

Differential Equations for Engineers
Doctor Srinivasa Rao Manam
Department of Mathematics
Indian Institute of Technology Madras
Lecture 22
Legendre Polynomials

So in this video we will continue to look at the solutions of Legendre's equation and we have seen that this Legendre equation which is the second order linear homogeneous equation has a polynomial solution and series solution if you fix your alpha as positive integer, okay. If it is not positive integer you can see that both the solutions are series solutions, we will have full series, okay, y_1 and y_2 both are series solutions.

So we will continue to look at the solutions when the parameter alpha is a positive integer, okay. So we will see when n equal to 1 we can see that when n equal to 1 or rather n equal to 0, okay. So let me start with n equal to 0, n equal to 0 if you take so what is the solution? This is even integer so you can see that y_1 of x is actually. What happens when alpha equal to, not n equal to, so alpha equal to 0. Let us say alpha equal to 0.

What is alpha equal to 0? So alpha equal to 0 so you have 0 solution is 1, right? So when alpha equal to 0 that means that is corresponding to n equal to 1, okay. So n equal to 1 itself is 0 so alpha equal to 0. So what you get is y_1 of x is 1, okay.

(Refer Slide Time: 01:54)

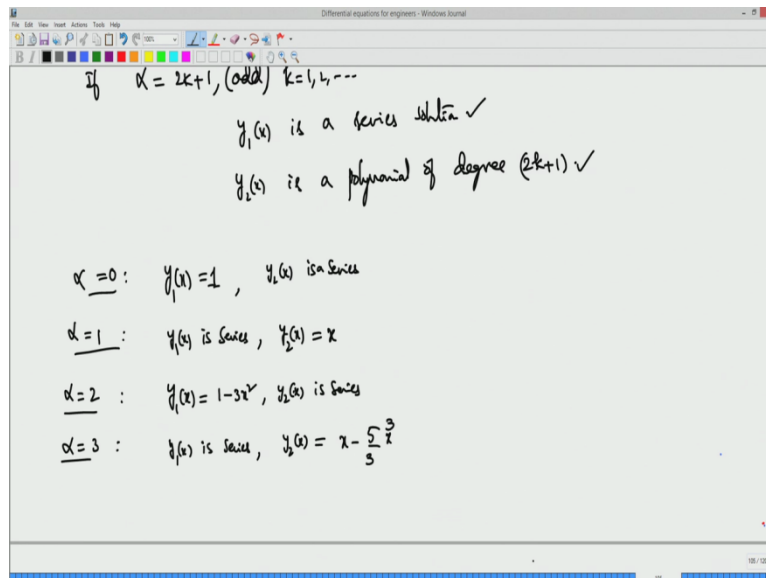
If $\alpha = 2k+1$, (odd) $k=1, 2, \dots$
 $y_1(x)$ is a series soln ✓
 $y_2(x)$ is a polynomial of degree $(2k+1)$ ✓
 $\alpha = 0$: $y_1(x) = 1$

And then alpha equal to 2 you see that either solution y_2 is series solution. And if alpha equal to 1, start with 1, so what you get is y_1 is series and y_2 is a polynomial solution. You can look at this y_2 . So when alpha is equal to 1. When alpha equal to 1 so n equal to 1. You can look at n equal to 1 you see that alpha equal to 1. What you get is all the terms involving alpha minus 1. So when alpha equal to 1 it will be 0. So you have y_2 is actually x , okay.

Like this you can get alpha equal to 2 and so on. In each case y_1 will be solution. You can say that I am just writing directly when alpha equal to 2, okay. Let us work out what is alpha equal to 2? So alpha equal to 1, alpha equal to 2, alpha equal to 3. So I can directly write so in this case it will be $1 - 3x^2$. Alpha equal to 3 you get y_2 . In this case y_2 is a series, y_2 of x is a series. In this case y_1 is a series and y_2 is actually alpha equal to 3 x minus 5 by 8 x^3 . Something like this.

Alpha equal to sorry it should be 3, 5 by 3, okay. When alpha equal to you see this is third order so third degree polynomial alpha equal to 3, okay. So you can see that $1 - 5x^3$, fine, y_2 at 1.

(Refer Slide Time: 04:04)

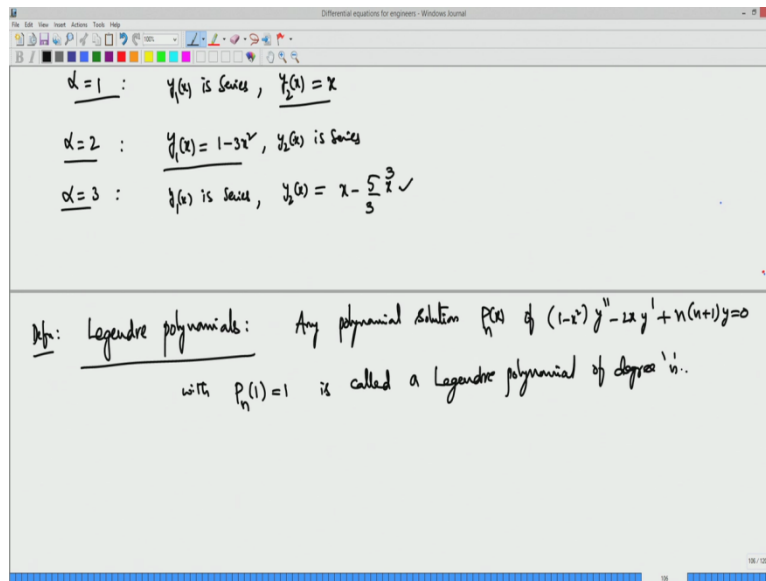


So you can see in each case when alpha equal to 0 so y_1 at 1 equal to 1 and y_2 at 1 equal to 1. But here y_1 at 1 when alpha equal to 2 if you want so Legendre polynomial is value at 1 but its value is equal to $1 - 3$. So that is going to be minus 2, okay. So you have to divide it with that number so that value y_1 at 1 should be 1, similarly here. So this is the polynomial

solution directly from the solutions you obtained y_1 and y_2 . Now we will define what is the Legendre polynomial.

The polynomial solutions Legendre polynomials. So what is the definition? So the definition is you consider any polynomial solution P_n of x for α equal to n , okay, α equal to let us say any polynomial solution P_n of x of the equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ with $P_n(1) = 1$. It is called Legendre polynomial of degree n .

