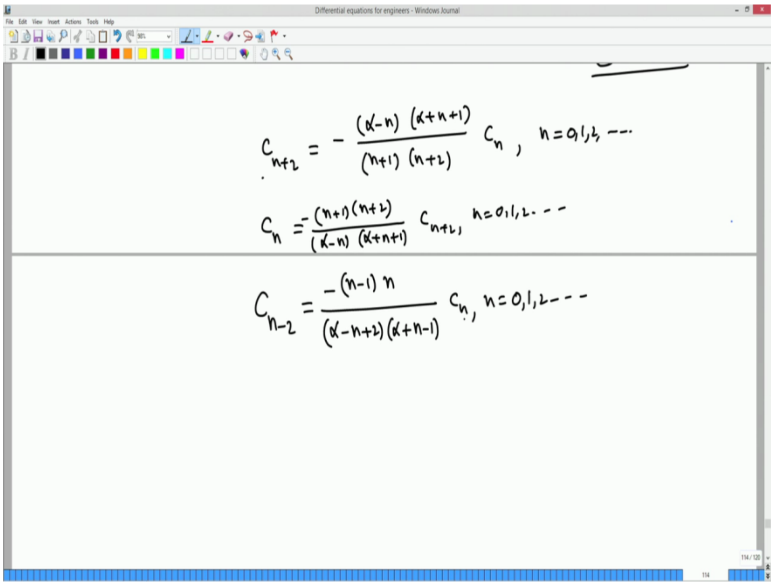

So once you fix your n you can write lower coefficients in terms of C n, fixed n, okay. So we use this C n. This C n is the coefficient of, so if you try to get all this C n minus 2, C n minus 4 and so on, let us see what is my C n minus 4? This is going to be minus n minus 2 here, so n minus 3, n minus 2, okay, divided by C n minus 2. This is going to be alpha minus n minus 2, okay.

Alpha minus n plus 4 and this is going to be alpha plus n minus 4, n minus 4 is n minus 3. So like this you can go on. You can get C n plus, so you already know C n minus 2 from the earlier expression that you can put it. So you can get C n minus 4 in terms of C n, like that you can get, okay.

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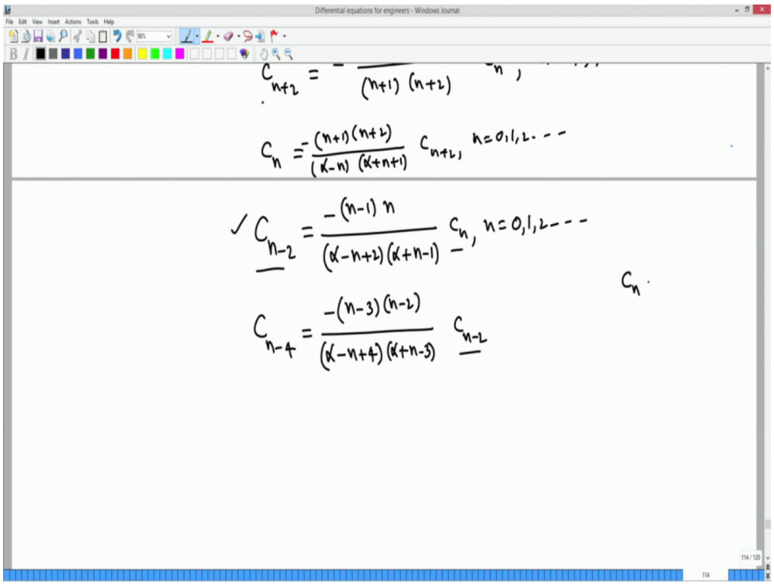

So once you substitute all these C 0, if n is even you can get C 0, C 2, C 4 up to C n minus 2. all this things are in terms of C n. If C n is odd you can get C 1, C 3, C 5 up to C n minus 2, okay. Yes C n minus 2. So if n is odd you are going up to, so C 1 you are not able to calculate, right, normally. So C 3, C 5, so n is odd means so minimum n equal to 3 onwards, right, n equal to 3. So all these things you are writing in terms of C n in the odd case, okay.

