Differential Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
Department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-12.
Abel's Formula - Demonstration.

We were discussing about the properties of solution of the second-order homogeneous linear ODE. So we have seen 3 properties, simple properties, one is linear combination of any 2 solutions we give y1 and y2, linear combination C1 y1 plus C2 y2 is also a solution, simple property, and 2^{nd} one is, if you are given to solutions y1 and y2, when can you say that they are linearly independent? So they are linearly independent if and only if Wranskian is nonzero.

We define a quantity called Wranskian of these 2 functions which is a function of x which should be nonzero for every point of x. And 3^{rd} property is the Abel's formula that tells you that if the Wranskian is 0 at one point, implies Wranskian is0 at everywhere. Same is the case is Wranskian is nonzero at one point, Wranskian is nonzero at every point implies y1, y2, the functions y1 and y2 solutions are linearly dependent or independent according to Wranskian at one point is 0 or nonzero, okay.

We have also seen, we have also deduced 2^{nd} solution from the Abel's formula, if you a priory know one solution y1, if you are given one solution y1, you can always find the 2^{nd} solution y2, okay as y1 into some integral from x0 to x into some exponential function divided by y1 square, okay. So since y1 is a, you are given y1 nonzero solution, one linearly independent function, solution, so the 2^{nd} one y2 is also linearly independent because y2 by y1 is non, it is not constant, okay. That is what we have seen.

Before I give you the example, I will just briefly discuss what we do next. So we have some more properties, so we have one more property of the solutions, what you know is that if you are given one solution, so so far and you do not know how to find solutions, okay. I give you, if I give one solution, you can find the other solution, so one of the natural questions is how many solutions, linear second-order homogeneous equation will have, okay. This will always have, linear second-order homogeneous equation will always have only 2 linearly independent solutions.

1^{st} of all you have to show that, you have to prove that it has only 2 solutions, it has 2 solutions y1 and y2, 2 linearly independent solutions and there is no other solution, any solution I write in terms of this y1 and y2. That means you have only 2 linearly independent solutions, okay. Only this I am stressing, only 2 linearly independent solutions for the second-order linear ODE. Same is the case for high order equations. Higher-order, Nth order equations will have N linearly independent solutions, okay, only N linearly independent solutions, independent solutions, okay.

So we will have only, only N linearly independent solutions you will have for the Nth order linear ODE, homogeneous linear ODE. So before I give you those 2 properties, let me give the example, if I give one solution y1, how do you find the other solution, I will give you example. Example, example of the property 3, so Abel's formula.

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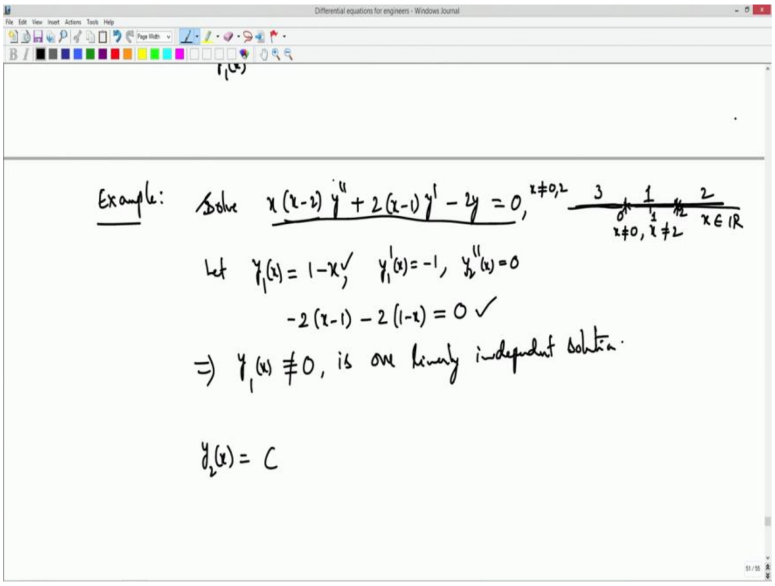

We will take some simple example where you have this x into, solve, solve x into x minus 2 y double dash +2 into x minus 1 y dash minus 2Y equal to 0. Okay. So you see this, this is your A0 of x, A0 of x, what is your domain? So domain is x belongs to our because x cannot be equal to 0 and x cannot be equal to 2. So 0 and 2 are, this is 0, this is 2 are singular points, these are the points at which your differential equation is not defined. So x should not be equal to 0 and 2.

