Differential Equations for Engineers.
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Lecture-10.
Properties of Solutions Of Second-Order Homogeneous ODE's.

So we have seen second-order linear differential equation, ordinary differential equation, we have seen how it can be converted to a system, system of coupled 2 equations, first order linear equations, so we have seen how it is done. We will see, we will study today the properties of the second-order linear ordinary differential equations. So we will start with the equation. So let second-order equation is, second-order linear ODE, linear ODE.

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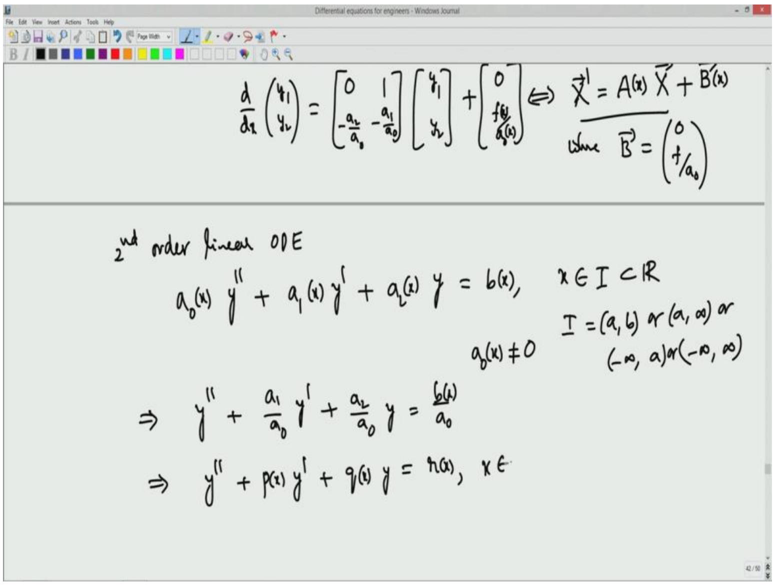

This is A0 x y double dash plus A1 x y dash plus A2 x y equal to some B x. So x belongs to the domain, some interval which is in the real line. So I can be, I can be finite interval or semi-infinite interval. So I can be equal to A, B open interval or A to infinity or minus infinity, some A and some minus infinity to infinity, it is a full R. Okay. So this is how, this is how I can be. Since we have seen that, whenever you have this one, the highest coefficient of the highest derivative should not be zero, so those are the x values when A0 F x should not be zero.

So you have seen that A0 F x should not be zero, that means the domain should be having all the values for which A0 of x is nonzero. Because this is nonzero, I can rewrite, so A0 plus A1 by A0 into y dash plus A2 by A0 into y equal to B by A0, B x by A0. Where A0, A1, A2, B, these are all given functions, so this is equal to say y double dash plus p x y dash plus q x y equal to some you can say R x this is R x. R x is B by A0. So x belongs to I. So what are the properties we have, properties of the solutions of second-order linear ODE which is homogeneous when R equal to 0.

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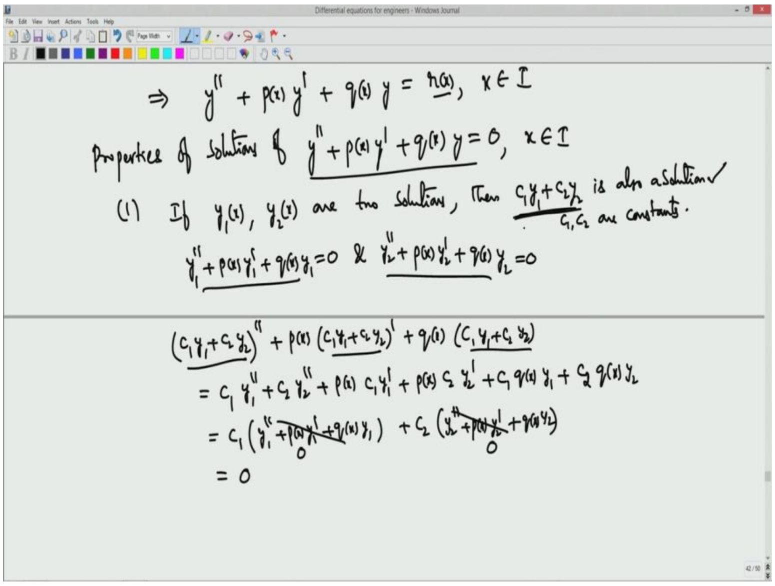

For the properties, we will, we will find some properties. properties of solutions of homogeneous equations. y double dash plus p x y dash plus q x y equal to 0. So we consider only homogeneous equation, that means the right-hand side, R x is zero. So such an equation, so we look at the solution. Suppose I give you, one, property one is, if I give you 2 solutions, y1 y2 are 2 solutions of this equation, then its combination, linear combination is also a solution. So let me write what exactly it is.

