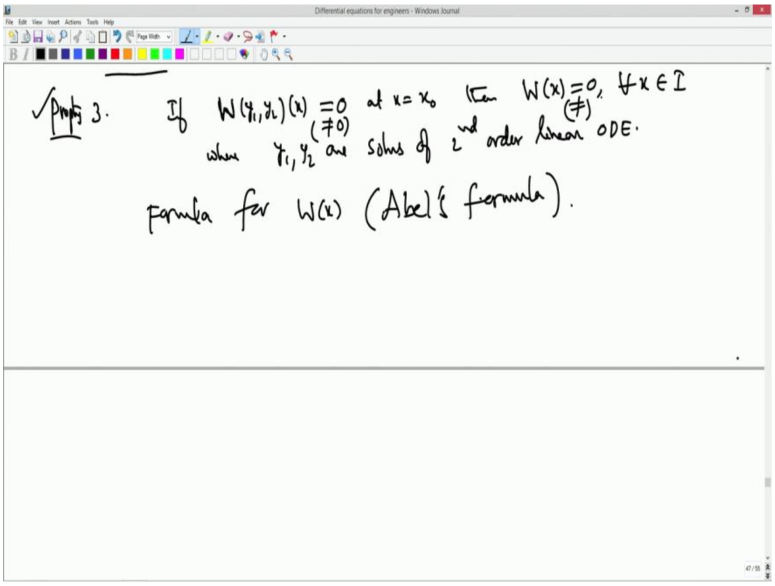
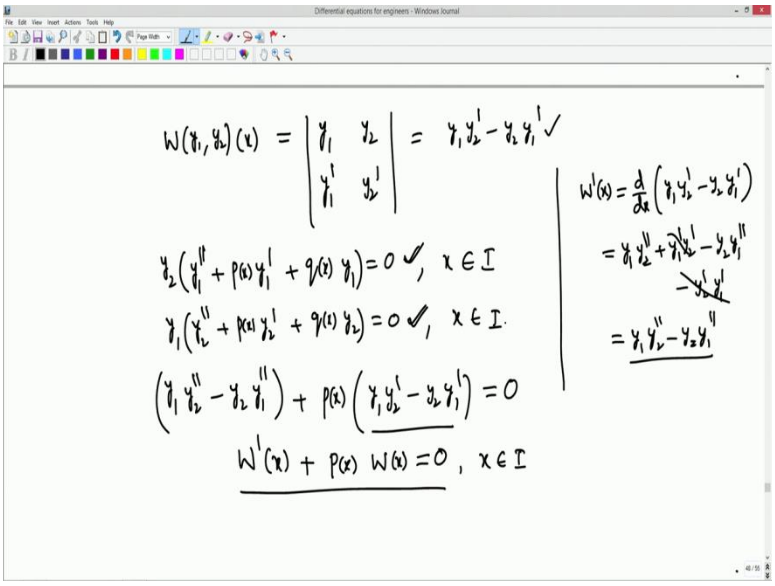
Differential Equations for Engineers.
Professor Dr. Srinivasa Rao Manam.
Department of Mathematics.
Indian Institute of Technology, Madras.
Lecture-11.
Abel's Formula to Find the Other Solution.

So we are looking at the properties of the solutions of the homogeneous second-order linear equation. So first property is that if you are given 2 solutions, y1 and y2, you can make many solutions, as many solutions as you want as a linear combination of these 2 solutions. If y1 and y2 are given, you can make C1 y1 plus C2 y2 as solution where C1 and C2 are constants. Okay, that is a first property. I have given simple property, 2nd one is if I give you 2 solutions, y1 and y2, they are linearly independent.

(Refer Slide Time: 1:10)





When they are linearly independent? They will be linearly independent if and only if certain relation between them should be nonzero, that is, that is what we define as the Wranskian, Wranskian of those 2 functions should be nonzero. Okay. So that is what you seen the 2nd property. They are linearly independent if and only if the Wranskian is non-zero for all values of x, that is what we have seen. So today we give, we give property 3, that is the property 3 tells you that if the Wranskian is, Wranskian is 0 at some point, that means Wranskian is zero everywhere, for all values of x. Okay.

