Commutative Algebra
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Lecture - 9
Isomorphism Theorems and Operations on Modules

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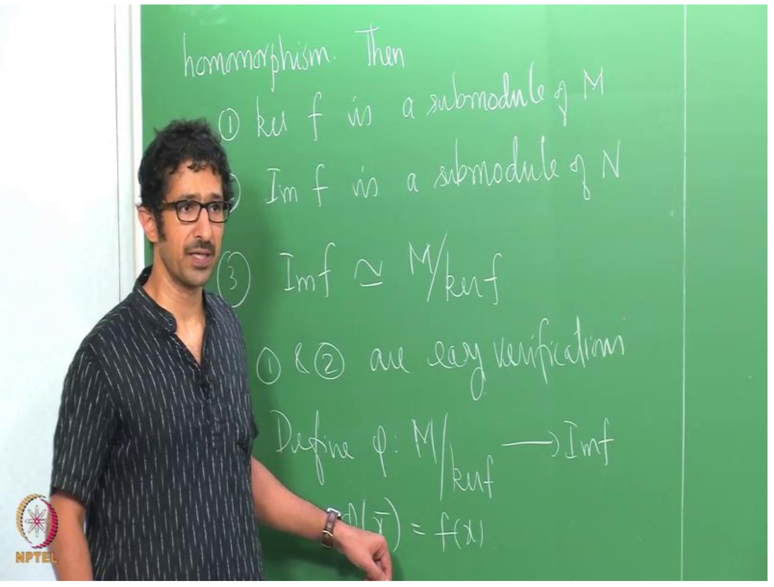

So, let f from M to N be any module homomorphism. Then one kernel f is a sub module of M, image f is a sub module of N, and third one image of f is isomorphic to M module of kernel of f so I think the 1 and 2 are easy verification. A proof for the third one you have seen for groups probably for rings as well, but let us quickly recall. So suppose you have define a map phi from M mod kernel f to image f by phi of x bar equal to f of x. So in this case we have to prove that it is well defined because there could be many representatives for x bar. So if x bar is equal to y bar what does that say.

Student: X minus y belongs to kernel f.

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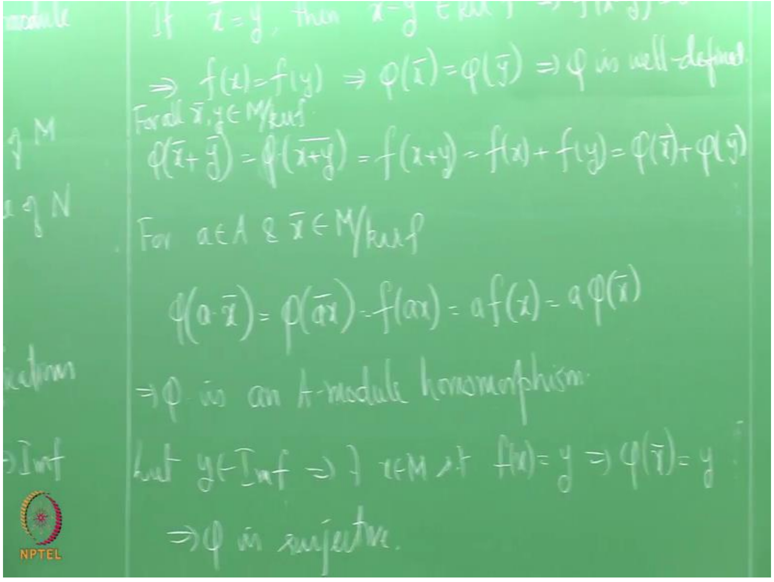

So these 2 equivalence classes are same implies that x minus y belongs to kernel f. Which implies that f of x minus y is 0. Since f is a module homomorphism, this says that f x equal to f y, which implies that phi of x bar equal to phi of y bar. This implies phi is well defined. Now why is phi a homomorphism, because f is a homomorphism so phi of x bar plus y bar is f of x plus y. So this is x bar plus y bar by definition of addition in quotient module this is same as x plus y bar, so let me phi of x plus y bar which is equal to f of x plus y, which is same as f x plus f y, which is same as phi x bar plus phi y bar so for all x bar y, bar in M module of kernel f.

