Differential Equations for Engineers.
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Lecture-2.
Methods for First Order ODE's – Homogeneous Equations.

In this period we willexplain what is a singular solution, we will demonstrate this with an example that is the solution of 2nd differential equation, ordinary differential equation which you cannot get it from the general solution. Okay. So I will give an example, so simple example is y dash, y equal to x plus C whole square, I will give you an idea what is envelope, okay. So you simply take, for this, what are these, these are simple parabolas parallel to x axis, just touching x axis, okay, this is your x-axis, like this.

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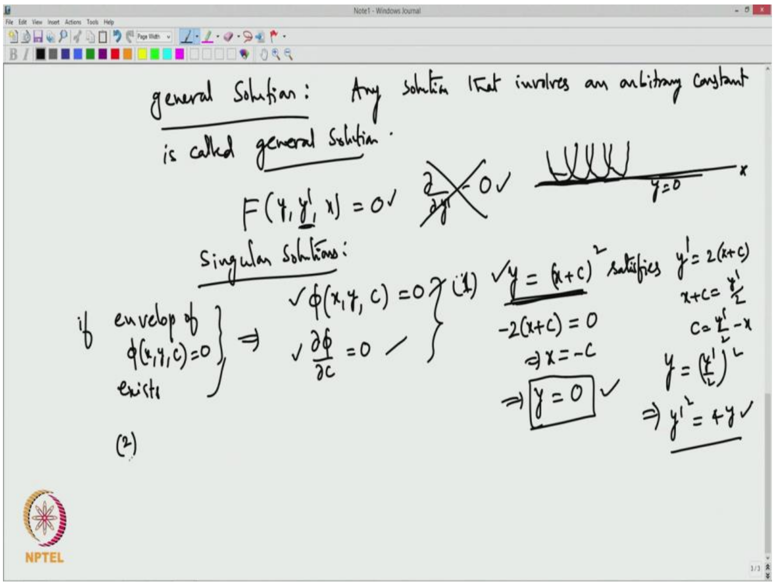

Envelope, any envelope is a curve, at every point it will touch one, one of the parametric, one of the solution curves, okay. One of your solution curves for certain C value. So you can series other things, at any point, you can see what is the envelope here, this is the x axis that is y equal to 0. So how do you get this? This satisfies, this is the general solution that satisfies the equation. If you actually how do you eliminate? You calculate its y dash which is 2 times x plus C, y dash is dy by dx, so that will give me my x, what is my C value.

So that what is my C value, C is equal to y dash by 2x, okay y dash by 2x. So this is x plus C equal to y dash by 2, C equal to y dash by 2 minus x. So this you substitute into the equation, you will get y equal to x plus C, that is y dash by 2 whole square. So this is nothing but you get y dash square is equal to 4y. So if y is there, it satisfies this equation. So any parametric family of curves, one parametric family of curves, if you eliminate the constant, what you end up is the differential equation like this, okay.

So alternatively you take this differential equation, you find its solution, okay, because we do not know. So this is clearly, you can see, dy, what is this one, dy by, so dy by dx, so we have, if you solve this equation, you will get this parametric, this is the general solution we get like this. You can see this one, this is 1, this is Phi, this is a real solution, one equation and you differentiate this general solution with respect to T, that is with respect to C, that is 2 times x plus C is equal to, so y, y you are differentiating with respect to C0 - 2 times x plus C and you are differentiating with respect to C, that is equal to 0.