If, if I give you 2 solutions y1 x and y2 x are 2 solutions, then linear combination C1 y1 plus C2 y2 is also solution. Where C1 and C2 are constants, C1, C2 are constants. So how do we see this? you know that y1 is a solution. So you are given y1 and y2 are 2 solutions, so you substitute, so you see that y1 double dash plus p x y1 dash plus q x y1 equal to 0 and y2 double dash plus p x y2 dash plus q x y2 equal to 0. Okay. Now we want to see C1 y1 plus C2 y2 double dash, substitute, just simply C1 y1 plus C2 y2, y2 into the equation, just verify whether it is zero or not.

So left-hand side I am the substituting. We have a p x, C1 y1 plus C2 y2 dash plus q x C1 y1 plus C2 y2 equal to, what is it, this is actually equal to, you can rewrite C1 because it is a constant, if I differentiate this, this will be y1 double dash plus C2 y2 double dash plus p x C1 y1 double dash, y1 dash plus p x C2 y2 dash plus q C1 q x y1 plus C1 plus C2 q x y2, this I can write, take the coefficient of C1, coefficient of C2, what you have exactly is y1 double dash plus p x y1 dash plus q x y1 and here y2 double dash plus p x y2 dash plus q x y2.

And we know that this is zero because these are known equations, these are known solutions. y1 and y2 are solutions of the equation, so this has to be zero, this has to be zero, okay. So that makes it completely zero. So that implies this one if I substitute in the place of y, is also satisfying the equation. Okay. So that implies this is also a solution. Okay. So this is how you show that. If I give 2 solutions, linear combination of, this is called the linear combination C1 y1 plus C2 y2 is a linear combination, this is a linear combination of able to solutions y1 and y2. This is also solution.

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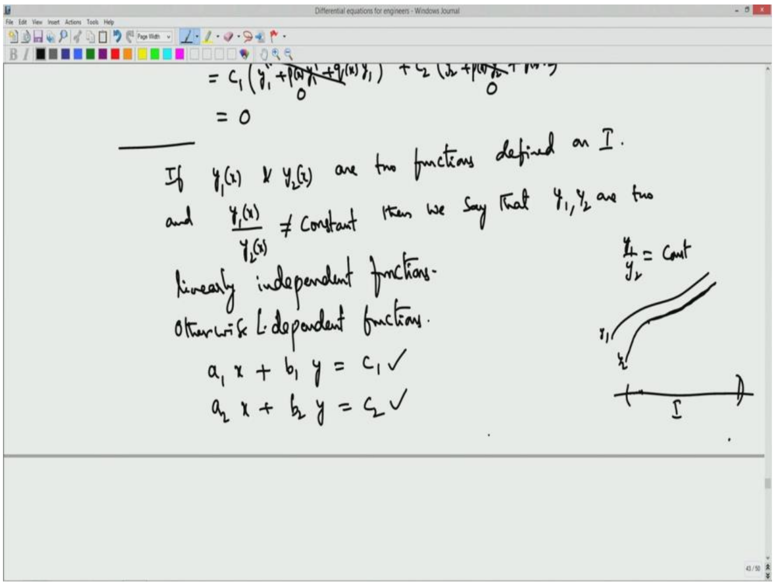

So before I proceed to give you some other properties, I define what is, what is called independence. If I give 2 functions, if y1 x and y2 x, they are not solutions of the equation, y1 and y2 x are 2 functions, 2 functions defined, defined on I. That means x belongs to I. I can be any interval, open interval. And they are and, let us say y1 of x divided by y2 of x, you divide, you divide it 1 over it. If this is equal to, not a constant, if this is not equal to constant, then we say that y1 and y2 are 2 linearly independent, 2 linearly independent functions.

So that means 2 functions are linearly dependent if their y1 by y2 equal to constant, that means one is constant multiple of other. That means 2 functions like this, y1 is this, y2 is this, if there exactly parallel, wherever defined over I, I is from between this and this. If they are parallel, they are linearly dependent. If they are not parallel, say, let us say like this and then goes parallely, since they are linearly independent because because y1 and y2 is not a constant. It is constant upto here, upto here is constant but here, from here to here it is varying, it is not a function of, it is just, it is just function of x.

At this point, this divided by this is different, okay. So at this point you have this divided by this, this is your y1, this is your y2, y1 by y2, everytime everytime you pick up the value here, so you have a different value, so it is not a constant. Okay. So we will say otherwise they are dependent, linearly dependent functions, okay. Like this A1 x1, A1 x plus B1 y equal to C1. And then A1 x plus B2 y equal to C2. Suppose you have the system of linear equations. When do you have a solution exists for these linear equations?