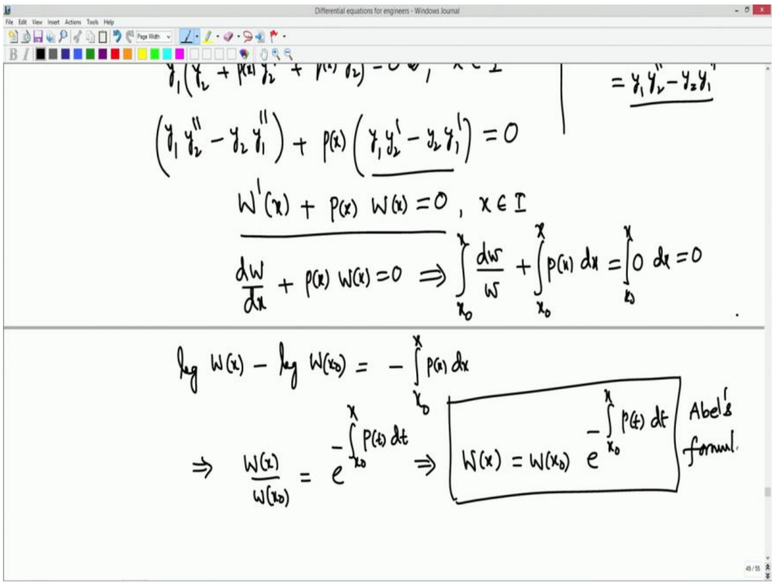
Where x is in I, that is whether differential equation is defined, linear second-order ODE whose solutions are y1 and y2, okay. So if the Wranskian is at some point is non-zero, then Wranskian is non-zero for every point, that is also true, okay. So this actually tells you that Wranskian is zero or non-zero, zero or non-zero at one point, then the Wranskian is nonzero at every other point, okay, at every point it is well-defined.

So this is, this is the formula we derived, this is called as Abel's formula, so for that we just start with, start with the Wranskian, Wranskian of 2 functions, y1 and y2 which is a function of x, which we define it as y1, y2, y1 dash, y2 dash, this determinant which is actually y1 y2 dash minus y2 y1 dash as a function of x. Now we simply calculate what is, what are the given, so y1 and y2 are solutions, solutions of the second-order linear ODE. So it satisfies y1 dash, double dash plus p x y1 dash plus q x y1 equal to 0.

And y2 is also solution of the second-order linear ODE homogeneous equation, so it also satisfied this equation, so y2 dash plus q x y2 equal to 0. What I do is, I take these 2 equations, I multiply one with the y2, right-hand side is zero, so if I multiply the function y2, the right side is zero, this I multiply with y1, 2nd equation I multiply with y1. Now I take this difference, 1 minus other, so you have 2nd equation minus 1st equation, so this minus this, if I take, it will have y1 y2 double dash minus y2 y1 double dash is the 1st term. So this minus this, plus p x which is common in both places of each of the 2nd terms. So we will have y1 y2 dash minus y2 y1 dash plus y1 y2 q minus y2 y1 q, so both are same, so it is 0.

So that is 0, the difference is 0, so this is equal to 0. So this is exactly Wranskian, this is your Wranskian, so I have p x into Wranskian of x which is equal to 0. So what happens to this term? So you can see this one has the side, W dash of x, d dX of the Wranskian, if you actually calculate, you will have a d dX of y1 y2 dash minus y2 y1 dash. So you can easily see that this will be y1 y2 double dash plus y1 dash y2 dash minus y2 y1 double dash minus y2 dash y1 dash. So this gets cancelled, what you are left with is y2 double dash minus y2 y1 double dash.