So your domain can be this or they saw this or this, okay. So when you say x is not equal to 0 or 2, that is what is the case, okay. So let us take any, any of those domain. Let us take either this or 2 or 3, any one of this. It should be given that this differential equation at either one of these domains, 3 domains, okay. So you can also, you can also verify by just trial and error, by trial basis, you can verify certain simple functions and solutions because you are given one solution, why should I give one solution, you can also find sometimes, it has some simple solutions, for example sometimes simple x, simple E power x, Sin x, cos x, x square, x cube, like that you can find some simple form.

If it is a solution you can simply verify that it is a solution, so that means you know one solution, okay. I do not have to give one solution, so you can also verify by trial and, by trial, by, by commonsense you can see some simple functions as a solution. If it satisfies the equation, that means you know one solution, okay. In this case I give you y1 x, this is the difficult, it may not be difficult but you can verify that 1 minus x if we put it here, okay, let y1 x is this, what is y1 dash of x, which is minus1.

What is y2 dash, double dash of x, that is 0. If you put it here, this is 0, y1 x is, now if you put these 3 into the equation, 2 into x - 1 into minus1 - 2 into y is 1 minus x. It is going to be, this is actually 0. Right. So this is going to be 0, this is actually 0. So that means, it is verified. So this is the solution. So this is one solution, okay. This is one solution that implies y1 x which is not equal to 0 completely the solution, is, is one solution, one linearly independent, nonzero solution implies linearly independent, okay, independent solution.

So you can see that 1 minus x is0 at x equal to 1. 1 at x equal to 1, so it is not your domain of the differential equation, should not be your domain, domain of the differential equation. So your domain, either this or this, okay, that point 1 should not be your domain of the differential equation, okay, we will see at the end, okay. So this is what is your y1, assume that y1 is 0, wherever it is defined, it is completely nonzero for all values of x at which the differential equation is defined.

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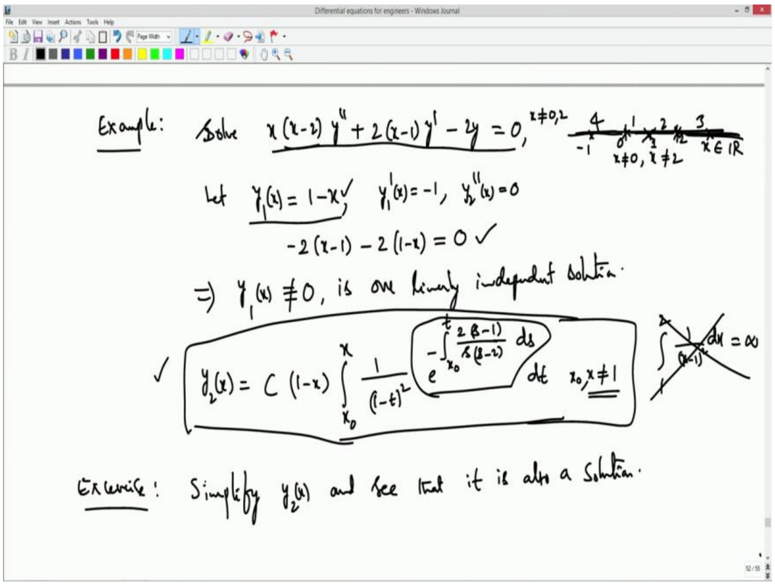