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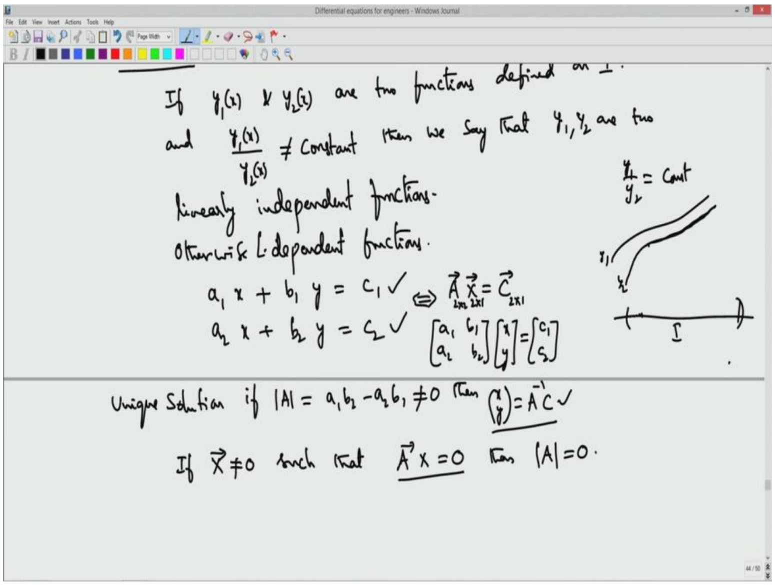

You have solution exists, solution is, unique solution is possible if the determinant, what is the determinant, if I can rewrite equivalent me, the system of equations, you can write AX equal to some C, these are, this is matrix, this is vector, this vector and this matrix, 2 by 2, 2 by 1, finally 2 by 1. So what is A, A1 B1, A2 B2, x is x, y, equal to C is C1 and C2. If the determinant is, if the determinant of this matrix not A, okay, so the determinant of A that is A1 B2 minus A2 B1, if this is non-zero, I know that I can invert the equation, I can invert this matrix to get the x, y.

X, y anything but A inverse of C, okay, this is how you get. So this is the condition. So and also another way of saying is, suppose if I can find some non-zero, if x nonzero matrix, it is a vector which is nonzero, such that A x equal to 0, okay, so that means if I choose my C1 and C2 zero, 0, 0, then I have a unique solution, when A is having inverse, that is zero. Okay. If this is nonzero, so that means if this, if I have a system for which you have a nonzero solution, then I must have determinant of A should be equal to 0. Okay.

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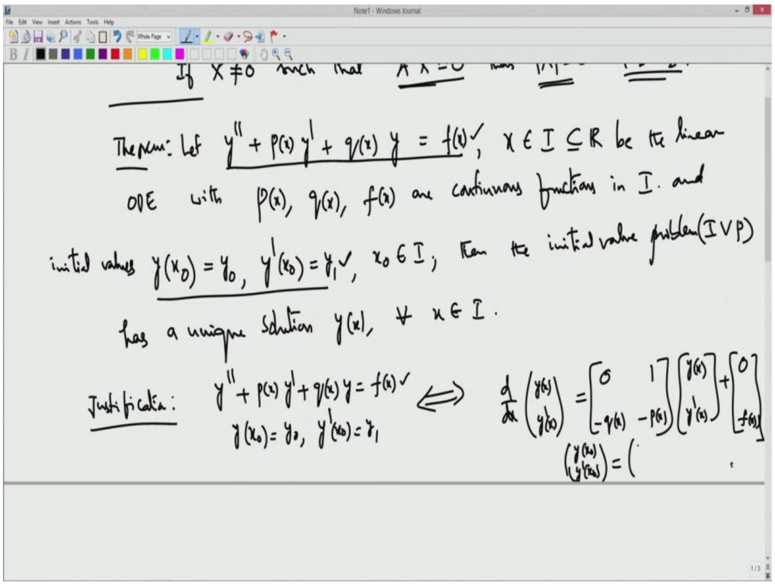

If this is non-zero, suppose this is non-zero, then I can invert it, I will get only x equal to 0 but it is given that it is x, non-zero. Okay. So before I give you property number 2, so I will explain some existence and uniqueness theorem for the linear second-order ODE, so let me write as theorem. So your equation is y double dash plus p x y dash plus q x y, let us take zero, it cannot be zero, you can take any F x, right-hand side, non-homogeneous equation, where x belongs I, I is open interval on full real line.

So let be equation, be the ODE, it is a linear ODE with p x, q x and F x are continuous functions. Suppose p x, q x and F x, differences are given and if they are continuous functions and what you have is, you have a second-order linear non-homogeneous equation and then once you see these coefficients, p x, q x and right-hand side F x, if they are continuous in I, so you have a unique solution. This equation will have a unique solution, okay. Unique solution if you provide initial conditions. So and you provide the initial conditions, that is y at x0, x0 belongs to I.

So x0 is given as y0, y dash at x0 equal to y1. So where x0 belongs I. This will be with initial values, initial values. So function value, unknown function value and its derivative is given at some point x0. Then this, the initial value problem, so that consists of this and these 2. We will have then the initial value problem, I will write this initial value problem, initial value problem, that means this one and this one together. So IVP, you can call this IVP as a unique solution. Unique solution y x for every x in I. So we can say this one.

So this is actually, it is, so we can somehow, the proof is beyond the scope of this course, what we do is, we will just give the justification why this theorem is true. Okay. So basically proof is by converting this initial value problem this and this, you convert this as a first-order system that as we have done earlier. So if you write, some justification we can give, justification of the proof. So y double dash plus p x y dash plus q x y equal to FX and you have these initial values y1. So this is actually, you can put it as equivalent form as d dx of first-order system, system of equations are first-order equation for a vector valued function.