And for a in A and x bar in M module of kernel f phi of a times x bar, again by definition of scalar multiplication on M mod kernel f this is same as a multiplied with x bar is same as a x whole bar. By definition of phi this is since f is a is an A module homomorphism this is equal to a times f of x which is a times phi of x bar. Therefore, phi is an A module homomorphism why this injective.

First let us see why is a surjective, can you immediately see that it is surjective, every element of image f is of the form f of x for some x in M, so therefore take it is image in M mod kernel f x bar phi of x bar will be f of x. So let y be in image f this implies there exists x in M such that f of x is equal to y and that implies p of x bar is equal to y that implies p is surjective.

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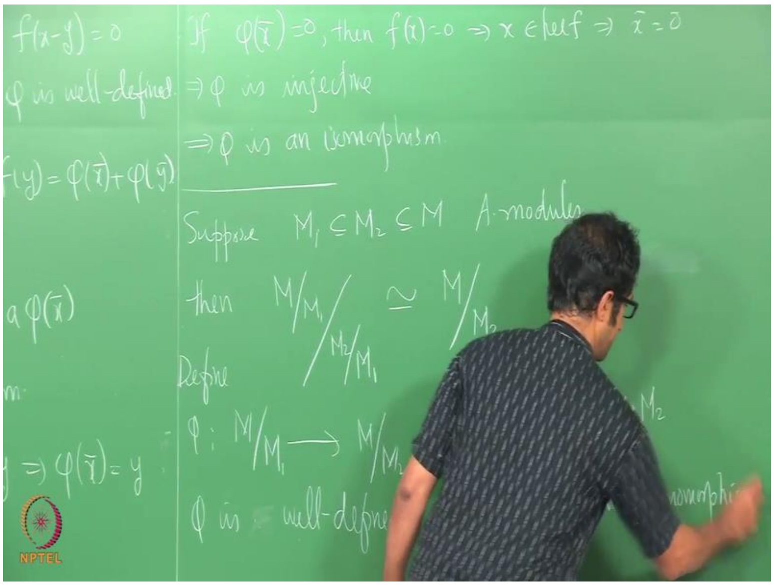

Why is phi injective if phi of x bar is 0 what does that mean.

Student: (Refer Time: 06:20).

Sorry.

Student: (Refer Time: 06:28).

That is true so one can directly go back from. Here this implies this implies this or you know this implies f of x equal to 0 which implies x is in kernel f which implies x bar is 0. So therefore, so this phi is injective this implies phi is an isomorphism and that proves the third statement. So this is the proof is pretty similar to the proof that you have done for groups, and for rings. There are some interesting consequences as in the case of groups or as in the case of rings and so on.

Suppose I have M 1 contained in M 2 contained M so A modules which means M 1 M 1 is an a sub module of M 2 and M 2 is an a sub module of M and say and hence M 1 in an a sub module of M then M mod M 1 mod M 2 mod M 1 is isomorphic to M mod M. There is you have seen in the case of groups as well as rings right. If you have in the groups what did we prove if H if you have something like this, then g mod H module of k mod H is a isomorphic to g mod k.

So and r mod I if you look at J contained in I contained in r J and I ideals of r then r mod J module I mod J is r module I so this is an exactly similar statement. Now how do you prove this and I said this is a corollary of first isomorphism theorem.

Student: From mapping m mod m 1 to m mod m 2.

So from M mod M 1 define let us say phi from M mod M 1 to M mod M 2 by phi of so to talk about different representations. Let us write x plus M 1 this equal to x plus M 2. First of all, this should be well defined, is this well-defined. If I have 2 different representatives for let us say x plus M 1; that means, x plus M 1 is equal to y plus M 1 which is same as saying x minus y belongs to M 1, but M 1 is contained in M 2 therefore, x minus y belongs to M 2 and hence x plus M 2 is same as y plus M 2.