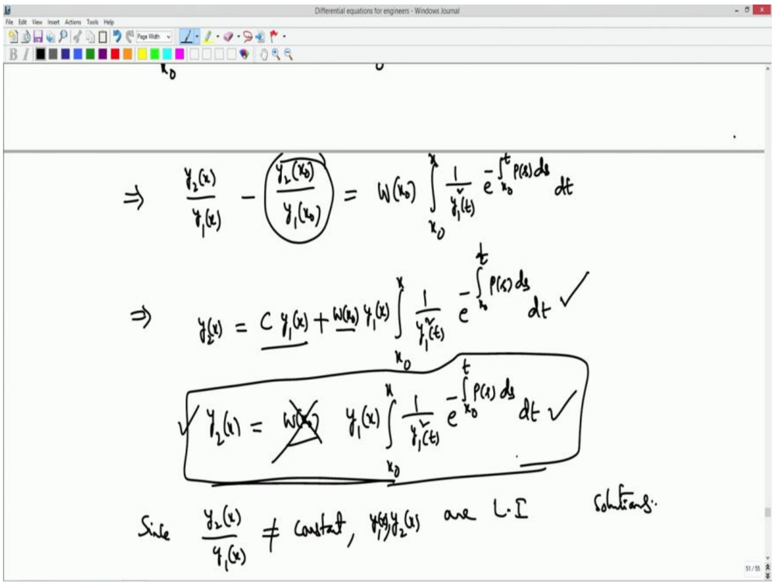

Now to find your y2, so what is your y2, y2 x is some constant times, this constant is coming from my WX0, so WX0 is constant into y1X, that is 1 minus x, so I simply substitute here in this formula. y1 x I substitute 1 minus x, now I can integrate from x0 to x, what is your domain? So you have to take your domain x0 to x, 1 by y0, y1 square of T that is 1 minus T whole square, okay, 2 into E power minus integral x0 to T, pS, what is pS? pS is this divided by x into x minus… So you have 2 times x - 1 divided by x into x - 2, okay.

This is what is the, x minus 1 you should write divided by s into x -2 ds. Okay. This with dT, so this is your solution. So in order to make sense, this integral, T should not be equal to 0, equal to1, okay. That means x0, even if you take x0 equal to 1, 1 to x, this is, this integral is diverging. So you can see that 1 to some 2 for example, 1 by x - 1 square dx, this is infinity.

Okay. So you can see that, this, this is infinity, so does not make sense. It is diverging integral, so, so you have to see that x should not be equal to 1 or, okay.

So that is what is your domain, so once your domain, our x and x0 should not be equal to 1. So that is how x is,, you should not take the domain as either between, so this 0.1 also you should not, you should avoid. So your domain is either 1, 2, okay or 3 is this, 3 or 4. So you can take any one of these 4 domains, okay as your differential equations, your solution is y1 x that is defined in the domain which is nonzero everywhere, so that you can find your 2nd solution like this, okay.

So it is just what is left is simply integrating this, finding this exponential function, this integral and then you divide with 1 by T square. x0 is not equal to1, so if you are in this domain between 0 to 1, you can take it as half to x, x0 as half. If you are in the 2nd domain, you can take say 1.5, say 1, 3 by 2 to x, okay. You can choose the way you like, okay. You should not take it as 1. Similarly if you are here, you can take x in the 3rd domain you can take x0 as 3, here you can take it as minus1, okay.

Whatever, so x0 you can choose, only it belongs to either of the domains, we just have to, there is only calculations are left. This is how you get your 2nd order, 2nd linearly independent solutions, okay. So I think this calculation will take more time, so you can try as an exercise. Calculate or evaluate or simplify, simplify y2 x and see, see that it is independent, and and see that it is also a solution. That means you just verify, as an exercise you take it. So we can do some simpler problems later, okay.

This is how you can find, if you are given one solution, you can find the other solution or you simply by commonsense, you, verifying that one solution, you can guess, you can guess and verify it so that you know one solution and implies you can find the other solution and that, then these 2 solutions, 1st solution when you guess, it should be, because that y is 0 is always a solution, I do not want 0, okay. For this 0 is always satisfying, y equal, y1 equal to 0 is always one solution.

So if I, you should know one nonzero solution, so linearly independent solution, only linearly independent solution so that you can find other linearly independent solution from this formula, Abel’s formula, okay. That is how we derive this. This is exactly the formula you have. So y2, y2 is this, final y 2, okay. From this I directly road, okay, this is how you can calculate. So there is another way of doing it, that you can also do that, okay. So we will do that later on.

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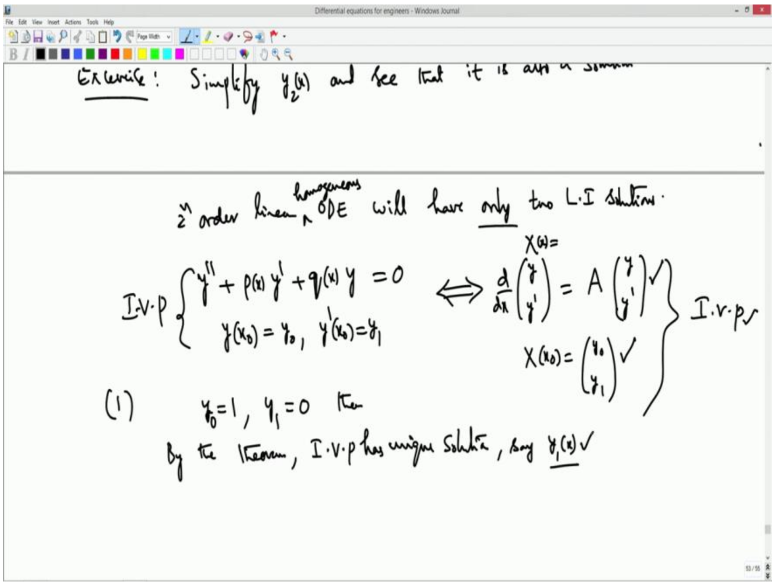