So once you substitute back what you get is C 0 in terms of C n. Some C n will be common in both the cases on the polynomial of degree n. This is what you have. So this C n I fix as, now I know why my P n is, what is my P n of x from the Rodrigues' formula is 1 by 2 power n into n factorial, n derivatives of x square minus 1 power n, okay. And this one we have already seen that coefficient of x power n is, what is the coefficient of x power n?

Coefficient of x power you can actually get it as 2 n factorial divided by coefficient of x power n in P n of x is actually equal to 2 n factorial divided by 2 power n into n factorial square. This is not difficult to see. If you actually differentiate x power 2 n so you have x power 2 n as the polynomial that you are differentiating n times, right? So if you differentiate n times this becomes what happens?

So how do you see this? Coefficient as actually coefficient of x power n in P n is equal to 1 by 2 power n into n factorial which I write here. So n derivatives of x power 2 n is 2 n, 2 n minus 1 up to 2 n minus, if you do twice you get into n minus 1. So you will get up to n minus 1, right, n plus 1 rather. Up to n plus 1 you will get, right?

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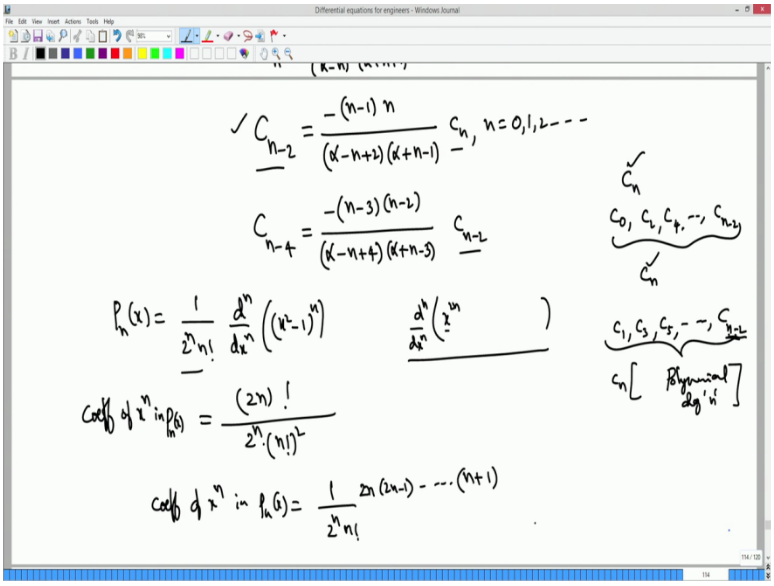

If you do x power 2 n, n times, okay, then what you get is x power n coefficient, okay. So that is the x power n coefficient, this terms will be simply this and this is equal to I have 2 n, 2 n minus 1. So if you divide and multiply n factorial so what you are getting is 2 n factorial divided by 2 power n into n factorial square. So like that you can get this. So you use this as your C n, okay. So let C n be 2 n factorial divided by 2 power n into n factorial square. If I choose this what happens to my C n minus 2, C n minus 4 and so on, okay.

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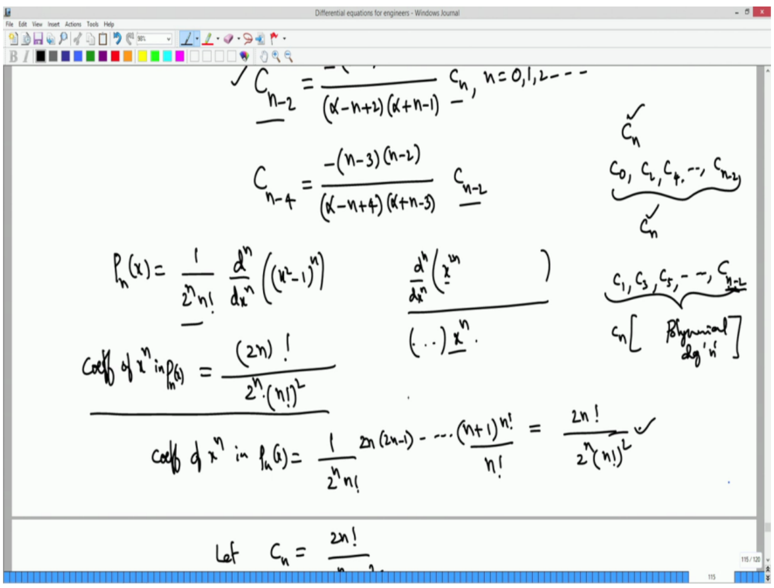