So this is well defined map and one can check that this is a homomorphism. So phi is phi is well defined and phi is an A module homomorphism.

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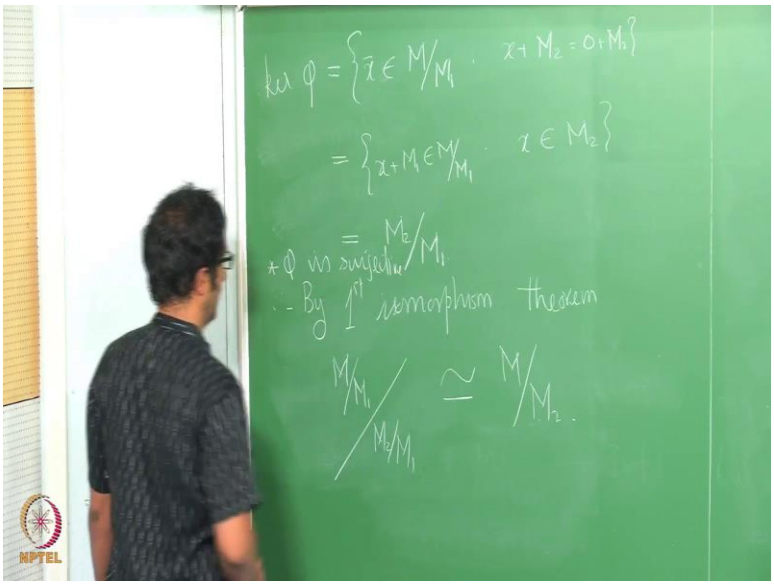

So, now what is the kernel of phi? Kernel of phi by definition at a set of all x bar in M mod M 1 such that x plus M 2 is 0. Or in other words I mean to say 0 plus M 2 or in other words the setup are so let me use this notation x plus M 1 and M mod M 1 such that x minus 0 which is x, x is in M 2. So this is precisely by definition this is M 2 mod M 1. So by a first isomorphism theorem, M mod M 1 module of the kernel which is M 2 mod M 1 this is isomorphic to M mod M 2 now suppose you have.

Student: Do not we need to be shown surjective to adjective?

Yes, correct so this is phi is this is again another phi is surjective, why is phi surjective, if you take any x plus M 2 in M mod M 2 then look at x plus M 1 and phi of x plus M 1 will be x plus M 2 therefore, it is surjective, but yes this needs to be shown.

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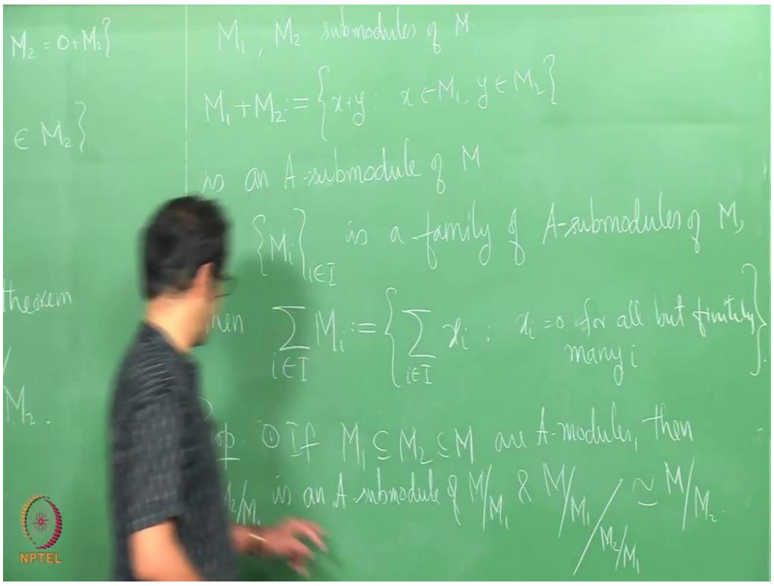

Now suppose I have 2 sub modules M 1 and M 2. Sub modules of M then we can you know as in the case of groups and ideals and so on, one can define M 1 plus M 2 this by definition this is set of all x plus y x belongs to M 1 y belongs to M 2. This is called sum of 2 modules. And this is again in A module this is again an A sub module of sub module of M. And take if I have, if this is i in I is a family of sub modules of a sub modules of M.