So while doing problems we can do that, so, now we will move onto the properties, what is the other properties we are left with, that the second-order, second-order linear ODE will have only, I am stressing this only 2 solutions, 2 linearly independent solutions. Okay. So how do I show this? 1st of all I show that it has 2 linearly independent solutions, okay. So let, how do show this? So what I do is, I take the second-order equation 1st of all, so y double dash equation is pX y dash plus q x y equal to 0.

So this is your second-order homogeneous ODE, homogeneous ODE. This should be homogeneous, homogeneous ODE, we will have only 2 linearly independent solutions. So how do I find 1st solutions? So I am not actually calculating the solutions I say that it has a solution, I give, I guarantee that there is a solution, okay, so how do I say that? I have fix my initial values, y at x0, the initial value at x0 equal to y0, y dash at x0 equal to y1. So y0 and y1 are constants, you are given.

So this we have seen, we can easily put it as an equivalent system as y and y dash d dx equal to some matrix A, matrix A involving only p and q, okay, into x is y and y dash. So what is your vector, this, this is your vector x, x is this. So at x, x0, actually, your y at x0, that is y0, y dash at x0, that this y1. So this is what is given. So if this is given, this is the 1st order linear ODE for the vector x, y1. So this is equivalent. This first-order system of equations, okay, this is the second-order equations.

Second-order equation I put it as the 1st order equations for the vector, 2 by 1 vector, y and y dash, okay, with this (())(15:09). So you can see that what I do is, I choose my y0 equal to 1 1st, okay 1st, 1st. That is I choose my y0 equal to1, y1 equal to 0. Okay. Then by uniqueness theorem, by the, by the theorem, without proof I have given existence and uniqueness theorem, okay. And this, this equation, this initial value problem, this is also initial value problem, this vector value, initial value problem, this is scalar value initial value problem.

Initial value problem for the scalar function y x, this is the initial value problem for the scalar function, for the vector function, vector valued function x. Okay. For the vector valued function x of x, okay. Because y is the function of x, this vector should be x of x. So from the theorem you know that you have a solution, this has unique solution. IVP has unique solution, either of this, either you choose these or this one, okay. It has a unique solution, this is one.

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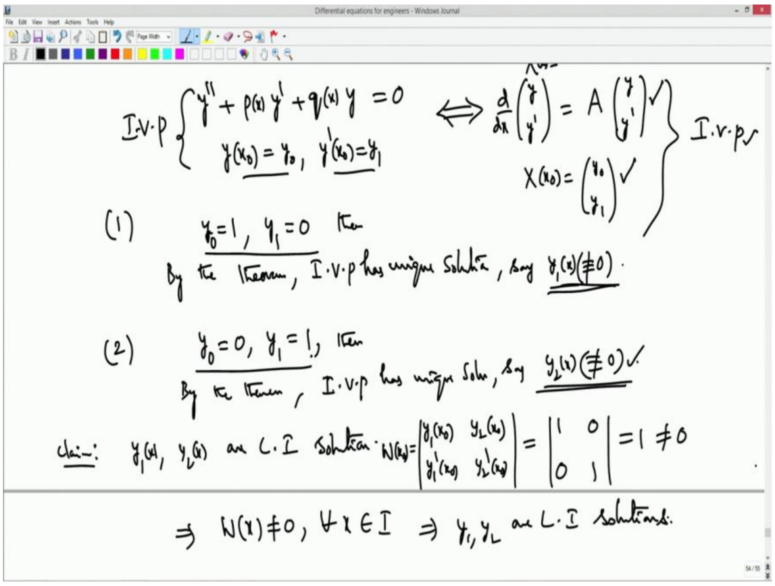

So call this, say y1 x is the function, okay, this is your 1st solution. And whose value at x0 is initial value is y0, okay, y0 is 1, which is nonzero, okay. Right. That means if it is 0, if it is nonzero at one point, you have a solution, that means it should be linear, it is nonzero function, as a function it is nonzero. At one point it is nonzero means you know certain function is a solution and you know that it is nonzero at one point, that means the function is nonzero function which is a solution. So you have, you know that y1 at x0, y1 is the solution of this let IVP, so y1 and x0 is y0. y0 you have chosen as 1, that means it is nonzero, okay.