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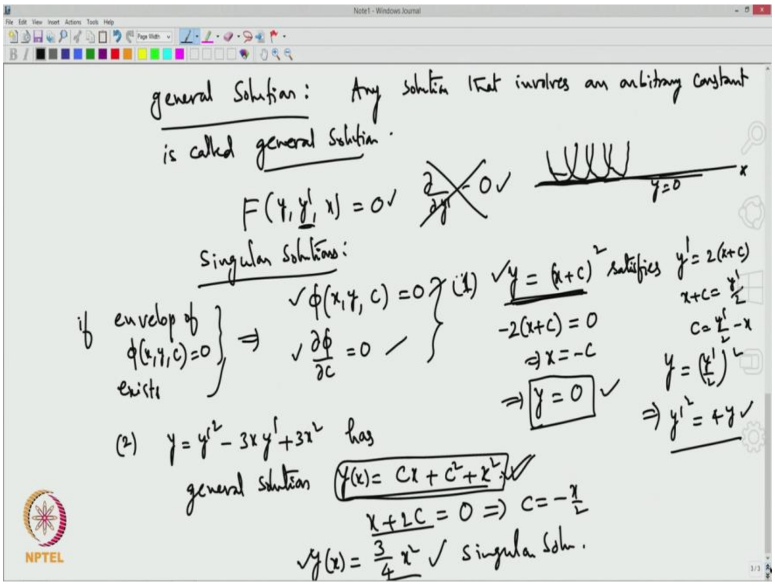

So that will give me -2 cannot be 0, so x equal to minus C, when you x equal to, C equal to minus x, you get the equation, 1st equation. So you eliminated C, so what you end up with is 0. So this is exactly or envelope, that is the singular solution. Okay. Now I do not want this trivial solution, if you want, I can give you another example, okay. This is one example, the 2nd example is you take this differential equation, so y dash square minus 3x, y dash plus 3x square, okay.

Do not ask me why, so this is, this is the OdE which is non-linear, which has the solution, which has general solution y of x equal to some constant x plus C square plus x square. Okay. So you have a general solution here, you have a general solution here, how do you know that this is a general solution? It has a parameter one, one parameter C. Okay. One arbitrary constant, so it is a general solution. You substitute it into the equation and C, verify it, it satisfies. So this is the general solution. Right. Because you do not know any method how to solve, now do you say that, if I say that this equation has a solution, which means you can only simply substitute and see, okay.

So you substitute and see, it satisfies. So this is the solution of the equation. General solution of the equation. So if I want a singular solution, have to calculate, this is one equation like Phi x, y, C is equal to 0, this is one and I calculate its derivative with respect to C, that is x plus 2C is equal to 0, okay. So that gives me C equal to minus x by 2. So we substitute, go and substitute, what you end up is 3 by 4 x square. We put this C equal to minus x by 2 into the solution, general solution C, that means you eliminated C.

How did I get this one, x plus 2C, y minus Cx minus C square plus x square, that are differentiated with respect to C, this is what is the result, okay. So you end up finally, once you get this C, C you substitute into this, what you end up is this one. This is, this you cannot get it from any C value, you put any C value, you cannot get 3 by4 x square. Whatever you put C value, if you put 0, x square only you get. So cannot get the solution out of this general solution, okay.

So that is why it is called singular solution. Anyway in this course we do not deal with the singular solution, you can learn the, in the reference books which I said in the syllabus. You can always get, certain always solutions, you can always, so for example if you have solution curves, envelopes here, this is the parabola, all these solution curves if you draw, their envelope is actually parabola here. Like 3 by 4 x square, y equal to 3 by4 x square is a parabola.

So envelope generated by these solution curves, parameter, one parameter family of solution curves, that is the singular solution. Okay. Let us give the, we do not deal with the single solutions, we only want general solutions, I gave you already the method of solution, one method of solution, solving the 1st order ordinary differential equation by that is called separation of variable method. I will give an example.