So for a vector y at x, y dash at x, so this is your vector. So for this if you take the derivative, what you get is the matrix, 0, 1 and this will be your, y of x, okay. So this is your y of x and y dash of x and then see you have minus q x minus p x. This is the matrix, if you actually do this, y dash, first element is derivative. So y dash equal to, if you actually take this element, this multiplication, you will that y dash. y dash equal to y dash, so the right-hand side should be 0.

And what happens here, the 2nd element, y double dash, that is coming from this equation. That is F x, so I should have F x here, okay I have F x and then minus p is x y dash and minus q x y, so that is how you get the system. And what is the initial value, y at x0, y dash at x0. So this vector at x0 is already known. That is given as y0, y1, where y0, y1 are the real numbers. So this is how you see, view the second-order system like this.

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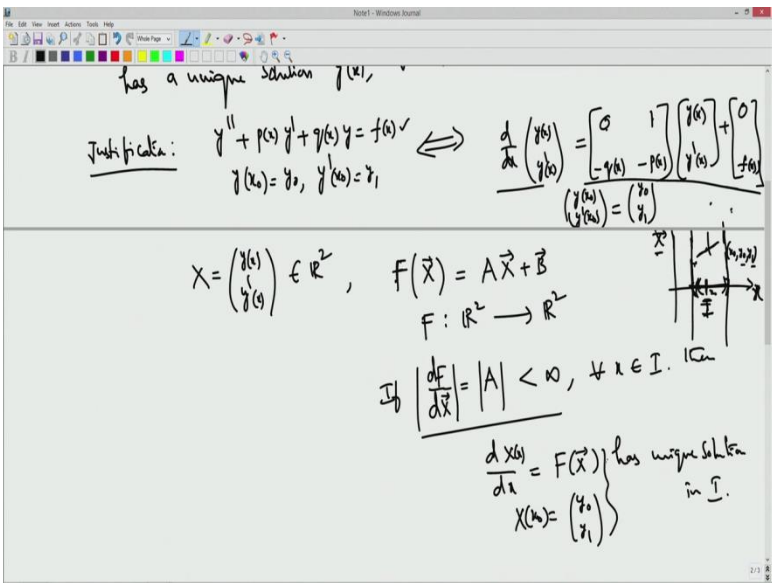

So if you see this as a vector valued function y x, y x and y dash of x, okay, so this belongs to R2. And what is your right-hand side, so what is your right-hand side, d dx of this, this is the left-hand side, remaining is the, this is what is the right-hand side, okay. This is your right-hand side, the right-hand side is actually the function of, some function of, function of, let me write this as a function of, this is you call this x, okay. So F x, it is small x, okay. It is a function of x, so F x is A x plus B, that is A is, if I call this vector as x, x of x plus B.

This is a vector, this is a vector, this is a matrix, A is the matrix, okay. This is how we have with the initial value, so first of all this is what it is. So this is, this is a function from x is here in R2. So R2 to, where it is going, so this is from 2 by 2 multiplying with 2 by 1. So this is going to be 2 by 1, okay. So 2 by 1 vector, 2 by 1 vectors together is 2 by 1 vector. So so it is basically again and R2. So what is the derivative F dash, okay, if I want dF by dx, this is actually, you can, you can view this as a scalar equation as the derivative is actually the matrix itself, it is an operator, okay.

So this if you do not know, this is from R2 to R2, any differential, any derivative, derivative of F with respect to the vector is actually, think of this as A x plus B, so its derivative is simply A. So you have this is an operator. So this is the derivative, derivative with an operator here because it is from R2 to R2, okay. So if this is, if this is known, so that means what is the modulus, what is, where is A, it is an operator, A is 4 elements, it is in R4.

So modulus of that is actually square root of, squares of sum of the squares of all the 4 elements in the matrix. Okay. If that is finite for every x in I, that is true because zero is continuous function, continuous function, q is continuous function, p is continuous function, they are defined on I. Square of this, p square, q square, they all consider functions on I, they are finite, so you can say this is finite. So this is the, this is a sufficient condition. Okay. Because you have chosen them as a continuous function, they are continuous functions and these are all continuous functions, so this condition is satisfied.

So this is can be, this is satisfied, then you have this system, the system dx by dx, okay, x of x equal to F of x as unique solution with, with x at x0 equal to some y0, y1, so that is what is given. So this has unique solution, unique solution in I, okay. So, so you can also view. So this is actually, this is more general theorem, so if you have a system like this, if this is true, this is bounded in a rectangle like this. So you take this x0, so initial value is x0 and your, this you think of R2. Okay.

This is your x, this is your x, this is your x and this is your capital X. So you are actually in the space. So at this point you will have some rectangle, so if at all the neighbourhood, the neighbourhood of x0, so let us say this is your I. x0 is here. So you will have a box containing x0 and that y0, y1, middle point is y0. So this middle point is actually x0, y0, y1. Because y0, y1 is in x. Okay. So in this neighbourhood, in this box, if these functions are continuous, you will have the unique solution.