So you can write first what is your C n minus 2? C n minus 2 is minus n into n minus 1 into n divided by, now alpha is actually n, okay. Now you fix your alpha. Alpha is n, n is n minus n plus 4. So you have 4 times, okay, not 4 so I think 2, okay. You fix it here so alpha equal to n you have 2 into 2 n minus 1 into alpha into C n. So what is C n? C n is now I fixed it which is 2 n factorial divided by 2 power n into n factorial square.

So this is equal to so you see that 2 n, 2 n minus 1 goes. So you have 2 n, 2 n minus, so you can rewrite minus n into n minus 1, 2 n into 2 n minus 1 and 2 n minus 2 factorial. This I am writing like that and we have 2 into 2 n minus 1 into 2 n into n factorial into n factorial. If you write this is equal to, so n into n minus 1 if you remove denominator this n factorial becomes, if I cancel this here what I am left with is n minus 2 factorial, okay. And we have 2 n minus 1, 2 n minus 1 goes and here n you cancel here one n.

So you get n minus 1 factorial. What you left with is 2 n minus 2 factorial plus, write 2 n minus 2 factorial into 2 power n and 2 2 goes here. That is what you have, okay. That is your C n minus 2.

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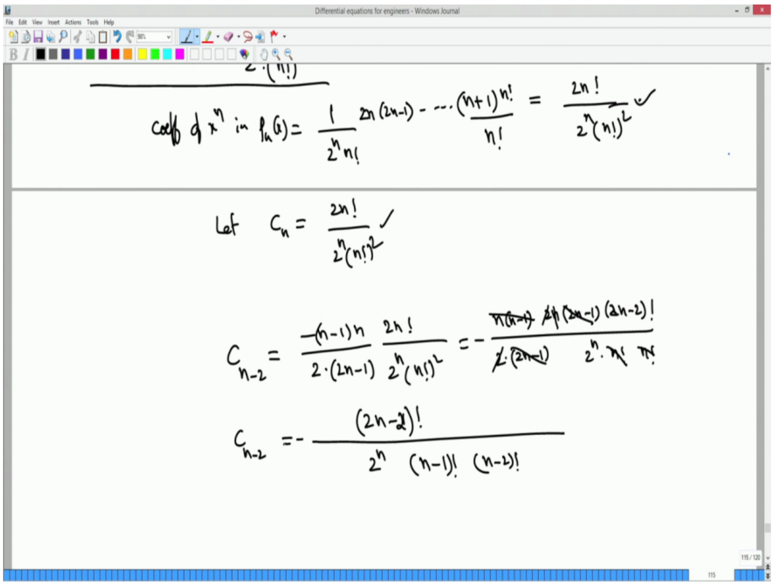

Now you get your C n minus 4. C n minus 4 is minus n minus 3, n minus 2, okay. Minus n minus 3, n minus 2 divided by, alpha is now we fixed here alpha as n, so n minus n plus 4, 4 into 2 n minus 3. Now what you get is C n minus 2. So C n minus 2 we already know what it is, right? C n minus 2 is 2 n minus 2 factorial divided by 2 power n, n minus 1 factorial, n minus 2 factorial. If you simply calculate it now, okay.

You already calculated C n minus 2 which you put it here so this will become, so what you do is n minus 2, n minus 3 you cancel here, okay. Of course you have a minus of this, right? So it is a minus minus plus here. So in this case what you have is in the denominator if you cancel here so it will become n minus 4 factorial. So this you can rewrite 2 minus 2, 2 n minus 3, okay, and then 2 n minus 4 factorial you can write.