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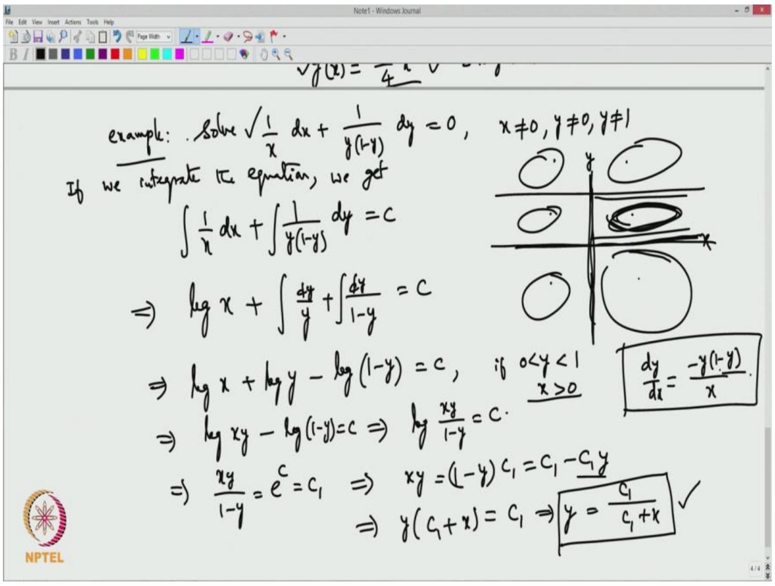

So we will solve one 1st order OdE one by x dx + 1 by x times 1 minus y dy equal to 0. Whenever you write this, now this is your 1st order differential, ordinary differential equation. So you see that x is, at x equal to 0, this function is not defined, one by x. So your domain, if at all you have with this equation, x equal to 0 is not part of the domain. And all those values, once you get the solution yx, okay, you have a solution yx, if it takes the value 0 or 1, for those values of x, that is also not in your domain. Okay.

So that means why is not equal to 0, y should not be equal to1. If you actually see the domain graphically, so this is your x, this is your y, x equal to 0 is y axis, y equal to 0 is x-axis, y equal to x is this one. So you should have a domain either here or here or here or here or here, so it should be, you can have your domain. You can say your equation is defined either here or here or here, not, not in this full domain, okay. You cannot take discontinuous domains.

So say my equation is valid here and here, so you take it here and you solve it, so the solution is valid only here. If you want solution here, you consider your domain this as this one, that you take it and solve. So that is how it is. So how do we solve this? This is already separated, variables are separated. Okay. So all the variables are separated, you can simply integrate 1 by x the x +1 by y Times 1 minus y dy equal to constant by integrating, of integration, okay.

If we integrate the equation we get, this is what we get, where C is the arbitrary constant. So if you know how to evaluate this, we are through. So this will be log x, log mod x, okay. Let us say we are only with, for simplicity you are in the domain x positive and y is actually between 0 and 1, okay, in this domain. So I do not need to write, x is positive, so a log function is defined, so I simply write log x plus this one, this integral I can write one by y minus I will see 1 upon 1 minus y minus y, so it should be plus.

1 by y plus one by one minus y is actually 1 by y Times 1 minus y. So this you integrate dy dy equal to C. So this will give me log x plus log y minus log 1 minus y, if y is between equals to C, if y is less than 1, positive and less than 1. So x is positive, in this domain I do not have to take integral 1 by x dx as log mod x, okay, so this will give me simply, I am doing this for certification. So log xy, calculations will be easier. So that is how we have to do.

So you have to, when you consider in this domain, I am considering this domain, so this is what it is. So log xy minus log 1 minus y equal to C. That implies log xy by 1 minus y equal to C. So this implies xy by 1 minus y equal to E power C, which is arbitrary constant, if C is arbitrary constant, C1 is also arbitrary constant, that will give me xy equal to 1 minus y Times C1, that is C1 minus C1 y, that will give me y common, you bring it this side, this term if you bring it, the other side, left-hand side, y is common, so you have C1 plus x equals to C1.

So that will determine my y. So y equal to C1 divided by C1 plus x. So this is your general solution of the given equation in the domain x positive and y is between 0 to 1. Okay. This is how you solve this equation which is already separated, separable form, nice separable form we have. What is this, where is this, it is not in, it is already in separated form it is given, okay. I can give you equivalently like this. This equation, I can give you, given equation is actually, if you write dy by dx equal to, so you take this other side, dy by dx equal to, divided by x minus y Times 1 minus y.