(Refer Slide Time: 05:59)



So you can see here. So if you want here α corresponds to 0, you want Legendre polynomial P_0 of x will be equal to 1 whose value equal to 1. So that is the polynomial. So α equal to 1 you can see that P_1 of, okay, corresponds to α equal to 1. So α equal to n , n equal to 1 if you consider, P_1 of x is also value x because whose value at 1 equal to 1. But in this case α equal to 2, P_2 of x is actually equal to, when I put y_1 of 1 equal to 1 made it minus 2.

So if you multiply with 1 by minus 2 into 1 minus that is $3x^2 - 1$ divided by 2. So constant multiple of this y_1 is actually a solution. So whose value at 1 is equal to 1, okay. Similarly here P_3 of x we can also write, what you get at 1, 1 minus 5 by 3. So 3 minus 2 by 3 so you have 3 by 2, 5 by 3, 5 by 3, 5 by 3, so let me see. So you can do like this, okay.

(Refer Slide Time: 07:17)

$y_2(x)$ is a polynomial of degree n .

$\alpha = 0$: $y_1(x) = 1$, $y_2(x)$ is a series ✓ $P_0(x) = 1$

$\alpha = 1$: $y_1(x)$ is series, $y_2(x) = x$, $P_1(x) = x$ 1-5/3

$\alpha = 2$: $y_1(x) = 1 - 3x^2$, $y_2(x)$ is series, $P_2(x) = \frac{(3x^2 - 1)}{2}$

$\alpha = 3$: $y_1(x)$ is series, $y_2(x) = x - \frac{5}{3}x^3$ ✓

Def: Legendre polynomials: Any polynomial solution $P_n(x)$ of $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ with $P_n(1) = 1$ is called a Legendre polynomial of degree 'n'.

So I am not sure exactly this is 5 by 3 or not. So we will see later. Like this you can get your Legendre polynomials, okay. So in this case you need first of all you have to see whether this is a solution. I am not sure whether this 5 by 3 or 5 by 8, okay. Maybe 5 by 3 only. So if you do this y_2 at 1 equal to 1 will give me 1 minus 5 by 3, okay. So this should be equal to minus 2 by 3. So we divide this with this so we get P_3 of x as $\frac{3}{2} \left(\frac{5}{3}x^3 - x \right)$. Something like this, okay.

(Refer Slide Time: 08:06)

$\alpha = 0$: $y_1(x) = 1$, $y_2(x)$ is a series ✓ $P_0(x) = 1$

$\alpha = 1$: $y_1(x)$ is series, $y_2(x) = x$, $P_1(x) = x$

$\alpha = 2$: $y_1(x) = 1 - 3x^2$, $y_2(x)$ is series, $P_2(x) = \frac{(3x^2 - 1)}{2}$

$\alpha = 3$: $y_1(x)$ is series, $y_2(x) = x - \frac{5}{3}x^3$ ✓, $P_3(x) = \frac{3}{2} \left(\frac{5}{3}x^3 - x \right)$ $y_2(1) = 1$
 $1 - \frac{5}{3} = -\frac{2}{3}$

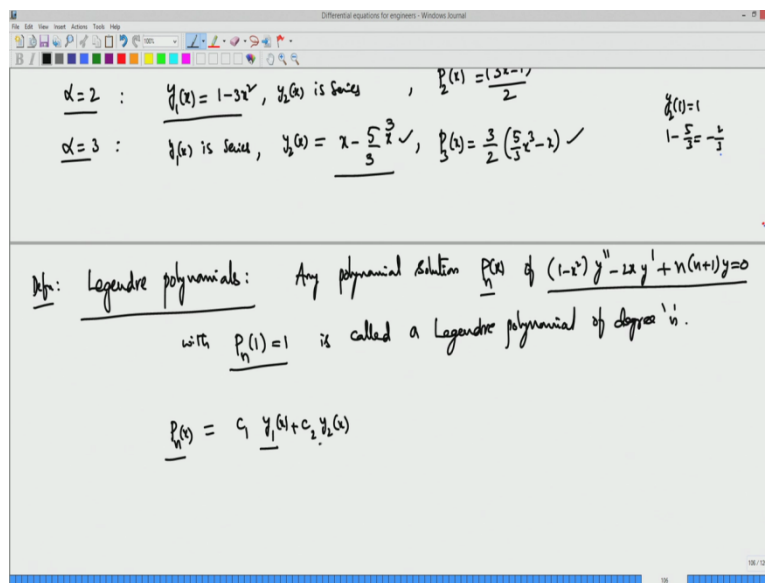
Def: Legendre polynomials: Any polynomial solution $P_n(x)$ of $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ with $P_n(1) = 1$ is called a Legendre polynomial of degree 'n'.

So you can get go on like this. So these are your Legendre polynomials. Basically these are solutions of this Legendre equation whose value at x equal to 1 should be 1, okay. So you can

easily say that because this is (sati) satisfying the equation. Any solution you know already that any solution $y(x)$ general solution is the linear combination of $y_1(x)$ plus $y_2(x)$, okay.

So if I choose this $y_1(x)$ as this polynomial solution $P_n(x)$ so where $y_1(x)$ is corresponding to $\alpha = n$, $y_2(x)$ is correspondence to $\alpha = -n$, two linearly independent solutions, okay. So you have a $C_1 y_1(x) + C_2 y_2(x)$, so C_1 and C_2 are arbitrary constants. You see that this is a polynomial if n is even, this is polynomial if n is odd, okay.

(Refer Slide Time: 09:00)



So in any case because $P_n(x)$ let us choose n is even. So if n is even, $P_n(x)$ is polynomial in any case. $P_n(x) = C_1 y_1(x) + C_2 y_2(x)$ is a polynomial, okay. But $y_2(x)$ is a series. That means a polynomial cannot be equal to a series. So that means C_2 has to be 0 in this case. Similarly when n is odd $P_n(x) = C_1 y_1(x) + C_2 y_2(x)$ is a polynomial. So this should be equal to $C_1 y_1(x)$ of x and left side is a polynomial, right hand side is a series.