Then you can generalize this definition of sum of 2 modules. Then one can define summation M i, i in I this to be set of all x i, i in i, but then, whenever we deal with rings modules groups and all these objects we do the addition operation only finitely many times. We do not talk about infinite edition. So therefore, what we do is what we say is that x i equal to 0 for all, but finitely many i.

Now, one can talk about another application of first isomorphism theorem. This is one application another application of first isomorphism theorem again similar to what you have seen in groups as well as in rings. That if M 1 and M 2 so maybe I will write this as a proposition. First let me just write this as a statement if M 1 contained in M 2 contained in M are A modules. Then M 2 mod M 1 is an a sub module of M mod M 1 and M mod M 1 module of M 2 mod M 1 is isomorphism to M mod M 2. That is a first statement that we proved earlier.

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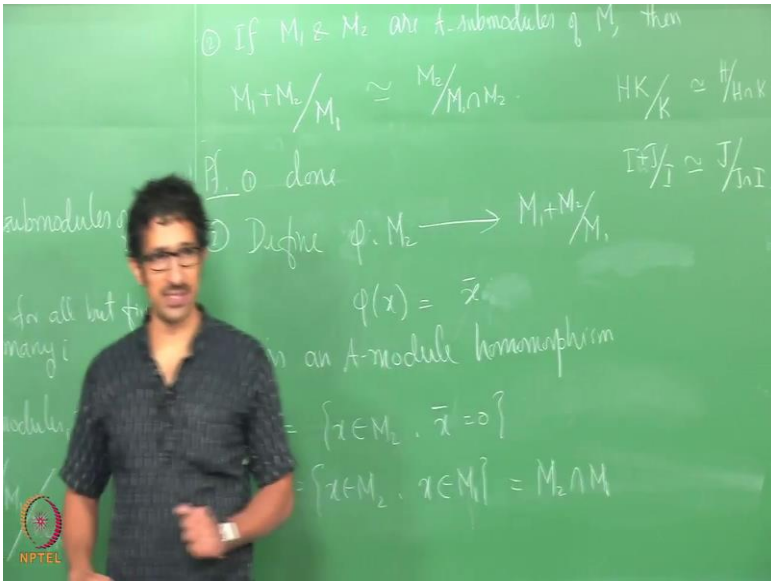

The second statement is if M 1 and M 2 are a sub modules of M, then what can you say about this g H mod H.

Student: (Refer Time: 18:37).

This is so are you know H K mod K or I J sorry I plus J mod i. Now these are something that you must have seen in your group theory group and rings theory course right. This is isomorphic to H mod H intersection K this is isomorphic to J mod J intersection i, similarly you want to say this is isomorphic to M 2 mod M 1 intersection M 2. So how do you prove this? Again think of as an application of first isomorphism theorem.

Student: (Refer Time: 19:50).

Define a map from.

Student: M 2 to m plus 1.

M 2 to plus 1, so define phi from M 2 to M 1 plus M 2 mod M 1 by phi of x equal to x plus M 1 or will just say since there is no confusion will just say x bar. Here we do not have to worry about well definedness because there is no question of 2 representatives and so on this is A module any element has it is own existence there is no representative issue here.

So this is a well-defined map. We need to say that this is a homomorphism. Is that clear it is a homomorphism. Again I will phi is an A module homomorphism. Phi of x plus y is x plus y whole bar which is x bar plus y bar therefore, which is phi of x plus phi of y and a times same thing. So therefore, it is a, A module homomorphism. Now if we prove one more and there are 2 more things what is kernel phi kernel phi is.

Student: (Refer Time: 21:32).

All elements in M 2 x in M 2 such that x bar is 0. Or in other words all elements in M 2 such that.

Student: X belongs to m 1.

X belongs to M 1 which is same as M 2 intersection M 2. So now, what remains to be proved do we have to talk about one-one?