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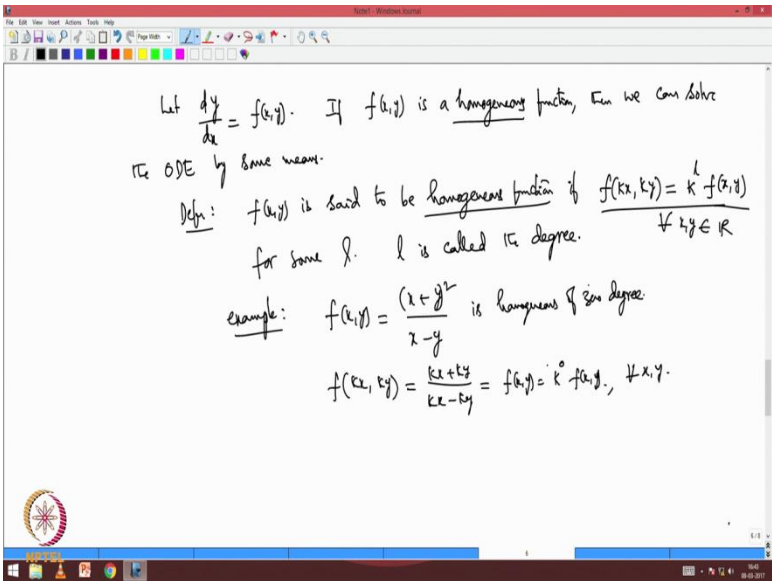

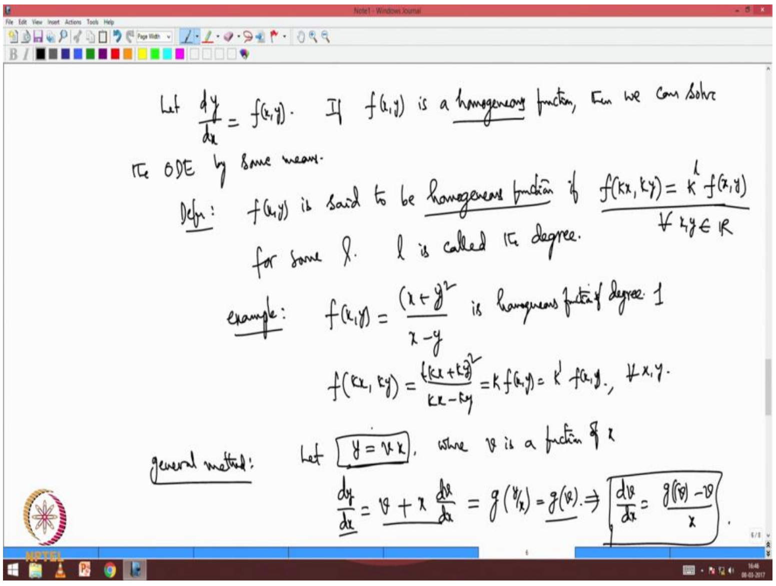

If I, if you have given equation like this, so this you can rewrite like this, and then which is already in separated form, okay. And then you integrate, simply integrate, you can get your solution. Okay. So next method is, if we, if f of x, y is having some new form, let us say it is defined what is called a homogeneous function. So let f of y dash, dy by dx equal to f of x, y, this is the differential equation. If f of x, y is a function, is a homogeneous function. That means what, then I can solve this equation, solve the ODE by some means.

What we do is, 1st of all before we solve this equation, so we should understand what is the meaning of homogeneous function. So the definition of homogeneous function is f of x, y is homogeneous, said to be homogeneous. Homogeneous function the definition, if you take any constant, multiply with both the variables x K, some Kx, Ky, I replace x, y by Kx and

Ky, then it will have some K power some constant, I say K power L f of x, y. Homogeneous function, if you get like this for some, L, L is called the degree.

So you say that f of x, y is a homogeneous function of degree L if you have this, this is satisfied for every x, y in R it is given, means full real numbers. So if you have such, generous functions, so what is the meaning of this? For example, this is our, these are not (())(15:34), you can have f of x, y, you can see that x plus y divided by x minus y are there. So you can have, these are the, these are the homogeneous, this is a homogeneous function of degree 0 because whenever I replace K here, K here, K cancels, finally K power 0. Okay.