So that is possible only if $C_2 = 0$. So in this case when $C_2 = 0$ so $P_n(x)$ when n is even is actually a constant multiple of $y_1(x)$ of x , it is some C_1 , okay. Similarly here when n is odd this is a constant multiple of $y_2(x)$ of x that is a polynomial. This is a polynomial, okay.

(Refer Slide Time: 10:06)

$\alpha=2$: $y_1(x) = 1-3x^2$, $y_2(x)$ is series, $P_2(x) = \frac{1-3x^2}{2}$, $P_2(1)=1$
 $\alpha=3$: $y_1(x)$ is series, $y_2(x) = x - \frac{5}{3}x^3$, $P_3(x) = \frac{3}{2}(\frac{5}{3}x^3 - x)$, $1 - \frac{5}{3} = -\frac{2}{3}$

Def: Legendre polynomials: Any polynomial solution $P_n(x)$ of $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ with $P_n(1) = 1$ is called a Legendre polynomial of degree 'n'.

$P_n(x) = c_1 y_1(x) + c_2 y_2(x)$
n even: $P_n - c_1 y_1(x)$ is a polynomial $\Rightarrow c_2 = 0 \Rightarrow P_n = c_1 y_1(x)$
n odd: $P_n - c_2 y_2(x)$ is a polynomial $\Rightarrow c_1 = 0 \Rightarrow P_n = c_2 y_2(x)$

So this Legendre polynomial is actually constant multiple of a polynomial solution of Legendre equations, okay. So how do I find this P_n instead of looking at this series, every time y_1 and then you fix your value at x equal to 1 as 1 and then you decide your P_n . Instead of that you can derive a formula for P_n , okay, formula for P_n . So it is called Rodrigues' formula. So Rodrigues' formula, okay. So we will derive this today.

(Refer Slide Time: 10:51)

n odd: $P_n - c_2 y_2(x)$ is a polynomial $\Rightarrow c_1 = 0 \Rightarrow P_n = c_2 y_2(x)$

Formula for $P_n(x)$: (Rodrigues formula)

So what is the idea? So this is actually, P_n of x is actually equal to some constant times, that constant you will see that 1 divided by $2^n n!$ and n derivatives of this function $x^2 - 1$ power n , okay. So that is what is the actual formula. So this is valid

between minus 1 to 1 because these are the polynomials. It is actually a polynomial which is defined everywhere. As a solution of Legendre equation it is actually valid here, okay.

So as a polynomial it is actually valid everywhere. So we can say that x belongs to \mathbb{R} . But as a solution of Legendre equation, this is the solution of the equation only in the range minus 1 to 1, okay. So we will see how you derive this Rodrigues' formula.

(Refer Slide Time: 11:45)

formula for $P_n(x)$: Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} ((x^2 - 1)^n), \quad x \in \mathbb{R}.$$

So we consider this n derivatives, this part. So so in this let u of x equal to x square minus 1 power n . So in this you take this as a function of x . You can differentiate this because it is a polynomial, okay, so what if you differentiate this u dash $d u$ by $d x$ equal to n into x square minus 1 power n minus 1 into $2 x$.

So this implies x square minus 1 times u dash of x equal to n times $2 n x$, okay, $2 n x$ x square minus 1 power n , okay. So this is actually your u . So I can write this as u of x . So this is what you have.

(Refer Slide Time: 12:46)

$n^{n-1} = 2^n n! \frac{d^n}{dx^n}$

Let $u(x) = (x^2 - 1)^n$

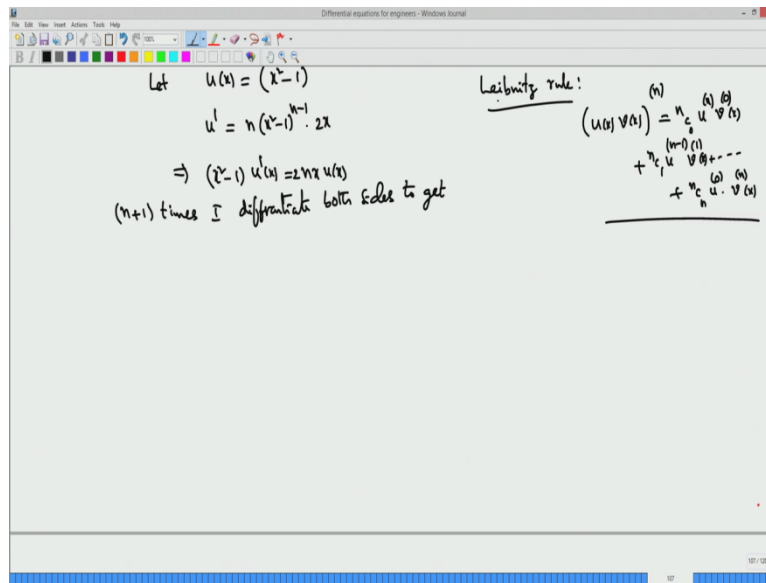
$$u' = n(x^2 - 1)^{n-1} \cdot 2x$$

$$\Rightarrow (x^2 - 1) u'(x) = 2nx u(x)$$

So u is polynomial and these are the product of polynomials both left and right hand side. So you can differentiate this $n + 1$ times, $n + 1$ times I differentiate, okay, both sides to get, what you get here? So this is the formula. If you apply $u \times v \times x$, you differentiate n times, okay. So if you write like this, this is the Leibnitz formula. This is actually in bracket if I write this is n derivatives, $n \ C \ 0$ u n derivatives and v 0 derivatives. So that means nothing, okay.

And plus $n \ C \ 1$ u $n - 1$ derivatives and then you have v 1 derivative of x . Like this you go on like a binomial formula so you can get $n \ C \ n$ u times n , u times here 0 , okay. So you have $n - n$. So that is 0 derivative with u and v n derivatives of x . So this is the formula. This is the Leibnitz differential rule, okay, Leibnitz rule.

(Refer Slide Time: 14:11)



You apply this here. If I differentiate both sides this is my u this is my v. Similarly here x and u x, this has two functions here, okay. So if you apply start with x square, n plus 1 C 0 that is 1 into u square x square minus 1. And if I differentiate u, n plus 1 times then I get n plus 2 of x, okay. So do not write this coordinate. So plus, now you differentiate n plus 1 C 1, you differentiate this you get 2 x and this will be I do only n times (differ) derivatives. So you have n plus 1.