Y1 x is nonzero, so you have a nonzero solution which is nonzero, okay. I can clearly say that this is not completely 0, this is nonzero solution. Now the 2nd step is, you choose y0 equal to 0, y1 equal to1, then, then again by the same theorem, by the existence uniqueness theorem, initial value problem, this linear initial value problem has unique solution, say y2 x which is also not completely 0 because what is the reason, now you look at, you may not be able to see from here.

Or here also you can say y2 at x0 is y0 that is 0, you cannot say anything. Now look at y2 dash x0, that is y1, that y1 is actually 1. So the derivative is nonzero for certain function, that means the function should, the function cannot be 0 function. If it is a 0 function, derivative is also 0 everywhere. So that means it cannot be 0 function. So that is how it is nonzero. So you have 2 nonzero functions y1 and y2, okay, I have 2 solutions by choosing initial values. Suitable initial values, choosing suitably initial values I, I obtain 2 solutions. I know that they exist, so you have 2 such solutions, okay.

So I will show that these are 2 linearly independent solutions. The y1 and y2 are, now I claim is y1 and y2 are y1 x, y2 x are linearly independent solutions. How? From property 3. At one point, so let y1 at x0, y2 at x0, y1 dash at x0, y2 dash x0. What is this one? This is exactly Wranskian at x0. This determinant is a Wranskian. What is this one, this is equal to 1, 0, 0, 1 which is 1. This is nonzero. As the Wranskian at one point is nonzero implies Wranskian is nonzero for every x in the domain. That implies the solutions y1 and y2 for which you calculate this Wranskian are linearly independent solutions.

So that is how we can always construct, so from these calculations we can see that, I can give you, given a second-order linear homogeneous ordinary differential equation, you have, I say that you have 2 linearly independent solutions, okay. But these are the only 2 solutions, no other solution exists. If as all were all there is a solution, I always write that solution as, in terms of these 2 solutions y1 and y2. That can also be done by just another simple calculation.

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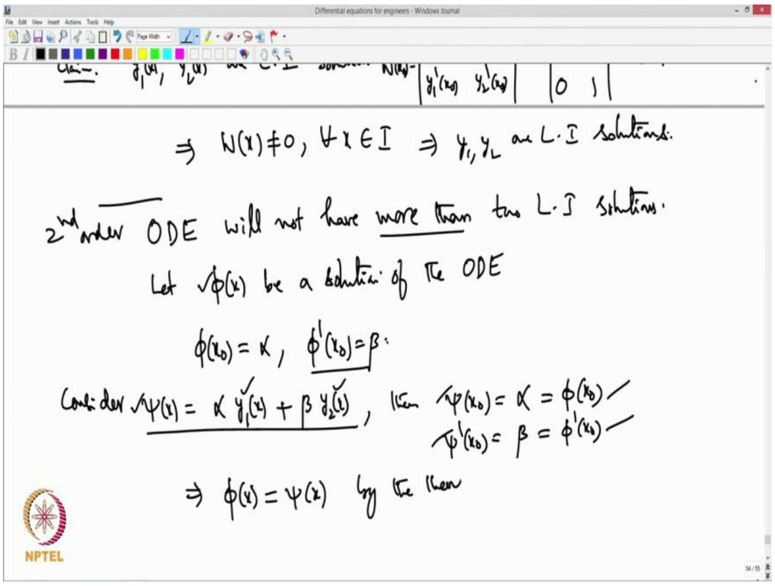

Let, so these are the only 2 solutions, these are the only linearly independent solutions. Actually precisely I cannot say, write like this, I cannot write like this, you cannot have more than 2 linearly independent solutions, okay. This equation, ODE, second-order ODE, second-order ODE, second-order ODE will not have more than 2 linearly independent solutions, okay. Still you do not know how to calculate. If I give you, so, so far if I give you one solution y1, you can find the other linearly independent solution, okay.