So this is 1, this is homogeneous 0 degree because f of Kx, Ky is equal to Kx plus Ky, Kx minus Ky which is equal to again f of x, y so that means we have K power 0 with one f of x, y. So it is degree of, it is true for every x, y, so it is homogeneous of degree 0. So you can have, if you make it square, 0 or 1 degree, it becomes homogeneous of degree one. Homogeneous function of degree one. Okay. So if you do this, simply replace with square, numerator with square, so we will have K comes out, so we will have K power 1, so we will have homogeneous function of degree one.

So these are common, you may get functions like this. This thing, these kinds of functions, whenever you have in your ordinary differential equation of first-order, you can convert this into earlier type, okay. So let us see how we do this. So what we do is, the general method is if, if you choose y by x, you try to write function of f of x, y, f of x, y, you try to write it as as functions of, function of y by x. Function of one x you take it out, you write y by x in this form. So it is a different, you try to write, so that means, I will say you, you define, you convert this function into new variable.

y by x, y equal to some Vx where V is, V is a function of x because y is a function of x, V is a new variable, that is also, should be function of x. Okay. So then dy by dx equals to V plus x into dV by dx. This is equal to f of x, y. So if you can write f of xy as some function of y by x which is G of V, this we can, so this implies, I can see, this is the one, this is equation, this is, this becomes, the given differential equation becomes V plus x into dV by dx equal to G of V, that implies dV by dx equal to G of V minus V divided by x.

You can see variables are separate it. Now the new variable that is V, x is the independent variable, so you can see this is like earlier type separated variables. So this you can solve by earlier method for the so that means any, whenever you have a function f of x, y, which is a homogeneous function, by this transformation you can solve this, you can convert this into, you can reduce this into an ordinary differential equation with right-hand side f of x, y, variables are separate it, for which variables are separated, okay.

Then you can proceed the same way how to solve this, you know to solve this equation. Once solve this, you can replace this V by y by x, that will give you actual solution y of x. Let us do some examples. Example of a problem, so we can understand.

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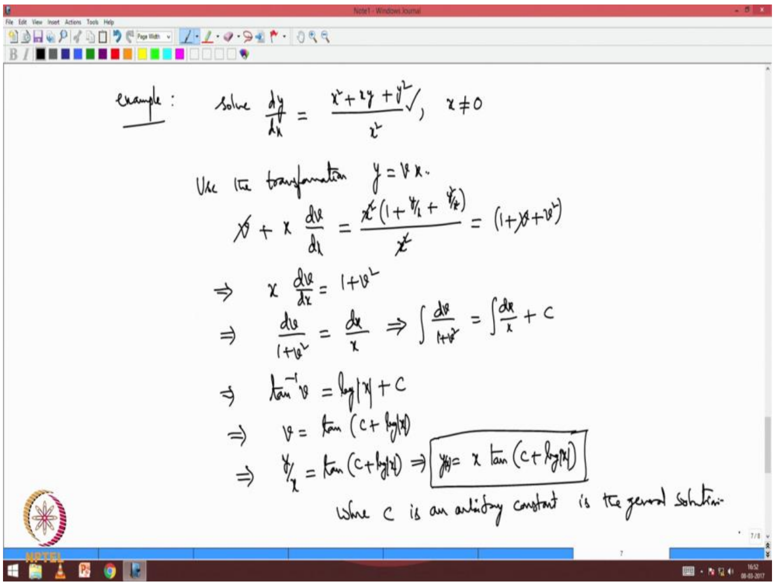

So solve y dash, that is dy by dx equal to x square plus xy plus y square by x square. Again you see this, this is an equation where one by x square is there. So that means it should not be 0. So you should have at any domain, this can be differential equation, it is defined in some domain where x equal to 0 is not part of the domain, it should not be part of it, okay. x equal to 0 should not be part of the domain in which this differential equation is defined. So you can see this right-hand side, all squares, this is square, square, square, so you can expect this is homogeneous function of degree 0.