Student: Point to surjective.

So what we need to show is that phi is surjective. How do you prove that it is surjective how does any element here look like? M 1 plus M 2 any element in M 1 plus M 2 looks like some x plus y bar or x plus y plus M 1 as a cosset.

Student: Y, y is 0.

It is y belongs to so if I take x in M 2 and y in M 1 y belongs to M 1 which means.

Student: (Refer Time: 22:43).

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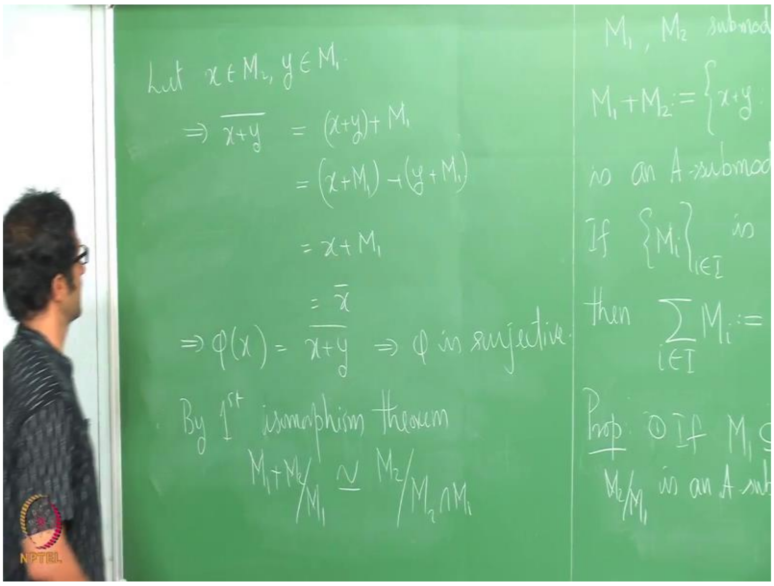

X plus so let x be in M 2 y in M 1 then x plus y whole bar this is same as x plus y plus M 1, but then this is same as by definition in the quotients this is same as x plus M 1 plus y plus M 1, but what is y plus M 1 which is same as 0 in quotient module. So this is equal to x plus M 1 or in other words this is same as x bar for that we need a representative x is in M 2.

So if I take any x in M 2 and y in M 1, this implies that phi of x is equal to x plus y bar. So this implies this implies that phi is surjective. So therefore, by first isomorphism theorem; M 1 plus M 2 mod M 1 is isomorphic to M 2 mod M 2 intersection M 1. We have already shown that the kernel is M 2 intersection M 1. Now when we talk about summation then next that should probably come to your mind is product, but in the case of modules there is no product right when we talked about ideals I and J. We talked about the product ideal I J, but if you talk about 2 sub modules of M say M 1 and M 2 we cannot talk about M 1 M 2 right because it is there is no product defined on modules.

But then, there is another product we have a product what kind of product scalar multiplication.

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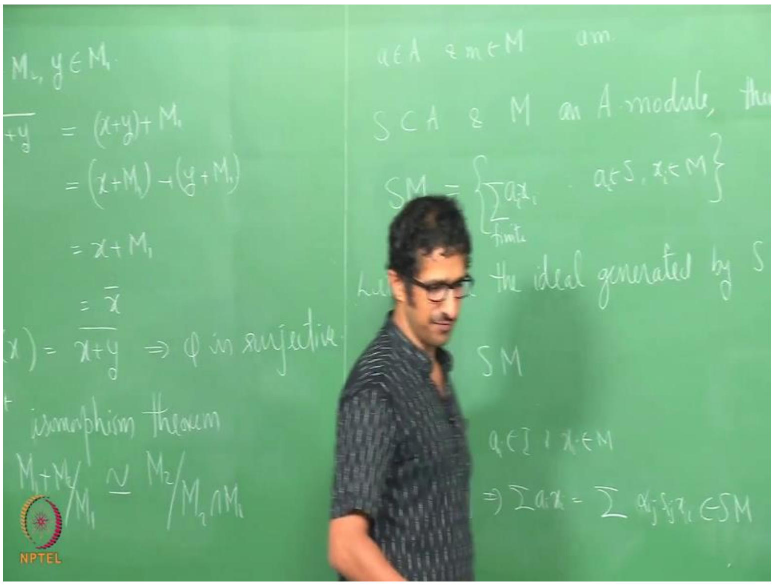

So, one can talk about what is called you know for any a in A and M in M A M is well defined. So what kind of product you can think of in this case. If I take any set S and subset of A and M and A module, then one can talk about S M right, but what would I mean, how would you define this to be.