Plus one more time you can do so this time n plus 1 C 2, n plus 1 into n divided by 2, okay. So divided by 2 so you have 2 x now I can differentiate, okay. Two times I am differentiating x square minus 1 so that will give you 2. So 2 and here u n derivatives, okay. So next time if you do n plus 1 C 3, three times if I differentiate x square minus 1 that will be 0 and so I do not have those other terms, okay.

So this is what happens for the left hand side. This should be equal to you can repeat the same thing with 2 n, n plus 1 so x times u n plus 1 derivatives plus 2 n differentiate n plus 1, n plus 1 C 1, differentiate x 1 as 1 and you differentiate u, n times. This is what you have both sides if you get.

(Refer Slide Time: 16:04)

Let $u(x) = (x^2 - 1)$
 $u' = 2x$

$\Rightarrow (x^2 - 1) u^{(n+1)} = 2nx u^{(n)}$
 (n+1) times I differentiate both sides to get

$$(x^2 - 1) u^{(n+2)} + (n+1) 2x u^{(n+1)} + \frac{(n+1)n}{x} u^{(n)}$$

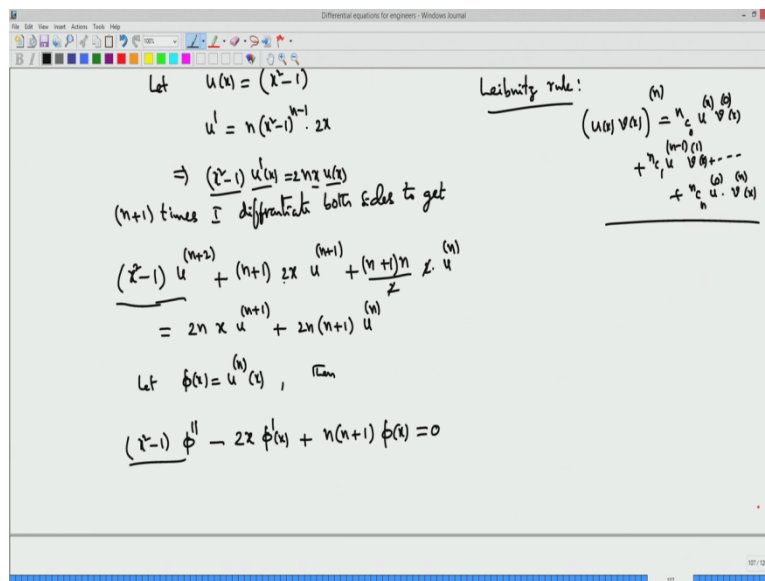
$$= 2nx u^{(n+1)} + 2n(n+1) u^{(n)}$$

Leibniz rule:
 $(u(x)v(x))^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)} v^{(n-k)}$
 $+ \binom{n}{1} u^{(1)} v^{(n-1)} + \dots + \binom{n}{n} u^{(n)} v^{(0)}$

So this will give you so if you choose this u_n . Let simply consider ϕx as you u_n of x , x derivatives of x , okay. Then what happens to your ϕx ? So what you get is $x^2 - 1$ ϕ , see two derivatives are left. Already n derivatives I have taken in ϕ so you have $n + 2$. So ϕ double dash plus, and you can see that $2nx$ so n^2 , n^2 , both sides will go. So this n^2 , n^2 and this n^2 , n^2 , okay. So two n^2 .

So if you see this what you get is so you can see that minus $2nx$, actually $2x \phi$ dash of x plus n into $n + 1$ into ϕx . Let us not do this, $n + 1$ into n . So this is anyways close. So if you see that this is what you will get equal to 0. So how do we see this? This is the first term so you can easily see this here and then u_n plus 1 derivatives. You have these two terms, $2nx$ plus 1 and $2nx$. So two n goes and what we are left with is only this, okay.

(Refer Slide Time: 17:40)



So what you get is actually plus, okay. This is plus, okay, so this is plus. Now u^n , u^n is n into n plus 1 and $2n$ into n plus 1. So when it comes this side, this if you take other side this will be minus, okay. This is what is you get. So this is nothing but your Legendre equation. So $\phi'' - 2x\phi' + n(n+1)\phi = 0$, okay. So that means and you see that ϕ , what is your ϕ ? ϕ is the n derivatives of this function u , okay.

What exactly your ϕ is? n derivative, so $\frac{d^n}{dx^n} (x^2 - 1)^n$ is a polynomial solution. And if I can fix this value $\phi(1)$, what is the $\phi(1)$? If you can directly calculate you can calculate $\phi(1)$, n derivatives if you do you can see that is going to be so $2n$ plus 1. So this is the polynomial for degree $2n$. u of x is actually $(x^2 - 1)^n$.

This if you differentiate n times, u^n of x is actually $2n$, $2n$ plus 1, right? So how do I put this? n into, so we will see that this is actually whose value at 1. If you directly look at this calculation you see that how do I see this equal to, so what we do is here again we apply this to get this we write $(x - 1)^n$, $(x + 1)^n$. so this I can write, okay. Now I differentiate n times both sides. So this whole thing if I differentiate n times, okay.

So this will give me $n C 0$, this is as it is, $(x - 1)^n$. You differentiate this you will get n times, okay, this is as it is and you differentiate this n times. If you differentiate you get n factorial, okay, plus $n C 1$, so $n C 1$ if you do differentiate this n into $(x - 1)^{n-1}$, when you differentiate this $n - 1$ times, okay. So you have $n - 1$ derivatives of $(x + 1)^n$ whose $n - 1$ derivative, okay.

(Refer Slide Time: 20:57)

Let $\phi(x) = u^{(n)}(x)$, then

$$(x-1)\phi'' + 2x\phi' - n(n+1)\phi = 0$$

$$\Rightarrow (1-x)\phi'' - 2x\phi' + n(n+1)\phi = 0$$

$\phi(x) = \frac{d^n}{dx^n}((x-1)^n)$ is a polynomial solution.