So 2 n minus 3 will go, okay, and here n minus 1. So this 2 n minus 1 this 2 you can take it out here, write n minus 1. This you cancel here it still become n minus 2 factorial, 2 power n as it is, 2 2 goes, you have 1 into 2. So you have 2 factorial rather, okay, it becomes like that.

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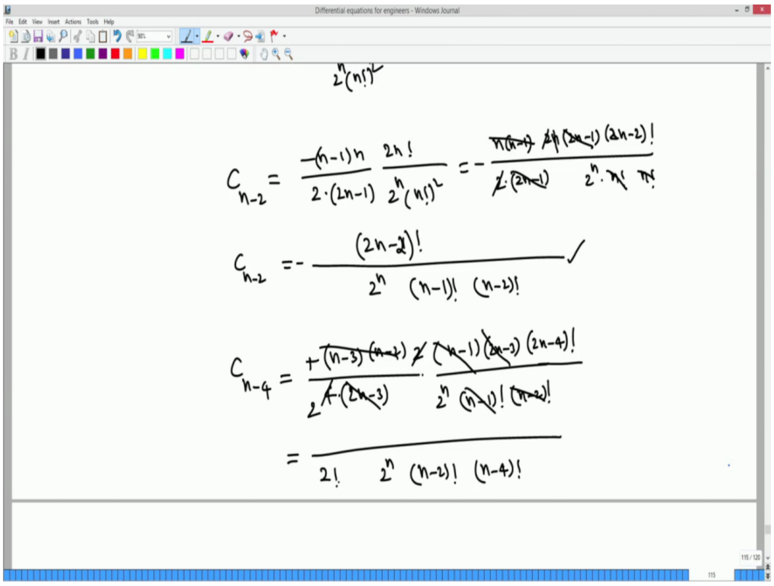

Then what is in the numerator? So we simply have 2 n minus 4 factorial. This is my C n minus 4, okay. So like this if you go on by induction you can write C n minus 2 k, how long you can go if n is even? If n is even so let us say 2 k, I can go up to k equal to n by 2, okay. If n is odd I can go only up to C 1. That means if n is odd you simply calculate n by 2 integral part of it, okay.

If it is n is 5, 5 by 2, so only 2. Integral part is that is 2, okay, like that. So you take like this if you write like this, this is the integral part. Bracket of n by 2. So k will go up to, k is from 0, 1, 2 up to n by 2 as an integral part, okay, (inti) (inti) integer part, okay, integer part is this.

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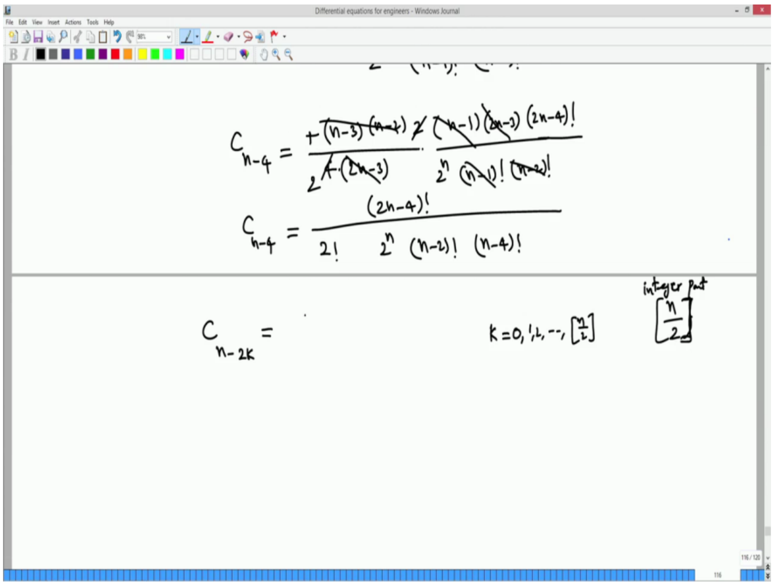

So by induction you can write 2 n minus. So you have a minus so when it is n minus 2 into 2 you have plus, so you have minus 1 power k, 2 n minus 2 k factorial divided by, this is going to be k factorial, 2 power n, n minus k factorial and n minus 2 k factorial. This is what you will get, okay. Now simply your solutions because I have already explained if you take C n which is same as the coefficient of P n, it has to be a same Legendre polynomial.