(Refer Slide Time: 5:28)

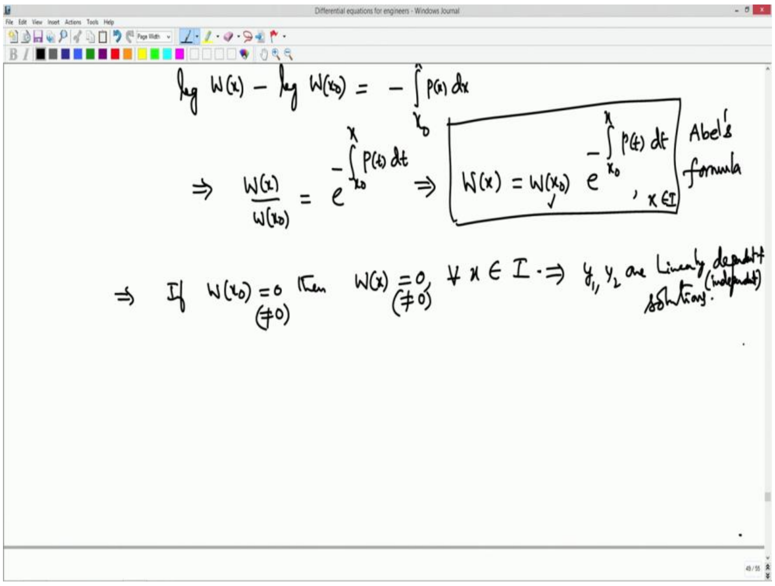


So this is exactly your W dash. So this is exactly your W dash of x. So I can replace this with W dash of x. So this is, this is the 1st order linear ODE, non-homogeneous linear ODE, okay for x belongs to I. Wherever this differential equation is defined, that is where this constant is also defined, so you have this first-order linear homogeneous ODE whose solutions we can simply calculate, this is exactly equal to dW by dx plus pX WX equal to 0. So this implies, I can split, they are already separable, you can separate the variables, all variables are separated, dW by dx equal to minus pX WX, so I can write dW by W plus p xD x equal to 0.

This I integrate both sides from x0 to x because W is the function of x, I am integrating from x0 to x, here also I integrate from x0 to x dx. So that will give me constant, okay. So let me call this constant log C. Sorry, this will not be, this you integrate from x0 to x, x0 to x dx, this will be 0 anyway. So this is equal to 0. So both sides if you do that, that will be 0. So what you are having is log W x minus log W at x0, this is W equal to minus integral x0 to x p x dx, okay.

So this will give me W x by W x0 which is equal to E power minus integral x0 to x p t dt you can price, okay. So this implies W x is equal to W at x0 times E power minus integral x0 to x, p is a function given, so here is dt. Okay. So this is called Abel's formula.

(Refer Slide Time: 7:40)



So this will give you a property that if the Wranskian is 0 at one point, see when I integrate from x0 to x, x0 to x, x0 is an arbitrary constant, so x0 is you can choose x0 from the domain of the differential equation so x0 belongs to I. You can choose any point, you can choose any point as x0 that belongs to the domain of the differential equation. So x0, W at x0, if it is at some point, W at some point x0 in I, if it is 0, if this is 0, WX is 0 for every x, okay. So this is true for every x belongs to I.

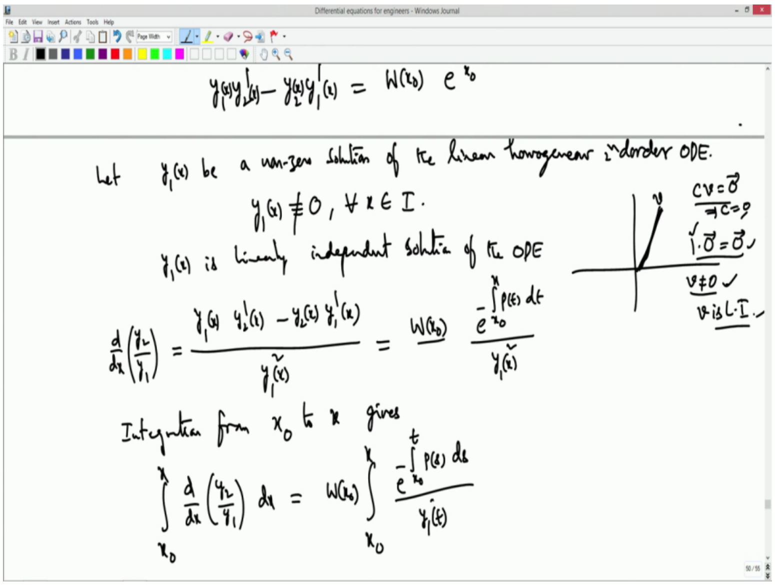
So this is actually true, so what you have is the relation for every x belong to I. So this implies is W at x0 is 0, then at some point x0, then W at x is also 0 for every x in I. Okay. So if this is non-zero at some point, then WX, WX is also non-zero for every x. So this gives you, so what is the meaning of W at x0? That is W at x0 is, if you look at this, this is your, this one. So this is your determinant. W at x0 is 0 means W at x is 0 everywhere. So W at x0 from the property 2, once the W x, W of x equal to 0 for every value, if and only if y1 and y2 are linearly dependent. Okay.