$\phi(x) =$

$u(x) = (x-1)^n$
 $u(x) = [(x-1)^n (x+1)^n]^{(n)}$
 $= (x-1)^n n! + n n(n-1) \dots (x+1)^{n-1}$

So instead you can also do it another way. So you differentiate you keep this as it is. So $n \leq 0$ plus 1 power n and you differentiate this n times. So you will get simply n factorial. And other things every time you differentiate $n \leq 1$, you differentiate this 1 time. So you get n into x minus 1 power n minus 1 and if you differentiate n , this is n th degree polynomial if you differentiate n minus 1 times, okay, you will get into some constant.

But it has like that you get every time. Every time you get, finally you differentiate $n \leq n$, okay. this n derivatives will have x minus 1 term.

(Refer Slide Time: 22:00)

Let $\phi(x) = u^{(n)}(x)$, then

$$(x-1)\phi'' + 2x\phi' - n(n+1)\phi = 0$$

$$\Rightarrow (1-x)\phi'' - 2x\phi' + n(n+1)\phi = 0$$

$\phi(x) = \frac{d^n}{dx^n}((x-1)^n)$ is a polynomial solution.

$\phi(x) =$

$u(x) = (x-1)^n$
 $u(x) = [(x-1)^n (x+1)^n]^{(n)}$
 $= (x+1)^n n! + n n(n-1) \dots (x-1)^{n-1}$
 $+ \dots + n^n$

Similarly $n C n$ you do not differentiate anything here, okay. You differentiate only this part so you will have n factorial and this part you keep as it is, x minus 1 power n . So you see that first term will have x minus 1, second term will have x minus 1 square up to last term will have x minus 1 power n . So you see that n th derivative at 1 this is actually your ϕ at 1 equal to $2^n n!$, n factorial because all the other terms are remaining x minus 1 and you put x equal to 1 this will be 0.

So this is how you can see that $\phi(1)$ is actually $2^n n!$. So this implies my P_n of x is a polynomial solution whose value should be equal to 1. So if I choose this $\phi(x)$ by $\phi(1)$ then $P_n(1)$ equal to $\phi(1)$, $\phi(1)$ cancel, it is equal to 1. So this is exactly you have this form d^n by $d x$ power n x square minus 1 power n , okay. So this is your Rodrigues' formula for every x , okay.

As a polynomial it is defined for all values of x otherwise it is a solution is valid as a polynomial solution, solution of Legendre equation in between minus 1 to 1. This is how you see Rodrigues' formula.

(Refer Slide Time: 23:25)

$$\phi(x) = \frac{d^n}{dx^n} ((x^2-1)^n) \text{ is a polynomial solution.}$$

$$\phi(1) = 2^n n!$$

$$\Rightarrow P_n(x) = \frac{\phi(x)}{\phi(1)} = \frac{1}{2^n n!} \frac{d^n}{dx^n} ((x^2-1)^n), \quad -1 < x < 1$$

$$u(x) = (x^2-1)^n$$

$$u'(x) = [2x(x^2-1)]^{n-1}$$

$$= 2x(x^2-1)^{n-1} + \dots + 2x(x^2-1)^{n-1}$$

$$\phi(1) = u^{(n)}(1) = 2^n n!$$

So you can from now from this formula you can calculate your P_0 , P_1 and so on, okay. So you can easily calculate. So x square minus 1 power n you take, fix your n and you differentiate n times. Now what you get is this one, okay. So if you do this we can see that P_0 of x is 1, P_1 of x is x . From this formula you can easily see P_2 of x is $3/2 x^2 - 1/2$. P_3 of x equal to $5/2 x^3 - 3/2 x$.

P 4 if you actually calculate you can see that 35 by 8 x power 4 minus 15 by 4 x square plus 3 by 8. So like this you can get all your Legendre polynomial solutions, okay. These are the solutions of the Legendre equation when alpha equal to n.

(Refer Slide Time: 24:24)

$P_0(x) = 1, \quad P_1(x) = x$
 $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
 $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \quad \checkmark$

So why do we need these solutions? You have a polynomial solution when you fix the parameter alpha equal to n. Why do we study this Legendre polynomials? Because these are the polynomials have certain properties, very important properties are actually they are orthogonal to each other. That means these polynomials P_0 , P_1 if you integrate you take P_0 and so let us say P_n of x and P_m of x , the dot product of this.

So simple as a vector dot product here you take the functions dot product is between minus 1 to 1 dx as a solution of this Legendre polynomial whose dot product, okay, this P_n dot P_m , okay. So dot product you can write like this, okay, if I differentiate. So the dot product you can write like this. If you write this is the dot product of the polynomial, okay, dot product of P_n , P_m .

So this is actually you will see that 0 if m is not equal to n and they are different, they are actually orthogonal to each other. So if it is same, if m equal to n then you have some number, okay. So you will see that is actually 2 divided by $2n + 1$.

(Refer Slide Time: 25:45)

$P_0(x) = 1, \quad P_1(x) = x$
 $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
 $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$ ✓

dot product
 $\langle P_n, P_m \rangle = \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$

So why do we need this? When you study partial differential equation in higher dimensions you convert this, you work with spherical coordinates instead of Cartesian coordinates. You convert the Laplace equation in terms of spherical coordinates. It becomes this Legendre equation you need. That is where if you actually solve the Legendre equation you get a polynomial solutions.

You may have to use and then there to get the constants have to use this dot product, this orthogonal property. This is called orthogonal property of Legendre polynomials. There are many properties for this function so we give the main important thing is this, okay, if they are different 0 otherwise it is 2 divided by 2 n plus 1 as fixed number, okay, P n square and m equal to n integral P n square d x value is this.

(Refer Slide Time: 26:37)

$P_0(x) = 1, P_1(x) = x$
 $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
 $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$ ✓

dit point
 $\langle P_n, P_m \rangle = \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$
Orthogonal property ✓

So we will show this property. This is the important property you can show. There are other very important properties but we need only this or some important we will see that as an exercise, okay. Property 1 is this. So how do I know that, so $P_n(x)$ satisfying the differential equation of Legendre equation, okay. So $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$ because P_n is satisfying that, okay.

Then and $P_n(x)$ is satisfying the other equation $P_m'' - 2xP_m' + m(m+1)P_m = 0$, this is the difference equation with n equal to m . So $m(m+1)P_m'' - 2xP_m' + m(m+1)P_m = 0$. What I do is I multiply this equation with $P_m(x)$ and I multiply this equation with $P_n(x)$ of x both sides, okay.