So the P n, okay, so let us say simply if n is even, okay, n is even what happens to your y 1 of x now? Y 1 of x is C 0, C 1, okay. Everything in terms of C k, okay. And C k I fixed now this number, whatever the coefficient of P n. Then you simply have n is even so n equal to 2 k, okay.

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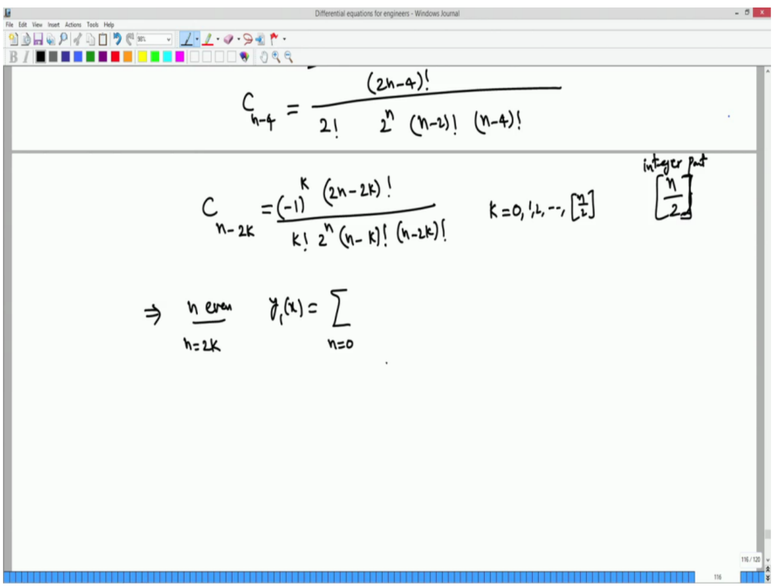

So if you do that so k is from 0 to n by 2. If n is even no problem, integer part also is n by 2, minus 1 power k, 2 n minus 2 k factorial divided by k factorial 2 power n, n minus k factorial, n minus 2 k factorial. This is what you will get into x power, if k equal to 0, what is this? If k 0 x power n, n minus every time. So when k equal to 0, x power n coefficient.

If k equal to 1, x power n minus 2 coefficient. So every time you have 2 k. So x power n minus 2 k. So this is your polynomial, okay. The way you fixed it when k equal to 0 this is exactly the coefficient of y 1 of x. Coefficient of x power n in y 1 of x is actually equal to coefficient of P n x power n in P n of x from the Rodrigues' formula, okay.

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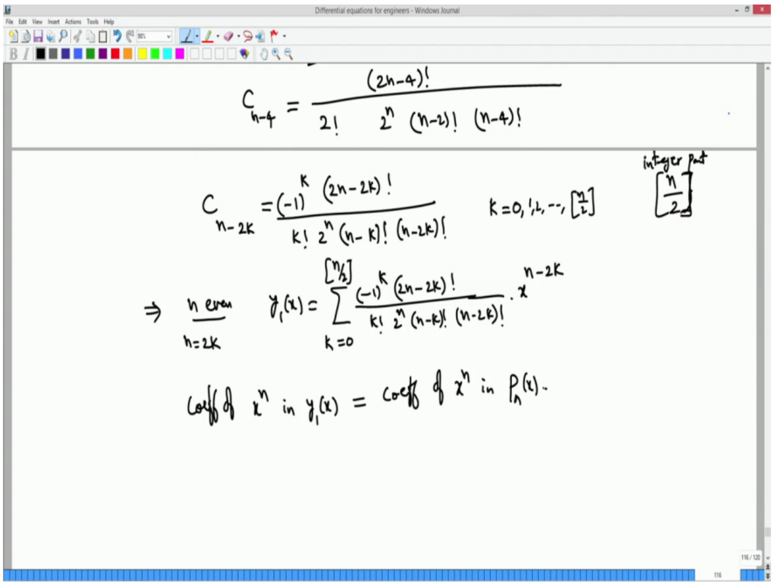