So what we have seen is W x is nonzero if and only if y1 and y2 are linearly independent. So if W at x0 equal to 0, then W at x0 for every x implies y1 y2 are linearly dependent. Okay. So implies y1 y2 are linearly dependent solutions. Okay. So if this is non-zero, if this is nonzero at some point, then W at x is nonzero for all points, that implies y1 y2 are linearly independent, independent solutions, okay, linearly independent solutions, dependent and independent. Okay. So that is your property number 3.

So we have shown the property 3, if it is 0, at one point, that means Wranskian is, Wranskian is 0 at one point means Wranskian is 0 everywhere, implies from property 2, okay, from property 2, you can say y1 and y2, solutions are linearly dependent. Okay, this is all you can conclude just by looking at this formula, Abel's formula. Okay. So you still do not know how to calculate, how to find the solutions y1 and y2 but if you know that your y1 and y2, if you know that you have to solutions y1 and y2, what we have is, you calculate the Wranskian, you can actually calculate Wranskian of these 2 solutions of the second-order homogeneous ODE, linear ODE as Abel's formula if these solutions at one point, if the Wranskian is 0, that means Wranskian is nonzero everywhere, okay.

That implies they are linearly dependent. So this is how we can see that whether given to solutions of second-order linear homogeneous ODE are linearly dependent or not, okay. So if you are given 2 solutions, you can easily, by looking at it is Wranskian, if it is 0, just one point is enough, okay. You have to, you can verify just one point, this solution. y1 at x0, y2 at x0, x0 can be any point in the domain of the differential equation. So you verify the Wranskian, if it is nonzero, implies, you can conclude that solutions y1 and y2 are linearly independent functions, solutions. Okay.

(Refer Slide Time: 12:54)



So we can use this Abel's formula to to use this Abel's formula, to get the, if you are given one solution of the linear homogeneous ODE, you can find the other, other linearly independent solutions by just, by using this Abel's formula. Let us see how it is done, okay. So we start with, so we write the Abel's formula, Abel's formula is this. Abel's formula is y1, y2 dash minus y2 y1 dash, okay. So this is all functions of x because this is of wx, this is wx equal to W at x0, this is simply a constant, W at x0. Okay.

So E power minus integral x0 to x pt dt. So what we do, so I know that if I divide this, if I divide this, if I divide this with y2 square or y1 square, okay, so both sides you divide y1 square. So if I can divide with y1 square provided y1 is, let y1 of x be a nonzero solution, nonzero solution of the ODE, linear homogeneous second-order ODE, second-order ODE. So what does it mean, so if it is a nonzero solution, y1 x is not completely 0 for every x belong to I. Okay. This is not 0, that is the meaning.

That means you are given, you can see that in the plane, you know the 2 vectors are linearly independent, they are independent okay. So you know what is the meaning of this. 0, 0 vector is always linearly dependent. So if you are, if you want one linearly independent vector, that means formal mathematically formal definition is, you can write some constant times that vector v equal to, if you want that to be equal to 0. If you can find such a vector for some nonzero C, then you say that v is linearly dependent. Okay.