(Refer Slide Time: 27:48)

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
 $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$ ✓

$\langle P_n, P_m \rangle = \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2m+1}, & \text{if } m = n \end{cases}$

Orthogonal property ✓

Property 1: $(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \times P_n(x)$
 $(1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0 \times P_m(x)$

So if I do this, this I multiply with P_m of x which is equal to 0. This I multiply with P_n of x which is equal to 0, okay. And then you take the difference when n is not equal to m you take the difference. When you take the difference what you get is n into n plus 1 minus m into m plus 1, lost terms into $P_n P_m$, okay.

This is equal to what is the difference here? $P_n P_m$ so what you get is 1 minus x square into $P_n P_m$ double dash of x into P_m of x minus $2x P_n$ dash into P_m minus 1 minus x square P_m double dash P_n of x minus, minus plus, okay, $2x P_m$ dash, okay, P_m dash of x into P_n . This is what you have, okay.

(Refer Slide Time: 29:18)

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
 $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$ ✓

$\langle P_n, P_m \rangle = \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2m+1}, & \text{if } m = n \end{cases}$

Orthogonal property ✓

Property 1: $[(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x)] P_m(x) = 0$
 $[(1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x)] P_n(x) = 0$

$n \neq m$: $[n(n+1) - m(m+1)] P_n P_m = (1-x^2) P_n'' P_m - 2x P_n' P_m - (1-x^2) P_m'' P_n + 2x P_m' P_n$

So now you try to integrate both sides. I have integrate this is a constant which is non zero because n is not equal to m. If you integrate from minus 1 to 1 d x both sides you try to integrate from minus 1 to 1, okay. This is your function d x. What you see is, so I can rewrite this as minus 1 to 1, 1 minus x square. First two terms I can write 1 minus x square P n dash whole dash. What is this one? What is this one, 1 minus x square P n double dash that is the first term and minus 2 x P n dash, this into with P m.

This is my this row, first two terms is this, minus second two terms also you can write the same way. So 1 minus x square P m dash whole dash into P n d x. Now one you integration by parts. One integral you integrate by parts. So this is your full integrant. The first term you do the integration by parts so you will get 1 minus x square P n dash P m with the limits minus, so integral minus 1 to 1, 1 minus x square P n dash P m dash, okay. I will do the same with the other things.

So minus 1 minus x square P m dash P n, this is from minus 1 to 1, and then this minus minus plus you get minus 1 to 1, 1 minus x square P m dash and P n dash. You see that these are same integrals with plus or minus. This gets cancelled and this is what is the remaining. So when you put x equal to plus or minus 1 this term will be 0 and this term will also be 0. So this together will becomes 0. This is what happens when n is not equal to m.

(Refer Slide Time: 31:47)

The image shows a handwritten derivation in a software window titled "Differential equation for engineers - Windows Journal". The derivation starts with the differential equation for Legendre polynomials:

$$[(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x)]P_m(x) = 0$$

For $n \neq m$, the derivation proceeds as follows:

$$\frac{n(n+1) - m(m+1)}{6} \int_{-1}^1 P_n P_m dx = \int_{-1}^1 \frac{[(1-x^2)P_n''(x)P_m(x) - 2xP_n'(x)P_m(x)]}{-(1-x^2)P_n''(x)P_m(x) + 2xP_n'(x)P_m(x)} dx$$

$$= \int_{-1}^1 \left[\frac{[(1-x^2)P_n']}{P_n} P_m - [(1-x^2)P_m']}{P_m} P_n \right] dx$$

$$= \left[(1-x^2)P_n' P_m \right]_{-1}^1 - \int_{-1}^1 (1-x^2)P_n' P_m' dx$$

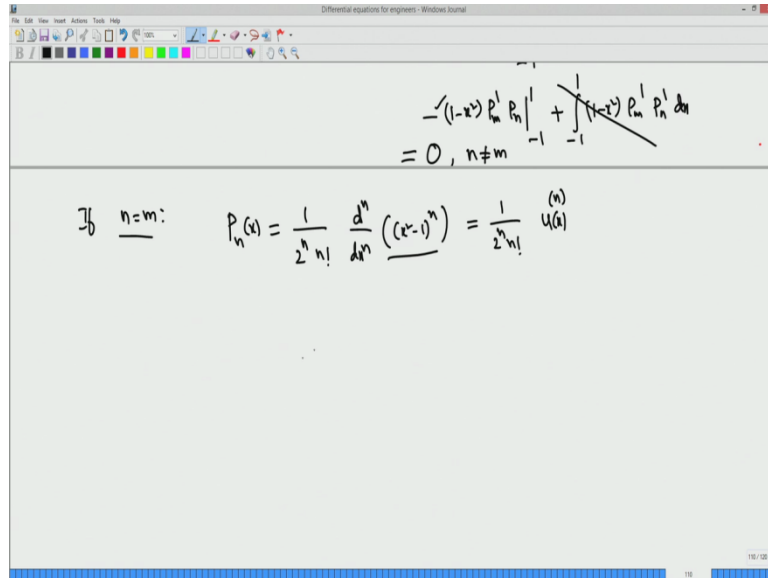
$$- \left[(1-x^2)P_m' P_n \right]_{-1}^1 + \int_{-1}^1 (1-x^2)P_m' P_n' dx$$

$$= 0, \quad n \neq m$$

When n equal to m, okay, if n equal to m so you cannot use the same procedure above. What we do is so we use the Rodrigues' formula, P n of x, n equal to m we consider P n of x is you have seen that 2 power n, n factorial into, that n derivatives of d n by d x power n of x square

minus 1 power n, okay. So you call this some u, okay. So you can write like 1 by 2 power n, n factorial, u of x, okay, for which you have n derivatives. You can write like this, n derivatives for u, u is this function, okay.

(Refer Slide Time: 32:38)



$$-\int (1-x)^{n-1} P_n' P_n dx + \int (1-x)^{n-1} P_n' P_n dx = 0, n \neq m$$

$$\text{If } n=m: P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^n n!} u^{(n)}(x)$$

So if you write like this you want to know what is this $P_n P_m$. So that means P_n square of x d x . So this is equal to minus 1 to 1. You can directly calculate using this formula. You have 1 by 2 power n, n factorial square into u n of x into u of x two times d x, okay. So this is just a constant. You can take it out minus 1 to 1. So let us evaluate this integral. This integral if you evaluate n derivatives of x, n derivatives of x d x. You just do the integration by parts, what happens, u n minus 1 of x, okay.