Both are solutions. These two are solutions. Y 1, P n x are solutions of Legendre equation. Y 1 x minus P n x is also solution of Legendre equation. What is this from this left hand side is now? If you have the same coefficient and you take the difference this will be n minus 1th degree. Equation with alpha equal to n. Is it possible? N minus 1th degree polynomial solution you will not get for the Legendre equation when alpha equal to n unless it is 0, okay.

That means it has to be 0 because either you have a series solution or a polynomial solution of degree n. Other solution is trivial solution 0, okay. Trivial solution is 0 so this has to be 0, okay. So that implies y 1 of x is nothing but your P n of x. So you have a formula. Now I can replace with y 1 as P n of x, okay. This is what if n is even.

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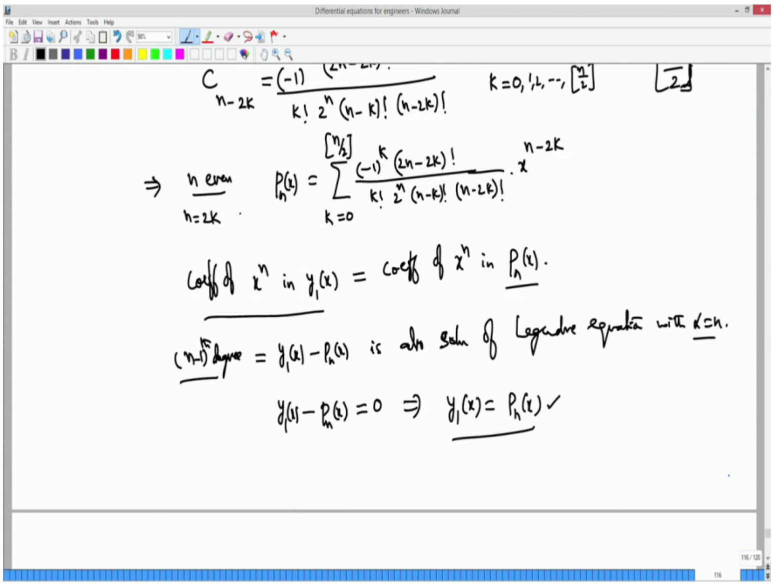

So if n is odd also it is the same thing. So n is anything. If n is odd it will repeat the same thing, 2 k plus 1. What you can do is you can go up to 2 k plus 1 minus 2 that is the first coefficient, okay, when you write this. N minus n is 2 k plus 1, okay. And put k equal to 0, n is odd that is the coefficient. Every time 2 minus so x power n minus 2, n minus 4, you come up to x, okay, because this is the coefficient of C 1 when it is odd. And finally so that is why you put this integral part n by 2.

So you do not make if n is odd, n by 2 so that is if it is n is 5, 5 by 2 is simply 2. So you will have two terms 5 3, 3 minus 2, so 1. So you have two more terms. So you go up to C 1, C 3, C 5. C 1 coefficient is x, x cube, x power 5. So that is the polynomial, okay. So in any case this is what is the case, okay. For any natural numbers you have P n as this. So this is the formula you can also use directly to calculate this Legendre polynomials, okay.