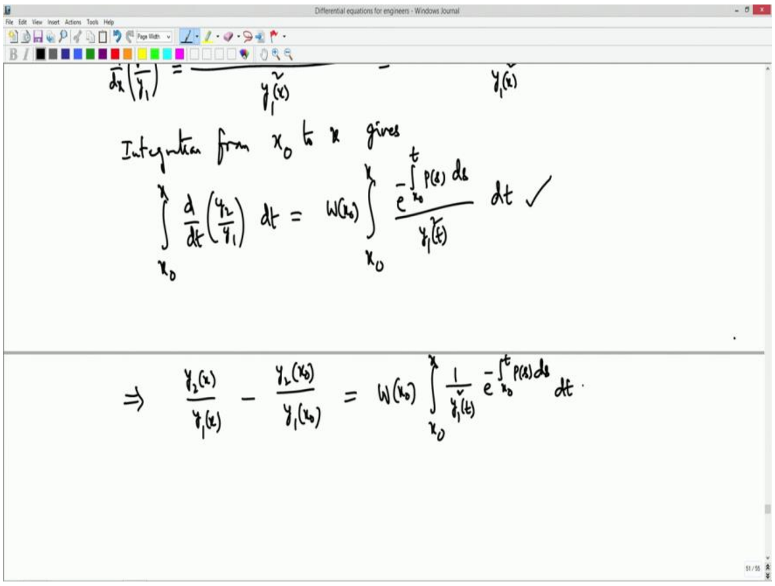
When this is possible, when this is 0, and C is nonzero, v has to be 0. So v, when v is 0, that means zero, 0, 0 that is linearly dependent because 0 vector into some constant which can be nonzero, I can take any, simply for the sake of example as 1, 1 into this is 0. So I can find some nonzero C, such that this vector equal to 0, okay, this is what is the meaning. Otherwise if you cannot make, if you cannot find such a C, so it implies, if you make this, Cv 0, implies C equal to 0, that means v has to be linearly independent.

That is possible only if v is nonzero. Okay. Only if v is nonzero, you can say that vector v is independent, v is linearly independent, okay, I write LI as linearly independent vector. So same thing is true here, so once it is nonzero, this is linearly independent function. So y1 is, y1 of x is linearly independent solution, independent solution, solution of the ODE. ODE means second-order homogeneous linear ODE, okay. So once it is given, if it is, because it is nonzero, I can divide it, assume that it is, you can divide, this is nonzero everywhere in the domain. So y1 x, y2 dash x minus y2 x, y1 dash x, I divide both sides with y1 of x square. Okay.

So W at x0, so W at x0 is actually constant, so some arbitrary constant. So you can fix any x0, W x0 is the solution, so it is an arbitrary constant. The way you solve this equation here, this W x0 is actually constant, that is actually an arbitrary constant, okay, W x0, it is a constant E power minus integral x0 to x pt dt divided by y1 of x square. I square both sides and this, this is exactly, this is now in the exact form. So the left-hand side I can write this as derivative of y2 by y1, right. So this is exactly equal to this.

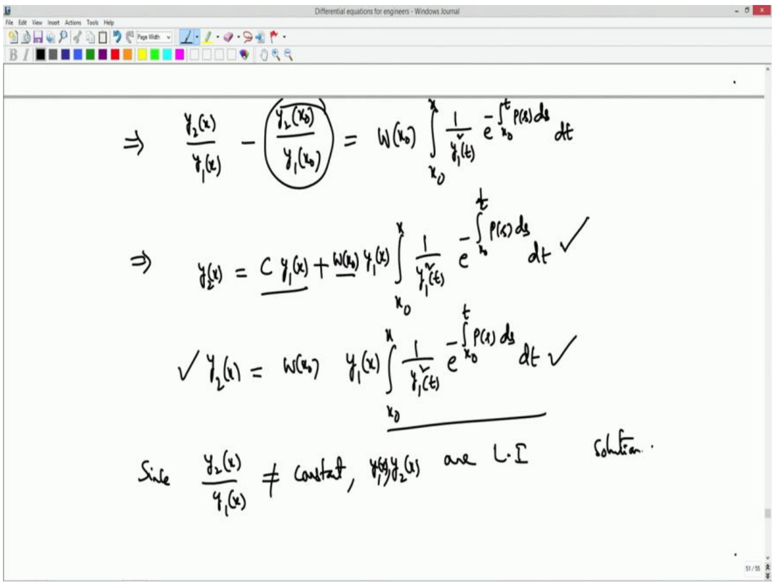
Right-hand side you see this is a constant and this is function of x, and this p is given, so this is a function of x, whole thing is a function of x. Okay. So I can now simply integrate from x0 to x, okay. So integrate, integrate, integration from x to x0, x0 to x, what you get, so x0 to x d dx of y2 by y1, okay. So this you are, this is a function of x, this you are doing it with respect to x, okay. So this is equal to, on the right-hand side is this, W x0 is constant, you are now integrating from x0 to x, this is now, x is a dummy variable, I can write simply E power integral x0 to some t p, now inside this integral I make t as a dummy variable.