This let us say this I integrated, okay. I bring n derivatives here, okay. So what is the integration by parts? First you write u n minus 1. This is derivative of this is this, okay. So this into u n of x and you have minus 1 to 1 minus, what exactly I did is, I take one derivative, bring it to here, okay.

So one derivative I bring here to here so when you do that you have u n minus 1 of x and then you have u n plus 1 of x d x, okay. And you know that u n minus 1 you see that what is u n minus 1? So n derivatives of this is this which is nth degree polynomial, okay, or rather n derivatives of x square minus 1 power n. This u x is nth degree polynomial in x square, okay.

(Refer Slide Time: 34:38)

$$\text{If } n=m: \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^n n!} u^{(n)}$$

$$\int_{-1}^1 P_n(x) dx = \frac{1}{(2^n n!)^{n+1}} \int_{-1}^1 u^{(n)} \cdot u'(x) dx$$

$$\int_{-1}^1 u^{(n)} u'(x) dx = \frac{u^{(n+1)}(x)}{n+1} \Big|_{-1}^1 - \int_{-1}^1 u^{(n)}(x) u(x) dx$$

And when you differentiate $n-1$ times you will still have x^2-1 . And you put $n-1+1$ that will be 0, okay. So what you have when you bring one derivative you have $n-1$ to 1 this, okay. You (div) integrate one more time this will become plus, okay. So you do the same thing, same technique.

So if you do the same technique so what you get is you bring one more derivative here to here. So you have u^{n-2} of x , u^{n+1} of x , this is from -1 to 1 . Again $n-2$ derivatives of you x will have x^2-1 . So this term also will be 0.

(Refer Slide Time: 35:26)

$$\text{If } n=m: \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^n n!} u^{(n)}$$

$$\int_{-1}^1 P_n(x) dx = \frac{1}{(2^n n!)^{n+1}} \int_{-1}^1 u^{(n)} \cdot u'(x) dx$$

$$\int_{-1}^1 u^{(n)} u'(x) dx = \frac{u^{(n+1)}(x)}{n+1} \Big|_{-1}^1 - \int_{-1}^1 u^{(n)}(x) u(x) dx$$

$$= - \frac{u^{(n+1)}(x)}{n+1} \Big|_{-1}^1$$

Minus minus now plus minus 1 to 1 u n minus 2 of x. I brought one derivative here so you have u n plus 2 of x d x. Like this you can go on, okay. So once it is placed one derivative you bring it (plu) minus two derivative you bring is plus. So it is like minus 1 square.

Two derivatives when I bring I have minus 1 comes out two times. Like that you can go on so you finally end up, this is same as minus 1 power n. I take all the n derivatives here to here so that is 0 derivatives here now and you have n plus n that is 2 n derivatives of x d x.

(Refer Slide Time: 36:15)

The image shows a handwritten derivation of the Leibniz rule for the nth derivative of an integral. The derivation is as follows:

$$\int_{-1}^1 u^{(n)}(x) u^{(n)}(x) dx = \frac{u^{(n-1)}(x) u^{(n)}(x)}{-1} - \int_{-1}^1 u^{(n-1)}(x) u^{(n+1)}(x) dx$$

$$= -\frac{u^{(n-1)}(x) u^{(n)}(x)}{-1} + (-1) \int_{-1}^1 u^{(n-1)}(x) u^{(n+1)}(x) dx$$

$$\vdots$$

$$= (-1)^n \int_{-1}^1 u(x) u^{(2n)}(x) dx$$

What is u? U is this, 2 n derivatives means you simply have it is a polynomial of degree x power 2 n plus something, right? This is the polynomial u of x. If you differentiate 2 n times what you get is simply 2 n factorial.

(Refer Slide Time: 36:35)

$= 0, n \neq 1$

If $n=m$: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^n n!} u^{(n)}$

$\int_{-1}^1 P_n(x) dx = \frac{1}{(2^n n!)^2} \int_{-1}^1 u^{(n)}(x) \cdot u(x) dx$

$\int_{-1}^1 u^{(n)}(x) u(x) dx = \left[\frac{u^{(n-1)}(x)}{1} u(x) \right]_{-1}^1 - \int_{-1}^1 u^{(n-1)}(x) u'(x) dx$

$= - \left[\frac{u^{(n-1)}(x)}{1} u(x) \right]_{-1}^1 + \int_{-1}^1 u^{(n-1)}(x) u'(x) dx$

$u(x) = x^{2n} + \dots$
 $u'(x) = 2nx^{2n-1}$

So what I get is this one minus 1 power n 2 n factorial, okay, and then integral minus 1 to 1. What is u x? X square minus 1 power n d x, okay. So this minus 1 power n if you take it inside, 2 n factorial minus 1 to 1, 1 minus x square power n d x. Now this you can evaluate in many ways. One way is minus 1 to 1, 1 minus x square power n d x.

If you put x equal to sin theta what happens here is 0, x equal to minus 1 this becomes x equal to sin theta if we put, 1 minus sin square that is cos square theta, cos 2 n, okay, cos 1 minus sin square theta cos square cos 2 n theta, d x is cos theta d theta. So cos power 2 n plus 1 theta d theta, okay. That is what you have.

(Refer Slide Time: 37:44)

$\therefore = (-1)^n \int_{-1}^1 u(x) u'(x) dx$

$= (-1)^n 2n! \int_{-1}^1 (x^2-1)^n dx$

$= 2n! \int_{-1}^1 (1-x^2)^n dx$

$x = \sin \theta$
 $\int_{-1}^1 (1-x^2)^n dx = \int_0^{\pi/2} \cos^{2n+1} \theta d\theta$

So what is your this one, minus pie by 2. So that is because this is minus pie by 2 to pie by 2 and x is minus 1, theta is minus pie by 2. When theta is x equal to 1, sin theta is pie by 2. So because this is even function of theta so you can have 2 times 0 to pie by 2 cos 2 n plus 1 theta d theta.

So if you know the formula for this you can apply directly which is going to be 2 n into n factorial square divided by 2 n plus 1 factorial. So this is what is the formula, okay. This is the formula for this integral. This is one way.