So passing through you will have the space curve in the x. That means you have a yx in the neighbourhood of x0, okay. That is what is the existence theorem. But if you see that there is a global existence, if you see that it is actually continuous, if you function is, it is, I remove this, lower, upper rectangle. So if this is full, okay, for every x0 belong to I, y0, y1, that takes all the values, let us take whatever all the vector is here and there is in this direction, okay. If in that if this is true, that is actually true, because there are only functions which are only depending on this, x variable.

So they are all bounded, so in that case you will have a global solution, not only the neighbourhood, so it will be, you will have full, after here, whenever it is defined in the full I. So that is what is the existence uniqueness theorem, you will have unique solution, you have a solution, it is actually unique. Okay. So you can assume that without proof, kind of justification is this is what you will see in the textbooks. If you are interested you can get into some reference books, you can see that existence theorem, you might see this form, so that this same as this form. Okay.

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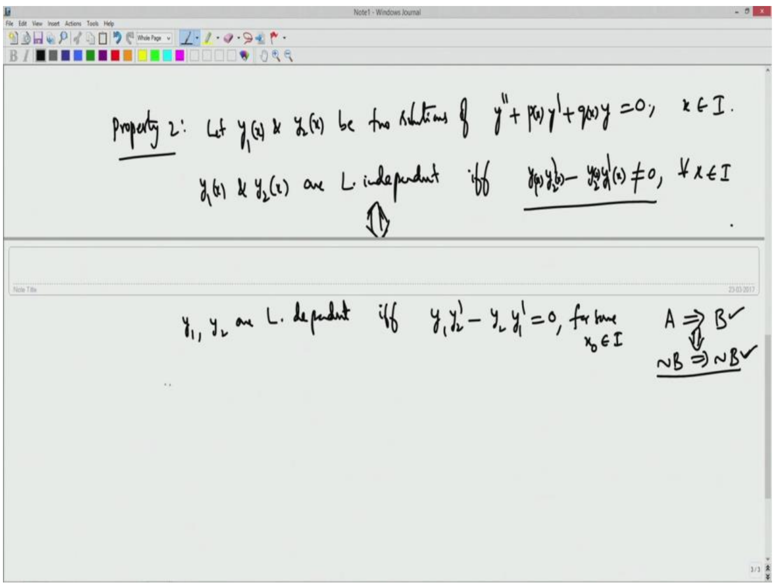

So with this I will give a property number 2, so property 2, property 2 is if you are given the homogeneous second-order linear homogeneous equation, that is let y1 and y2 be 2 solutions, 2 solutions of y double dash plus p x y dash plus q x y equal to 0 where x belongs to I. If it is given, when do I know that they are linearly independent? So y1 x and y2 x are linearly independent if and only if I will have some expressions to be satisfied by these functions y1 and, by the solutions y1 and y2. That is y1, y2 dash minus y2 y1 dash which are which are functions of x, which has to be not equal to 0 for every x in I. So this is what is the property number 2. So we can say if you are given 2 solutions which you do not know how to find but if you are, as you are having 2 such solutions, then you can say whether they are linearly independent or not, they are linearly independent if you can check this condition, okay.

So this is equivalent to, this is equivalent to put it in this way. So if you if you know the logic A implies B, counter positive is not B implies not A, so you can put it in that form. So if I write, if I use that logic, if this is, if these are equivalent, these 2 statements are equivalent, keeping this in mind, you can rewrite the property as y1 and y2 are linearly, y1 and y2 are linearly dependent if and only if y1 y2 dash minus y2 y1 dash equal to 0 for every x in I. Okay.

I can do like this, for some x0 belongs to I. Right. So if this is not true, that means there is some x0, if this is not true, there is, there will be some x0 that is true implies they are linearly dependent. If they are linearly dependent, so that means you are going from here to there. So if they are linearly independent, the not of this, not of this is there exists some x0 that belongs to I such that this quantity is zero. So we will prove this statement in a straightforward way.

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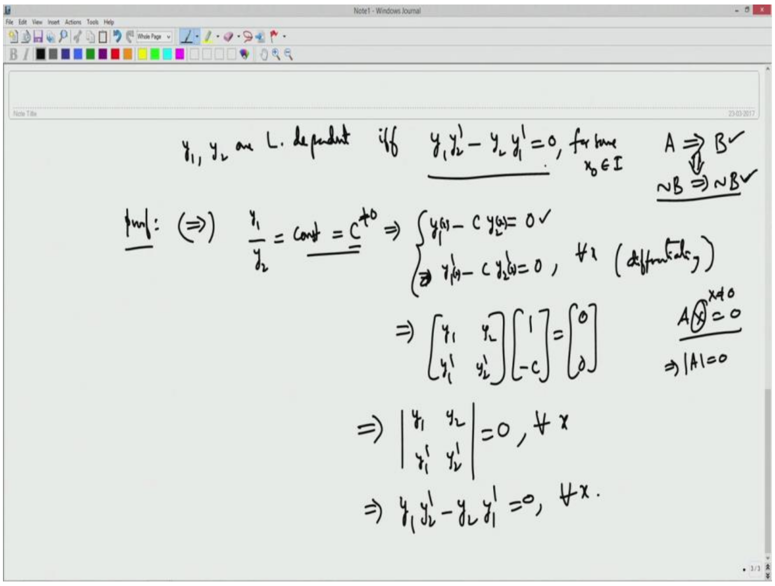

If you write, rewrite like this, this we can prove nicely. Okay. So start with they are linearly dependent. So what you mean by, so we will go from this side, okay. So we will prove from this side, assuming that y1 and y2 are linearly dependent, then we will show that this point is zero, some point. Okay. So they are linearly dependent means they are actually parallel. That means y1 by y2 is a constant, okay. Let us call this some C, that implies y1 minus C y2 equal to 0. Okay. So you can differentiate this because y1 and y1, y2 are satisfying, second-order equations, if you differentiate y1 dash minus C, this is a constant, y2 dash is also zero.