Student: Summation a I.

Again as in the case of ideals we saw, we are not only talking about a x where a is in I and x is in J, we have to talk about their summation. Then only you will have a structure on that otherwise it will be just a set it would not have a structure. We are interested in studying structured sets.

So here if I define this to be equal to summation and if I define to be a x where a belongs to s and x belongs to M, there is no problem, but that would not make any sense we will just be set so instead what we do is we take a i x i finite summation with a i from x. So most often we in this case a small observation that, if I look at let I be the ideal generated by S. What does what does that mean I take all finite linear combinations of elements of S with elements from scalars from r I mean the ring a so if I is the ideal generated by S can you relate I M and S M.

There is a immediately there is an inclusion which is this is contained here can we say this is contained here. If I take any element here how does any element here look like summation a i x i where a i belong to I and x i belong to M. Now how does any a i look like any a i look like summation let us say alpha i J S J some finite summation. Yes, here also I should say S i J because for each i there could be different sets different elements.

So therefore, this summation a i x i looks like summation alpha i J s I J x i now alpha i J x i is again in M. So therefore, you can think of this as s I J times some element in M so this belongs to S M so this S M and I M are same therefore, we will stick to most often we will stick to I M, but then there are some simple cases where we use the notation S M.

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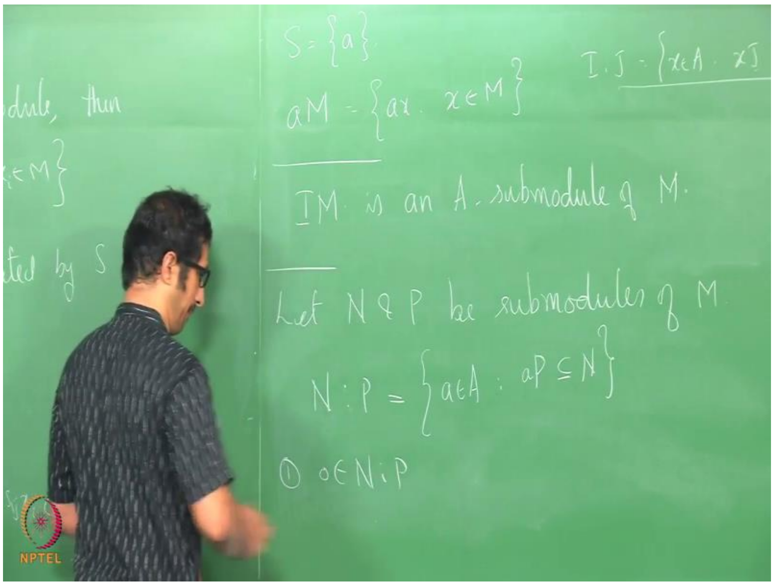

For example, if your set S is singleton a. Then we write a M instead of ideal generated by they are the same. So this is set of all a x, x belongs to M. So this is one such place where we use this notation S M.

Now, well this we define this set and we saw that if I take I M and S M. They are the same if I is generated by s is this an a sub module 0, belongs to it. And for any x here minus x is here x is of the form some summation a i x i, just take minus of a i for each right so additive inverse is there additive identity is there, the you know it is addition is a billion is kind of it comes from the module itself so therefore, this one can see that this is an scalar multiplication again it is straightforward verification, I M is an a sub module of M.