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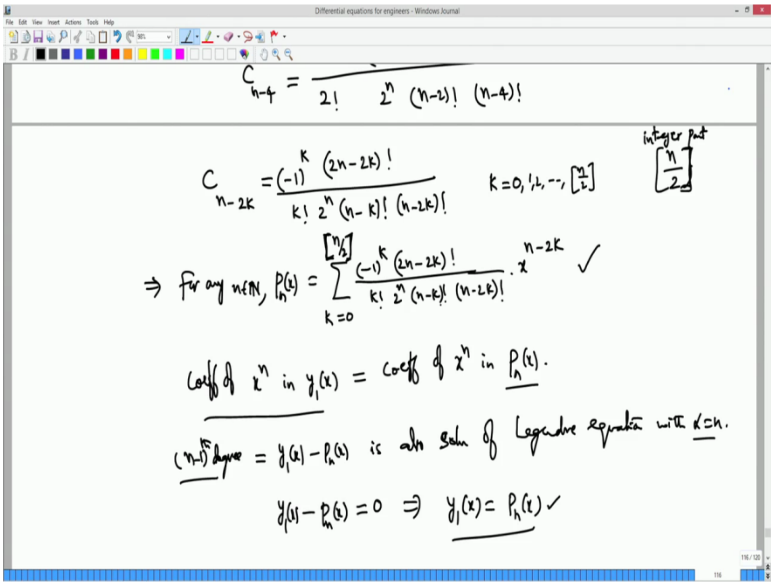

And one more property I give you now. It is just you have see that 1 I can write it as P 0 of x, x I can write it as P 1 of x, okay, Legendre polynomial, x square, okay. So n equal to 1 it is true, n equal to 0 it is true, n equal to 1 it is true. This polynomial 1 x I can write in terms of P 1, okay. What if I write x square? what is x square? So now what happens to x square?

So let us assume that x power n minus 1 is a linear combination of P 0, P 1, up to P n minus 1. So you see that 1 is linear combination of P 0, x power 0, okay. X is linear combination of P 1 that is actually x, right? So it is actually linear combination is 0 into 1 plus 1 into P 1, right? So that is again actually this is a linear combination of these two.

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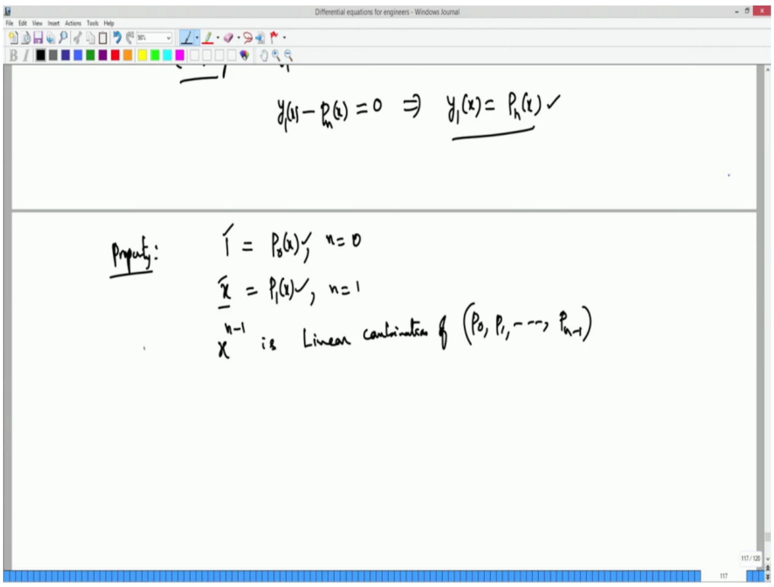

So this you assume by induction we can assume this. If you assume this we can now say x power n. What is x power n? How do I calculate this x power n? So you know that P n is a polynomial. P n of x is a polynomial and this is having x power n coefficient.

So coefficient also you know some and remaining x you already assumed that up to x power n are linear combination of P 0, P n minus 1, okay. And so you have x power n is simply linear combination of P 0 minus of linear combination of P 0, P n minus 1, okay, plus P n. This whole thing divided by 1 by C n, right?