You can easily see, put x equal to Kx, it becomes K square, here x equal to Kx, y equal to Kx, you can have K square comes out, y equal to K, you have another K square, so finally out of all the 3 terms, K square comes out here and the denominator also you have a K square when you replace x by Kx. So K square, K square goes, it comes out same as itself with K power 0. Which means this is a homogeneous function with degree 0.

So we use the transformation, use the transformation y equal to Vx, where V is function of x. y is the dependent variable, now V is the new dependent variable. Okay. So you can replace dy by dx, which you can calculate from this function, which is replace x into dV by dx, this is what is your dy by dx, which is equal to, you can take x square out, if you can take x square out, 1 + y by x plus y by y square by x square, that is the numerator, then we have a square as it is.

So you have, to cancel this, that will become 1+, which is coming from the transformation, y by x is V plus V square. Okay. So this implies, now you take this left-hand side and right-hand side, you can write that write dV by dx equal to 1 + V square. y, y cancels from both sides, we can have 1+ V square. Now the variables are separated, you can write one side all the V variables, on the other side all the x variables. So this implies, you can integrate both sides, integration will give you equal to dx by x plus C is an arbitrary constant.

So this is actually your general solution because you can integrate, you know how to integrate this. This is actually tan inverse V equal to log x plus C. So this is your general solution, that means V equals to tan C plus log x. Okay. So that implies V is y by x. So you have y by x equal to tan C plus log x, that implies y equals to x tan C plus log x, this is what is your general solution of the given differential equation. And of course when I write, when I integrate, integral dx by x, I write log x.

That means it is defined, what is log x, log x is defined only for x positive. So this, so this is, when you write like this, that means I am only going for x positive, you should actually write mod x, log mod x, then it works for all x, okay, including, anyways 0 is not defined. Even if x is negative, positive, does not matter, then this is true for every, so integral the x by x is log mod x. Okay. So this is your general solution y is the dependent variable, so it should be function of x as you can see, C is an arbitrary constant, C is an arbitrary constant, where C is the arbitrary constant is the general solution. Okay.

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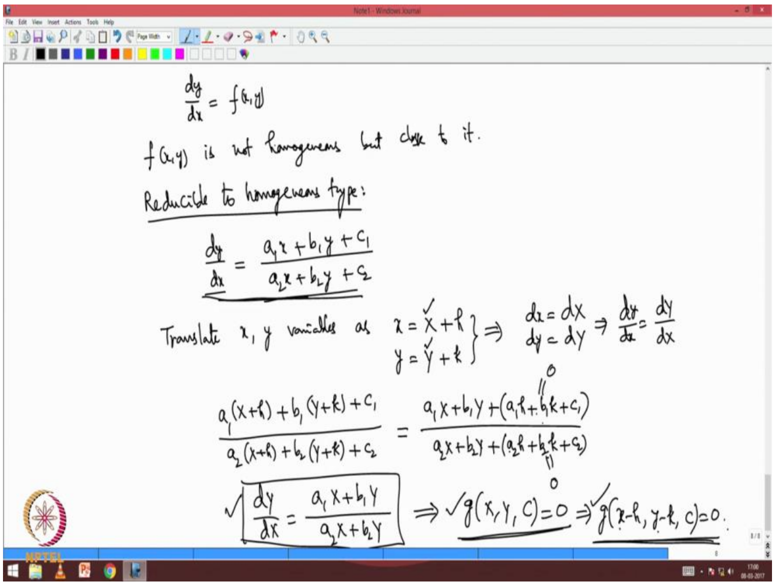

So we will do one more problem. So whenever you have this homogeneous function, so dy by dx equal to f of x, y, where f of x, y is a homogeneous function of certain degree, you can solve this differential equation. Certain equations, though they are not homogeneous functions but they are, they, they look like homogeneous functions but with constants, so you can actually make them homogeneous, certain functions, okay. So you can reduce certain F, f of x, y is not homogeneous but close to it. Close to… The form of f of x, y is close to homogeneous function.