(Refer Slide Time: 18:42)



So ds divided by y1 of t square, now dt. Okay. This if I integrate, this function of t with respect to execute to x. So same thing I can do here, so I can make this as a dummy, so dt, so dt. Because I am integrating from x0 to x, Inside integral I tried as a dummy variable. You can keep also x also common no issues. Okay. So this is what you have, so this will directly give me y2 x by y1 x minus y2 x0 by y1 x0, so this is equal to W at x0 times x0 to x, whatever this function, so one by y1 of , y1 square of T, E power minus integral x0 to t ps ds.

(Refer Slide Time: 19:47)



This is a function of T, so function of T, so you have dt. So this is what you have, so this implies y2 of x equal to, so this whole thing is a constant, so you can take it as a constant because you are solving differential equation for y2 by y1. So y2 by y1 at some point gives me arbitrary constant. So this will be equal to y2 by y1 x equal to some constant times, constant I take C plus W at x0 into as it is, x0 to x, 1 by y1 square of T, E power minus integral x0 to x, x0 to t pS dS into dt.

So if I multiply both sides y1, I take it to the other side, this will be C times y1 of x, this will be y1 x, so this means W at x0 into, this is constant times y1 x. So this is what you get as a solution, 2^{nd} solution y2. So I found my y2, y2 as because y1 and y2 are 2 linearly independent solutions in your Wranskian, we started with the Wranskian from the Abel's formula, so what is y2, y2 is actually solution, 2^{nd} solution, you can easily see, y1 is the solution, you can take, you can ignore this term because you already know that y1 is the linearly independent solution and what you are left with the 2^{nd} solution is this one.

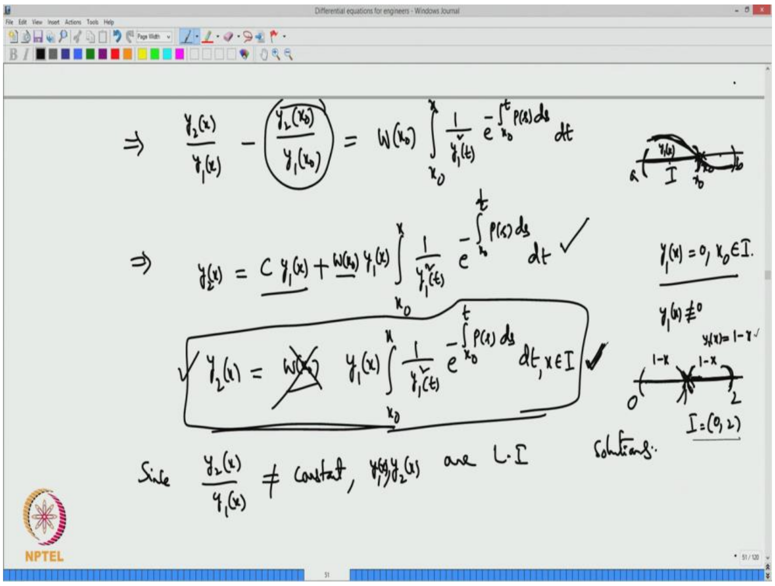
This W x0 is a constant, W at x0 times y1 x into x0 to x, 1 by y1 square of T, E power minus integral x0 to t pS dS dt. So if you look at this integral, this is the function of x, this will never be 0 because it involves integration with respect to the exponential function. Exponential divided by y1, okay, y1 square. So this will be 0. So if you see y2 by y1, if I divide y2 by y1, it will never be, and, it is not a constant. y2 by y1 is not a constant means y1 and y2 are linearly independent since y2 by y1, so from this, y2 by y1 is not a constant.