This is true for every x, okay, this is true for every x. So this means, this together means, this is by differentiating, differentiating the first, this equation you can get this equation. So this you can write as a matrix, y1, y2, y1 dash, y2 dash, matrix and 1, minus C equal to 0, 0. So I know that C is a constant and they are dependent means and once they are linearly dependent, I, you are given that they are nonzero solutions, 2 nonzero solutions, okay, you can take it as 2 nonzero solutions.

If there are 2 nonzero solutions, this constant should be nonzero, okay. So this constant should be nonzero, so you have 1, minus C like that you have nonzero solutions for a system A x equal to 0. So when do you have nonzero solutions for the system, that means the determinant A has to be zero. Okay. And x is non-zero satisfying this, implies non-zero. So that implies modulus of A has to be zero, that means here, what is the modulus of y1, y2, y1 dash, y2 dash equal to 0 for every x.

That means this is exactly your condition, y1 y2 dash minus y2 y1 dash equal to 0 for every x. So we only need at some point, so we can take some x0. So that means this is true. So if this is dependent, I prove this one, it is straightforward, okay.

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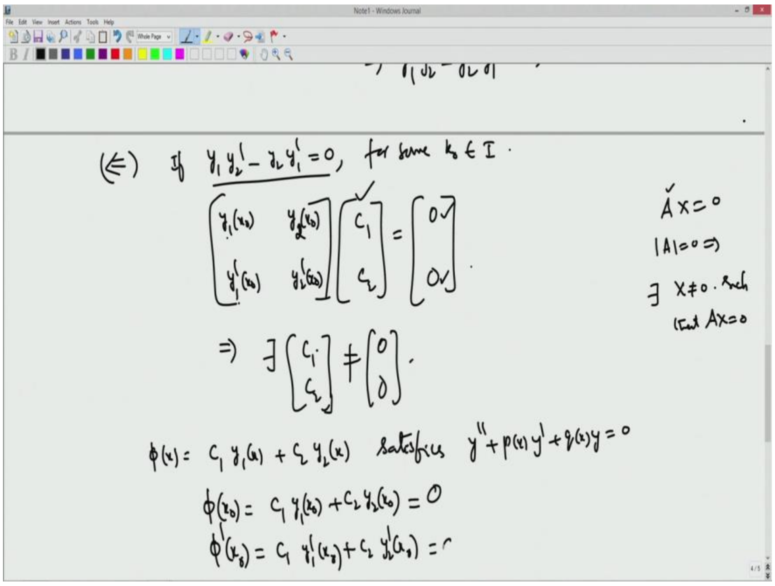

So let us prove the other way that if I assume the condition is or, okay. If I assume y1, y2 dash minus y2 y1 dash equal to 0 for some, okay, for some x0 belong to I, then I want to show that they are linearly dependent. So what you do is, you consider the same system, so whatever you have y1 but you consider at those values, okay. y1 at x0, y, y2 at x0, so the system you consider, okay. So y1 dash at x0, y2 dash at x0, this matrix into some C1 and C2, some constants, you take it, you do not know this, okay.

So you consider this system, so you have A, you consider AX equal to 0 where A is given like this with these values where y1, y2 have the values, y1 and y2 and its derivatives at the point at x0, okay, for which this quantity is zero. So you have, you consider the system and you know that this determinant is zero. So you know that this determinant is zero, that is what is given implies if the determinant of A is zero, implies I will have a nonzero solution, okay.

There exists nonzero x0, such that A x equal to 0. So you have nonzero solution for C1 and C2, so that implies there exists C1, C2 matrix, vector nonzero, so you will get such a nonzero solution, okay. Now we consider, we consider some phi x, which is, you make it C1 with y1 x, with these constant, with this constant, so let us assume that both are nonzero, okay. Now you will see that both of them has to be nonzero, one cannot be zero and one. So certainly this is nonzero means one of them is not zero. You will see that both of them are actually not zero. Okay.

So whether they are zero not, one of them or both of them, whether you know, was there one of them is zero or not, with these you know that there exists some C1 and C2 vector which are nonzero, as a vector it is not zero. That means it can be, one of them can be zero. Such a C1 and C2, you form some function phi like this, C1, y1 x plus C2 y2 x because y1, y1 and y2 are solutions of the homogeneous equation. So what happens, this satisfies, once you form this, this satisfies equation, the equation, the ODE, the ODE is, the equation is y double dash plus p x y dash plus q x y equal to 0.