(Refer Slide Time: 38:29)

The image shows a handwritten derivation of the integral $\int_{-1}^1 (1-x^2)^n dx$. The steps are as follows:

$$\begin{aligned} &= (-1)^n \int_{-1}^1 u(x) u'(x) dx \\ &= (-1)^n 2n! \int_{-1}^1 (x^2-1)^n dx \\ &= 2n! \int_{-1}^1 (1-x^2)^n dx \end{aligned}$$

On the right side, a substitution is shown:

$$\begin{aligned} \int_{-1}^1 (1-x^2)^n dx &= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} \theta d\theta \\ &= 2 \int_0^{\pi/2} \cos^{2n+1} \theta d\theta \\ &= \frac{2 (\pi^n n!)}{(2n+1)!} \end{aligned}$$

Another way is you can see that minus 1 to 1, 1 minus x square power n d x is actually this is again even function so you have 2 times 0 to 1, 1 minus x square power n d x. So this is actually if you replace x square with t so what you get is 0 to 1, 1 minus t power n, okay, into t power minus half d t. So this you can write like 2 times integral 0 to 1, 1 minus t power n plus 1 minus 1 as n, t power minus half as 1 minus half d t. This is exactly your beta function.

So this is actually 2 times beta of half and n plus 1. This is 2 times Gamma half Gamma n plus 1 divided by Gamma n plus 1 plus half. If you use this formula you get similar thing. So you expand this Gamma. Gamma half is root pie and this you use the formula Gamma n plus 1 is actually equal to Gamma alpha plus 1 is as alpha into Gamma alpha.

(Refer Slide Time: 39:54)

The image shows a software window titled "Differential equations for engineers - Windows Journal". The handwritten work is as follows:

$$= \frac{2 (2^n n!)^2}{(2n+1)!}$$

$$\int_{-1}^1 (1-x^2)^n dx = 2 \int_0^1 (1-x^2)^n dx$$

$$= 2 \int_0^1 (1-t)^n t^{-1/2} dt \quad x=t$$

$$= 2 \int_0^1 (1-t)^{n-1} t^{-1/2} dt$$

$$= 2 B\left(\frac{1}{2}, n+1\right) = 2 \frac{\Gamma(\frac{1}{2}) \Gamma(n+1)}{\Gamma(n+\frac{1}{2})}$$

Below this, there is a note: $\Gamma(k+1) = k \Gamma(k)$ with an arrow pointing to the right.

If you use this formula and expand this Gammas you will see exactly same thing. So that I will leave it (exem) for exercise. So if you use that so what you end up is finally we will have $2 n$ factorial into 2 into $2 n$, n factorial square divided by $2 n$ plus 1 factorial. So this is exactly 2 divided by $2 n$ plus 1 into $2 n$, n factorial square.

(Refer Slide Time: 40:26)

The image shows a software window titled "Differential equations for engineers - Windows Journal". The handwritten work is as follows:

$$= - \frac{u(x)}{n+1} \Big|_{-1}^1 + (-1)^n \int_{-1}^1 u(x) u'(x) dx$$

$$\vdots$$

$$= (-1)^n \int_{-1}^1 u(x) u^{(2n)}(x) dx$$

$$= (-1)^n 2n! \int_{-1}^1 (x^2-1)^n dx$$

$$= 2n! \int_{-1}^1 (1-x^2)^n dx$$

$$= \frac{2n! \cdot 2 (2^n n!)^2}{(2n+1)!}$$

$$= \frac{2}{(2n+1)} (2^n n!)^2$$

On the right side, there is a separate derivation for the integral $\int_{-1}^1 (1-x^2)^n dx$ using the substitution $x = \sin \theta$:

$$\int_{-1}^1 (1-x^2)^n dx = \int_{-\pi/2}^{\pi/2} \cos^{2n+1} \theta d\theta$$

$$= 2 \int_0^{\pi/2} \cos^{2n+1} \theta d\theta$$

$$= \frac{2 (2^n n!)^2}{(2n+1)!}$$

Below this, there is a note: $\int_{-1}^1 (1-x^2)^n dx = 2 \int_0^1 (1-x^2)^n dx$ and $x=t$.

This is what is this integral if you substitute here. So go back and substitute what you see is integral minus 1 to 1 $P n$ square x $d x$ is exactly 2 divided by $2 n$ plus 1 because 1 divided by $2 n$ factorial square you have now, okay. When you square it so you have this square that cancels with this, okay. So you are left with this. So this is what you have, n is anything from $1, 2, 3$ onwards. Even plus or minus 1 , okay. So you can say n belongs to Z , okay.

(Refer Slide Time: 41:05)

$$\begin{aligned}
 &= 2n! \int_{-1}^1 (1-x)^n dx \\
 &= \frac{2n! \cdot 2 \binom{2n}{n}}{(2n+1)!} \\
 &= \frac{2}{(2n+1)} \cdot (2^n n!)^2 \\
 &\Rightarrow \int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \checkmark, \quad n \in \mathbb{Z}
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_0^{\pi/2} \cos^{2n+1} \theta d\theta \\
 &= \frac{2 \binom{2n}{n}}{(2n+1)!} \checkmark \\
 &\int_{-1}^1 (1-x)^n dx = 2 \int_0^1 (1-x)^n dx \\
 &= 2 \int_0^1 (1-t)^{n-1/2} dt \quad x=t \\
 &= 2 \int_0^1 (1-t)^{(n+1)-1} t^{-1/2} dt \\
 &= 2 B\left(\frac{1}{2}, n+1\right) = 2 \frac{\Gamma(1/2) \Gamma(n+1)}{\Gamma(n+3/2)} \\
 &\quad \Gamma(k+1) = k \Gamma(k) \checkmark
 \end{aligned}$$

So this is what you have. So this is the important property, it is called orthogonal property of Legendre polynomials we have seen in this video.

So we will see some more properties and maybe in next class what we see is we will see in a different way different formula as a polynomial expression as a finite sum for the Legendre polynomial $P_n(x)$ so that normally you can directly derived from the polynomial solution $y_1(x)$ or $y_2(x)$ from which you can directly get your $P_n(x)$ by just fixing n th term, okay. By choosing in a different way we will try to get the formula for $P_n(x)$ in a different way, okay. So that we will see in the next video.