Again going back to the theory that we developed in the case of ideals, another object that we studied was if I take 2 ideals I and J then looking at their colon, right I colon J. So in this case if I have N and P be sub modules of M, what can you say about how would you like to define N colon P. How would you like to define this does this make sense in the case of modules of course, we do not have a multiplication in modules? But then we have a hint line here right what is the possible way of defining.

Student: If a x belongs to a subtract x in containing n.

So, I have this A M you know we have just defined if I look at an element a in A with see a P would be a sub module of m, but then we want to look at all elements which will multiply it and take it inside. See how did we define I colon J? This was set of all x in a such that x J is contained in I.

So here see any ideal is in A module. So this if I consider I as an module and A module this should be true. So what I want is this should be inside n. It is a sub module of n, but now what we have obtained as a set in a not in M. So the colon operation in modules give a, a set in A. So if you have a certain a, what question would you like to ask well where the first thing is see you are inside a ring. So whether it has a it has any of the ring structures is it just simply a set or whether it has it is abelian group where it is sub ring where it is ideal you know there are lots of questions.

So, what do you think about this set? If for example, think of the situation where your M is A itself your module is A itself. And you take 2 ideals I and J then they are sub modules and in that case the definition is same right. In that case this turned out to be an ideal. So in this case what would you expect? Would you expect it to be an ideal in A? So first let us see whether this is does this have 0, 0 is there right. So 0 belongs to N colon P if x and y are in N colon P.

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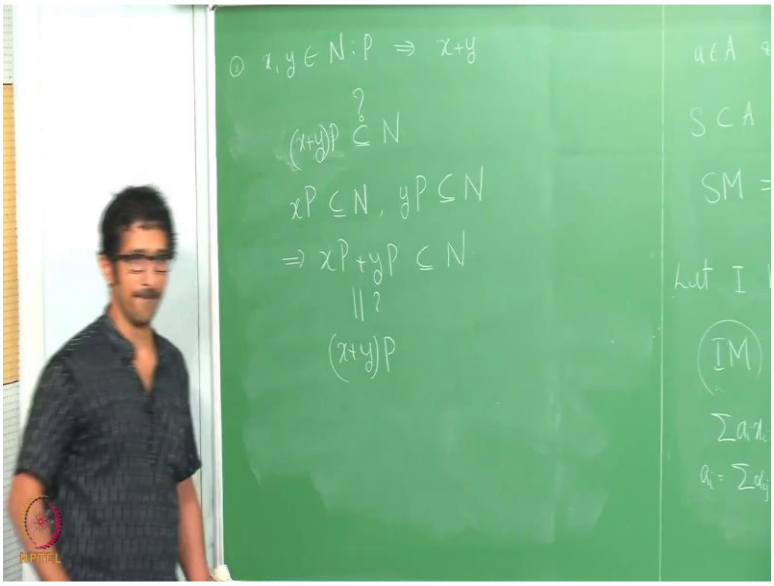

What can you say about x plus y? X comma y in N colon P does that imply x plus y is in N colon P x P is inside N y P is inside N.

Student: X P.

N is a sub module therefore, x P plus y P is in N. So here you know one should note that x plus y I mean x plus y P, the see our aim is to see whether this is contained in N. What we know is that x P is contained in n and y P is contained in N. So if I take x P is in N, y P is in N this is a sub module this is a sub module. So sum of both are sub modules of N. So their sum is also a sub module of N. This implies x P plus y P is in N, but is this same as x plus y P.