And what are its value at x0, phi at x0 is C1 y1 at x0 plus C2 y2 at x0. What is its value, this is simply, you multiply this vector with this. That is y1 C1 plus y2 C2 at x0 that is zero. So that is zero. And similarly phi dash at x0, if you actually from phi x is this, phi dash if you calculate, C1 at y dash, y1 dash at x plus C2 x y2 dash at x, that is the derivative at x. You put x equal to x0, and then this becomes y2 y1 at x0, okay. This is again, the 2nd element, y1 dash into C1 plus y2 dash C2 is also zero. Okay.

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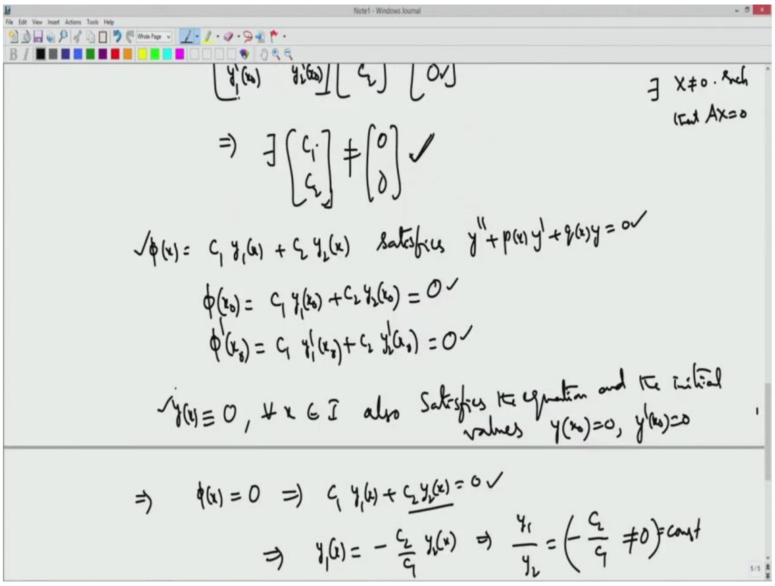

Now from the uniqueness because p and q are continuous functions and you do that, if we take this, so with this phi satisfying this equation and these are the initial condition phi at x and phi dash at x0 is zero. So that you have such a solution, C1 y1 plus C2 y2, okay. And you know that zero is also satisfying, y of x equal to completely zero for every x in I also satisfies the equation and the initial values, initial values. Y at x0 is zero, y dash at x0 is also zero. Because you simply differentiate yx equal to 0, you get that. Okay.

So I have 2 such functions, phi is also satisfying the initial value problem, zero is also satisfying the initial, by the uniqueness of the solution, for the differential equation with the initial condition, okay. By uniqueness of the initial value problem, this phi x should be equal to 0. So what is phi x? C1 y1 of x plus C2 y2 x should be equal to 0. So that means I can write y1 x equal to minus C2 by C1, okay into y2 x. That implies y1 by y2 minus C2 by C1. So this is not a constant, okay. Assume that C2 is zero, if C2 is zero, okay, we feel that this is not zero, this is, as a vector this is not zero.

So if C2 is zero and C1 is non-zero, then C1 into y1, because y1 is a nonzero function, C1 has to be zero. So both, that is not true. If C2 is zero, C1 is also zero why, from this, okay. Assume that you have got some C1 and C2 satisfying this, you are working with those C1 and C2. Assume that C2 is zero, then my C2 will be zero here, then C1 y1 will be zero. That means C1 also will be zero, okay. C1 is zero because y1 is nonzero solution. So both, if C2 is zero, C1 is also zero.

So that means both C1 and C2 should not be zero. If you want this vector C1 and C2, this is nonzero. So that means C1, C2, both are nonzero that means the ratio is also nonzero. So this is exactly constant, right. This is exactly constant, is a constant, that implies this is exactly the definition of y1, y2 are linearly dependent, they are parallel to each other. Okay. So that is how you show, so if you are given a 2 linearly independent solutions, 2 solutions, 2 nonzero solutions of the homogeneous equation, they are linearly independent if and only if this condition is satisfied, this condition is non-zero for every values of x.

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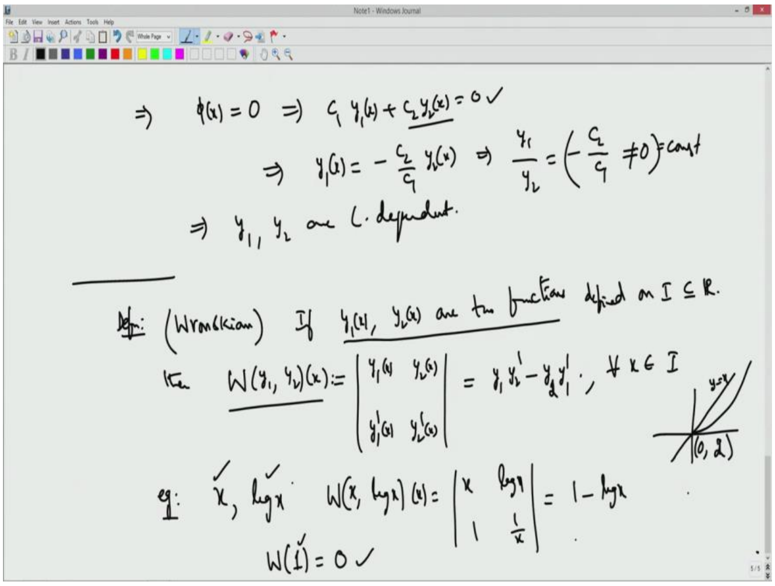