Y2 is the way you define, this y2 is the 2^{nd} linearly independent solution, 2^{nd} solution. That means that is y1, y1 y2 are or you can directly write, so constant y1 x, y2 x are linearly independent solutions. Okay. So you have found, so if you know one solution, assume that you can divide it, okay, assume that it is nonzero at every point of the domain of the differential equation. If we divide it, what you derived is from the Abel's formula, you get your 2^{nd} solution y2.

Because it is a homogeneous equation, constant times the 2^{nd} solution is also a solution. So you can take, you can ignore this constant, the reason is, if this is a solution, this constant, any constant times of this is also a solution. So, so we were discussing about the properties of solutions of second-order linear ODE, homogeneous ordinary differential equation, so, so far we have seen 3 properties. One simple property, linear combination of any 2 solutions if we give is also solution, 2 solutions if you give me which are, which I can say linearly independent, if and only if the Wranskian is nonzero.

Okay. The 3rd property is the Abel's formula, if the Wranskian at one point is 0, Wranskian is, if it is 0 at one point, Wranskian is 0 everywhere. If it is nonzero, Wranskian is nonzero everywhere. Implies you can conclude that this function, solution y1 and y2 are linearly dependent or independent. So we have seen what is Abel's formula by considering differential equation, linear second-order homogeneous ordinary differential equation for which if you know one solution, y1 x, you can find the other solution, the linearly independent solutions to this formula, Abel's formula that is y2 x is given in terms of y1 x into some integral that involves 1 by y1 of, y1 of t square.

(Refer Slide Time: 25:01)



So you have 1 by y1, so that means the 1st solution should not be 0, so all those points at which 1st solution is 0. So you, so that means it does not make sense, if it is 0, okay. This formula does not make sense, Abel's formula does not make sense if y1 of x is0 at certain points. Okay. So those points you can remove from the domain of the differential equation and consider the differential equation. For example, you see, you take the domain, you take this domain and then, so if y1 is 0 at some point, for example at y1, okay, that is what we will see later on with an example, domain of the differential equation is 0 to 2, then, suppose y1 of x is 0, so in this, for example your solution is 1 minus x for y1 at 1 is 0.

So in that case, this formula does not make sense. So what you do is you consider the differential equation in this interval and in this interval, 0 to 1 and 1 to 2, separately so that y1 of x is nonzero in those 2 intervals. So you consider the differential equation, consider your solution y1 of x in the domain 0 to 1 and get y2 in that domain. Similarly you consider y1 of x in 1 to 2, in the interval 1 to 2 and then you, from the Abel's formula you can get your y2 of x, in the domain, in the domain 1 to 2, in the open interval 1 to 2.

So what happens, if you consider what happens at x equal to1. y1 of x is 1 minus x in 0 to 1 and also y1 of x which is 1 minus x, same 1 minus x in the domain 1 to 2. That is actually continuous function. So at 1, y1 x is defined, it makes sense. So y1 x is actually solution we have seen already. Okay. Now same way, now y2, y2 we found, whatever you find here and here, okay. If it is continuous, throughout what you will see is that, because it is an integral, y1 x is continuous, you can see here.

So y1 x into some integral. So integral is a continuous function, integral, some fixed point to x. So this is actually continuous function, okay because of this integral, in fact differentiable. Okay. So because of this you see that after getting this your solution y2 in each of these intervals, you can see that this is a continuous function. So at 1, you simply take the limit. Limit x goes to 1 from left-hand side, it will be same as limits y2 of x and x goes to 1 from the right-hand side, both are same. So at 1 also it is defined, that is how you have to see this Abel's formula, it is valid in the full domain 0 to 2.

So this is how we will see in the examples when y1 is 0 at some point and how do we find y2, even though it makes sense, it does not make sense at those points, we will just put it in the formula and get your y2 so that actually works, okay. So this is how you can get one solution, 2^{nd} linearly independent solution, once you know one solution, one nonzero solution y1 of x. Okay and then how do you say that we have also seen that this is y1 times something which is not constant because integral x0 to x involved in y1 integral, so it is nonzero, it is non-constant, so that implies y2 by y1 is non-constant.

That means it is, they are linearly independent. So this is what we have seen, we will see with an example in the next video, we will try to give you, we will demonstrate in the next video with examples how to get the 2^{nd} linearly independent solution once we know one solution y1 of x, okay. Thank you very much.