So for example, those functions where f of x is such a function, we can actually reduce this equation to homogeneous, an ordinary differential equation with the right-hand side, dy by dx equals to, with a homogeneous function, you can reduce this into an homogeneous equation, okay. So equations, these are the questions, such equations are the equations that can be reduced, are reducible to equations, reducible to reducible equations, reducible to homogeneous type. Simple, let us take A1 x plus B1 y plus C1 by A2 x plus B2y plus C2.

So if you are having this type of function, you can reduce this into a homogeneous function. So right-hand side is not a homogeneous function, you can make it by simply translating these variables x and y. Translate x, y variables as x equal to some new variable plus H. I have introduced a new variable x plus H, okay. I will write y as capital Y plus some K. So these are the variables, these are the X and Y are the variables, H and K, you choose H and K in such a way that when you say to these small x, y into this, it will become homogeneous function.

Simply just take this as a transformation, you substitute into the right-hand side, you will have A1x equals to x plus H plus B1y plus K plus C1 divided by A2x plus H plus B2y plus K plus C2, this is equal to A1x plus B1y plus some constant, that is A1 H plus B1 K plus C1 divided by A2x plus B2y, I take it from here and here. So remaining is A2 H plus B2 K plus C2. So if I choose my H, K values in such a way that C A1, B1, C1, A2, B2, C2, they are given, they are known, they are known constants.

If I can choose my H and K, some numbers satisfying this, I make this 0. If I choose my H and K, so these 2 are 0, then this function, then what remains is in terms of this new variable X and Y, it is homogeneous. Okay. So right-hand side is a new function of X, Y, this is one. So it has to be a right-hand, left-hand side is also should be same, right. So what is is dY by dX, you can see that from this you can write, dx equal to because H is constant, d capital X. dy equals to d capital Y.

So this means dY by dX is equal to d capital Y by d capital X, both are same, right. So I replace this dY by dX with capital dY by dX. Right-hand side have chosen my H, K so that this, this is 0, this is 0, whatever in the bracket. So what you, what is left with is A1x plus B1y by A2x plus B2y. Now this is an equation with right inside is having a function that is homogeneous function. So you know the method how to solve the equation with homogeneous right-hand side, homogeneous function f of X, Y.

Once you solve this, you can replace finally, so what you get is the solution as some function of X, Y, C is equal to 0 where C is arbitrary constant. Once you apply the method to solve this homogeneous equation, this is how you will get a solution, general solution of this equation for some G. You replace this X, Y with this from this transformation, capital X you replace with x minus H, small x minus H, ultimately, H is known, H you have chosen in such a way that this is 0, okay. And Y you can replace, y minus K and C equals to 0.

So you can see that X, Y are your actual variables, actual variables are here small x, y, H and K are fixed constants, because that is how you fix them so that this is 0, these are the 2 equations, linear equations in H, K, which you can solve it. Okay. So if you want to have the solution, so 1st there is a question, you should have a solution. When do you have a solution, when do you have a unique solution for this H, K, for these 2 equations, this is equal to 0 and this equal to 0?

So you should have, so we will see that, okay. So this is how you get the solution, whenever you have a unique solution H, K, so that this equation is 0 and this equation is 0, so you can have this transformation, that reduces the given equations into this homogeneous type that can be solved this to get this solution, general solution which in the old variables is this. Finally this is the form. Okay.

So in this video we have explained a few methods to solve the first-order ordinary differential equation starting with the simpler one where variables are separated, it is called separation of variable methods. And when, when the function Y dash equals to f of X, Y and f of XY is a homogeneous function and we have explained the methods to solve the differential equation and some simpler form of non-homogeneous equation which can be reduced to homogeneous type. So as, then we integrate it using the homogeneous technique, homogeneous, just the technique that is used to solve the homogeneous equations. Okay, so that is it.