That is equivalently we are putting in this way and then we shown nicely, very easily just by considering a small system we can show that they are dependent if and only if that quantity is zero at some point x0, okay. So this naturally, so we can define naturally, so that is the definition, so there is, whatever you see on the right-hand side, the quantity, we can define as a Wranskian, this is a Wranskian of, Wranskian of 2 functions okay.

Wranskian of, so if y1 x, y2X are 2 functions, 2 functions defined on, defined on I which is in R. Then Wranskian of y1, y2 of x is defined as the determinant, that is y1 x, y2 x, y1 dash x, and y2 dash x. So this is your y2. This is, this determinant is nothing but your quantity. So y1 y2 dash minus y1 dash y2 or y2 y1 dash, so this is exactly, this called Wranskian, this is the definition of Wranskian. So if you are given, these are you see, you can, you can note that these are 2 functions, I define this Wranskian like this for every x in I. Okay.

So these are only 2 functions, they are not solutions of the linear homogeneous equations. If they are linearly, if they are solutions of the linear second-order homogeneous equation, then they are linearly independent if and only if this Wranskian should not be zero at every point, okay. So but this is not true for general functions. You can consider some example like I give you. So 1, log x, I consider these 2, okay. So what is the Wranskian, Wranskian of these 2 functions, Wranskian of 1, log x or x, log x.

If you think of x, log x, let me see and we define the domain later, so this is simply x, log x, 1, 1 by x. Okay. So what is the determinant, 1minus log x. So you can see this, you know that x, log x, x, log x is, x is, y equal to x is this, okay. Y equal to x, y equal to log x is some other things, right, it goes like this. Simply you define only between open into will 0 to 1, 0 to 2 let us say. You see that these functions are clearly linearly independent but they are not the solutions of any differential equation, right.

Because the Wranskian can be zero, Wranskian is this, Wranskian at x equal to1, okay, Wranskian of x, log x, so function of x at 1, this is zero. Okay. Wranskian is zero, Wranskian is 0 implies, Wranskian at some point, it is 0, right, so at 1, at 1 it is zero, so it implies they have to be linearly dependent. That means they have to be parallel to each other but that is not the case that is true, that means if you want the Wranskian, if you want the Wranskian is 0 at some point, there, then the functions y1 y2 are linearly dependent only if they are the solutions of, that is true, only if, that is true only when these are the solutions of second-order linear homogeneous equation, okay, not for other functions, this is the example, okay.

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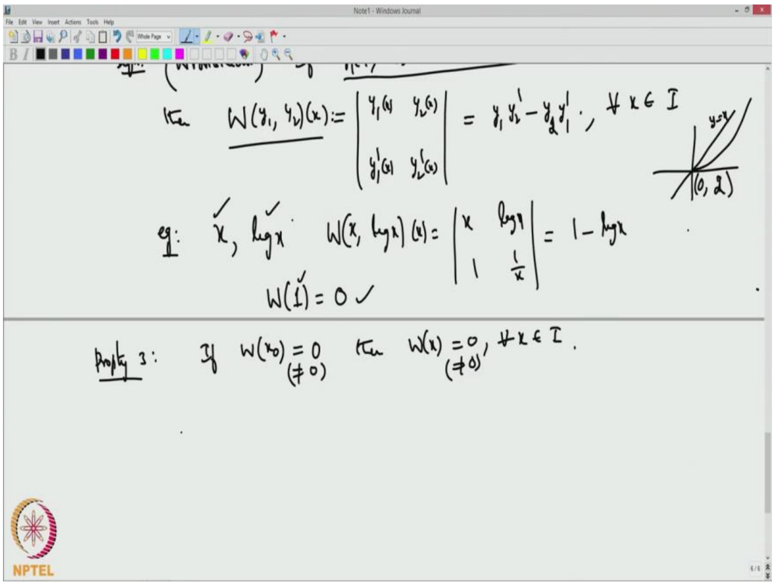

So this if you consider these 2, though the Wranskian is zero, they are not actually linearly dependent. Okay. So we will have a property 3, property 3 in the next video that tells you that if Wranskian of, Wranskian at some point is zero or non-zero, okay, then Wranskian at x at every point is zero or nonzero every point. So this is done by deriving a formula called Abel's formula. Okay. So next class we will derive Abel's formula and then we will see this property 3. So once the Wranskian is zero and the Wranskian is 0 Everytime.

The Wranskian is nonzero at some point, then the Wranskian should be nonzero at every other point. This is proved by looking at a formula called Abel's formula, we will see in the next video.