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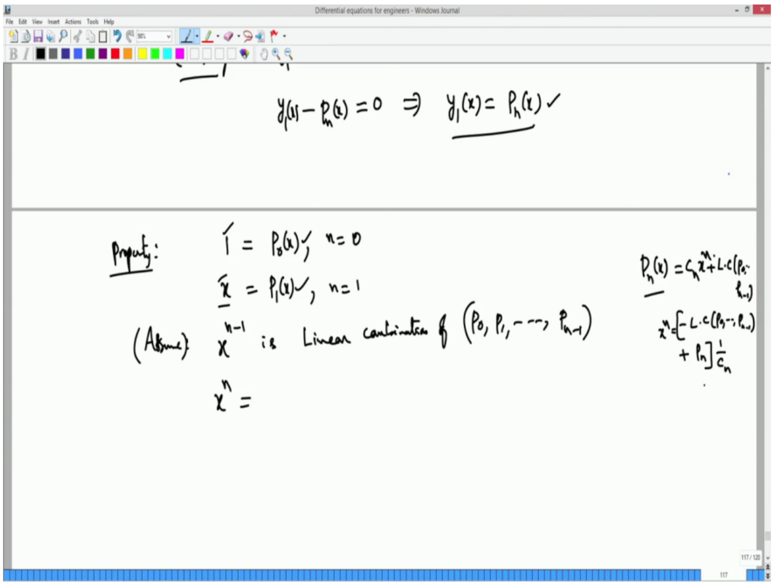

So what is this one actually? This is simply linear combination of P 0, P 1, P n, okay. So you can write like this. So x power n you can say simply P n by C n, okay, into C n by C n plus I should say minus, minus of 1 by C n times linear combination of P 0, P 1 up to P n minus 1. So this is nothing but linear combination of P 0, P 1, P n because P n minus 1, okay. So that is it. So this implies x power n is a linear combination of this. This is the proof, okay. It implies this is what you get.

So that means any polynomial of degree n is a linear combination of P 0, P 1 up to P n of x, okay. And these are all functions of x, these polynomials, okay. So this is the property 3. So I think there are many more properties of this polynomial which are really useful. So most important property is the orthogonal property which I have derived in the last video. So this is enough.

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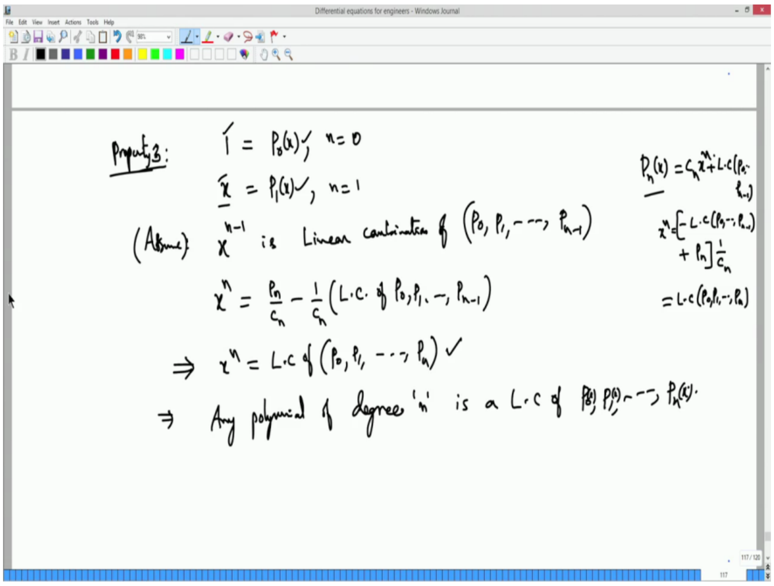

And move on to solve other equations when 0 is regular singular point, okay. So we have applied the power series method to the Legendre equation when 0 is an ordinary point, okay. So now we can look into some second order linear homogeneous equations where 0 is actually singular point, singular but it is a regular singular point. So in the next video we will do this Frobenius method to solve the second order homogeneous equation with 0 being (re) singular point but a regular one, okay.

So you know the definition of a regular singular point. So we will give the method. So such an equation if you have second order homogeneous equation with 0 being the regular singular point, so either this side or that side, x greater than 0 or x less than 0 we can find solutions. That is the method of Frobenius. So we will give that in the next video.