So you are just now seeing theoretically that you have, you have 2 such, always 2 linearly independent solutions exist, I will say not more than 2, okay. So let phi be, phi x be, be a solution, be a solution. I give you… Suppose you have a solution of the ODE, so that is this, this ODE, okay. If that is the ODE, so if it satisfies that equation, ODE, then you calculate, if you know the solution, you can calculate its value at x0. So you call this at some point if you know the solution function, you can also, you know the value, function value at x0, initial value.

So let us say x0 is the point of the domain, phi at x0 is say some Alpha. phi dash at x0 is some beta. This you can calculate if you know phi x. Okay. Now what happens, now I write, now I define some psi x is equal to Alpha times y1, y1 x plus beta times y2 x. I consider this, I consider this function. Then, because that y1 and y2 are 2 solutions, okay, like a, with earlier argument we had 2 solutions some other combination, linear combination is also a solution.

So psi x is the solution of the let ODE, linear ODE, linear homogeneous second-order ODE. And what is its value at x0, psi at x0 is equal to phi at x0, Alpha, right, this is Alpha. Because y2 at x0 is 0, y1 at x0 is one, that is how we have chosen, right. So this is Alpha, this is nothing but phi at x0, okay. Similarly psi dash at x0 is actually equal to beta because y1 dash at x0 is 0, y2 dash at x0 is 1, so that gives you beta which you, which is actually equal to phi dash of at x0.

So you have 2 solutions psi and phi are 2 solutions whose initial values are same. Okay, initially they are same, so you have same initial values by the theorem, that is existence and uniqueness of the theorem, second-order linear equation or in fact any linear equation, linear equation, homogeneous equation. So you have initial value problem, that initial value problem for the linear equation or linear system of any order, you have only unique solution. So that means because they are usually, whenever it is defined, for all values of x is defined, so that implies phi x should be equal to psi x by the, by theorem.

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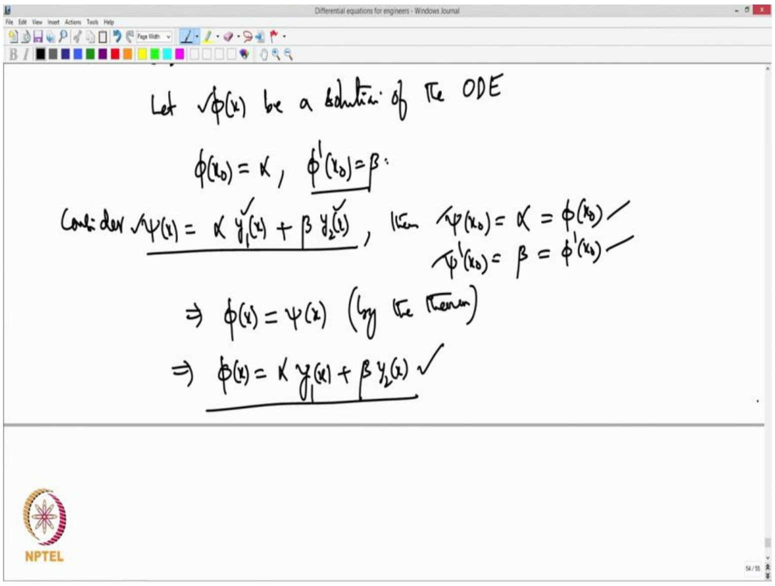

I will write theorem actually from the existence uniqueness theorem. So this implies, what is actually psi, psi is Alpha times y1 plus beta times y2 x. So that means any solution phi I can write in terms of y1 and y2. That means there is no other linearly independent solution of the second-order linear homogeneous ODE. If at all there is one solution, that is actually combination of this y1 and y2. That means these are the only, let y1 and y2 are the only 2 linearly independent solutions. Not more than 2 linearly independent solutions you do not have, that is what it says.

So these are the properties of second-order homogeneous linear ODE, okay. So having known this, so now you know that, if you are having a second-order linear ODE, you have only, only 2 linearly independent solutions. So if I give you 1 solution, you can find the other linearly independent solution, okay. So in the next class we will choose different problems for which, starting with constant coefficients, constant coefficients will take and then we will try to find the solutions, okay, we will try to find the solutions.

If they are 2 linearly independent, once you find the 2 linearly independent solutions, because you know that there are the only 2 linearly independent solutions should have, you simply take a linear combination of those 2 solutions, that will give the general solution, okay. Take C1 into one solution plus C2 into another solution, you add them up, that should be the general solution of the second-order linear homogeneous equation, okay. So we will see later.