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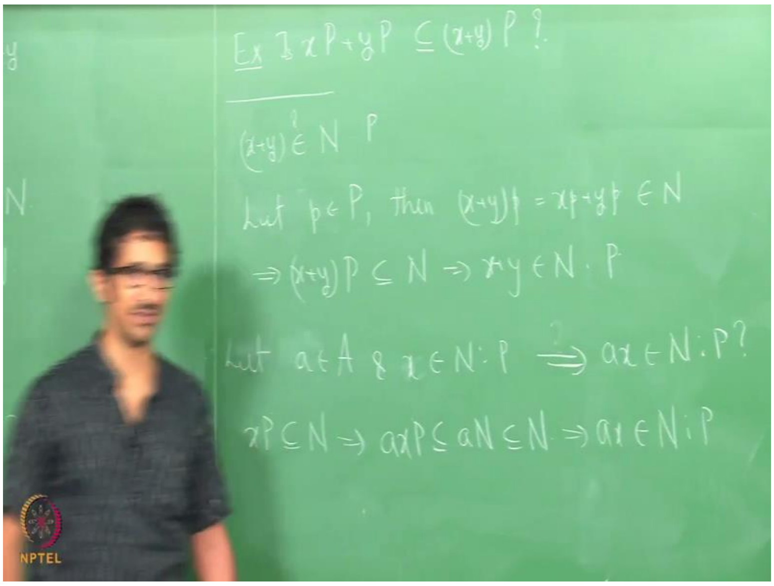

There are few things that you should is x P plus y P contained in x plus y P is convince yourself.

But we are not really bothered about that. See what is that we need to prove we need to prove that x plus y belongs to if x and y are in N colon P whether x plus y belongs to N colon P. If x and y are belonging to P so I take any x plus y P so let p be in P, then what can you say about x plus y times P. This is equal to x P plus y P x P is in N y P is in n therefore, this is in n so this implies that.

Student: Since x plus y P is the subset of x plus y P so it will be contained.

This is contained here this is contained, that is true, so we have the other inequality is true so therefore, this is p is contained in N so this implies x plus y is in N colon P. So therefore, N colon P is closed under addition. Therefore, this is an abelian group what about whether it is if I take an element a in A, and r in x in N colon P is it true that a x is in N colon P x P is contained in n therefore, a x P is contained in a n which is contained in N right. X P is contained in n this will imply that a x P is contained in a N which is contained in N. Therefore, a x is contained in N colon P.

So, what have we proved? N colon P is an ideal. So this is an ideal that plays a lot of role in in the structure of modules in in various roles.

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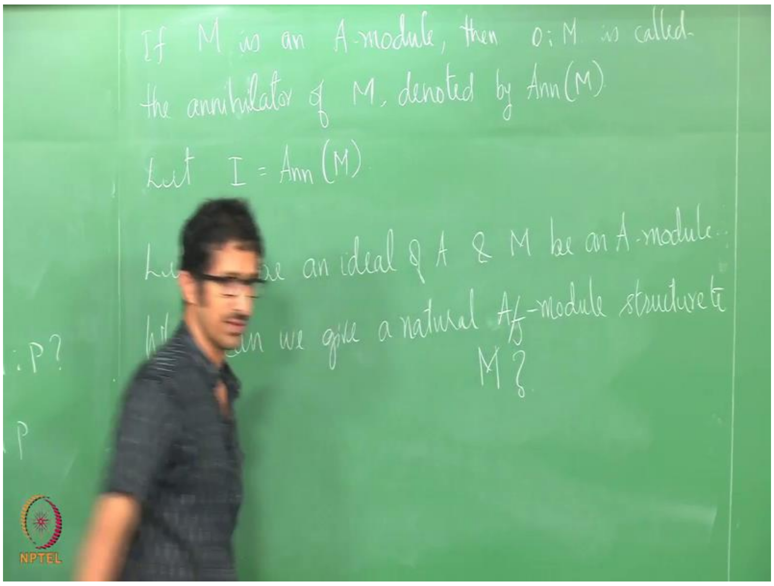

For example, if you take if M is an A module then 0 colon M. So take n to be 0 and p to be M. This is called the annihilator of M.

Now, if I take I to be so denoted by if I is the annihilator of M. Then see in general A mod I is an A module, but so A mod I is also a ring, but you cannot I mean in general you cannot give A module structure to A over A mod I over quotient ring you cannot give A module structure to the ambient.

Similarly, if I have if M is an A module then M mod N is an A module, but if I take I to be an ideal and A to be a ring and M is an A module in general we might not be able to say M is an M mod I module. Can we say? So let me put it this way let J be an ideal of A and M be an A module. When can we say? Give a natural module struck A mod J module structure to M.

Think about this question, we will take it up next time.