Commutative Algebra

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Lecture – 08
Modules and Homomorphisms

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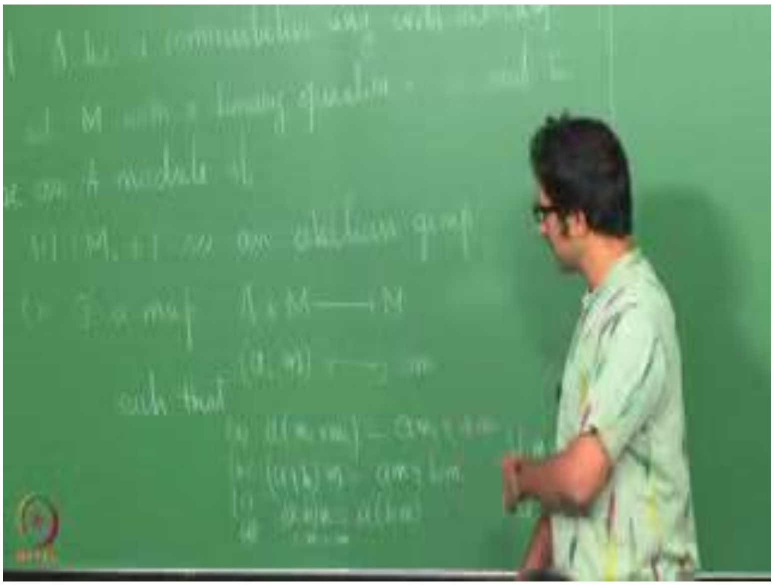

We shall begin with the definition of modules. You must have seen vector spaces over a field. This is exactly the generalization of vector spaces over fields to the case of objects over rings so in our situation, in general one can define what is called a left module a right module and so on, because your ring need not be commutative. We would not go into those details. We will stick to commutative rings and hence the definition involves only modules. Left modules and right modules are all the same. So let A be a commutative ring with the identity. A set M with a binary operation is said to be an A module, if what should be the conditions. I said it is a generalization of vector spaces into the case of modules so what should be the conditions.

Student: A plus.

The first is the intrinsic property that this this should be an abelian group. And then there are you know interactions with the ring. That is there exists a map A cross M to M with, so A comma M going to A M, map such that what are the properties that are essential.

Student: A is distributive.

It should be distributive over the summation a m 1 plus m 2 should be equal to a m 1 plus am 2, then if I take a plus b multiplied with m it should be a m plus b m.

Student: A comma.

The associativity multiplication that is a b multiplied with m is same as a multiplied with b m.

Student: One dot.

Yeah.

Student: One dot m becomes.

1 dot m is m so this for all m, m 1 m 2 in m and a comma b in a.

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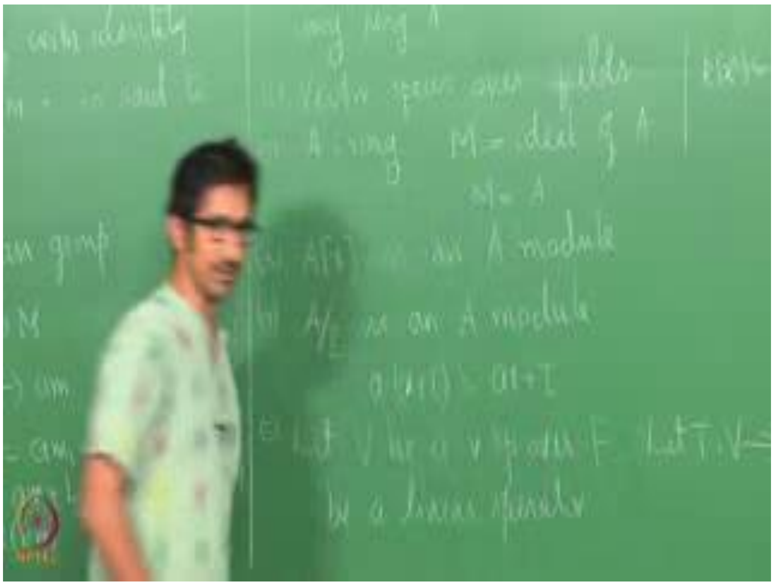

So, these are basic properties that need to be satisfied. Let us look at some examples. What should be the zeroth examples?

Student: 0 comma.

If you take any ring and take M to be 0 is an A module. For any ring A, then as I said this is and you can see that it is a generalization of vector spaces into the case of rings. So if you take any vector space over fields they are all modules A modules f modules.

Now, can you think of start with the ring A can you think of an M.

Student: Ideal of a ring.

M equal to ideals an ideal of A if you take a sub ring.

Student: (Refer Time: 06:46).

If you take a sub ring which is not an ideal then it need not be a module it need not be closed under this map might not be there. For example, if you take sub ring k x square of k x right. If you take k x square in k x, then k x square is not a sub ring with respect to the standard multiplication. So, but if you take any ideal of a in particular the most trivial example is M equal to A itself, every ring is a module over itself.

Student: A x.

A x. If you take a any ring this is an A module with the standard addition multiplication and so on what more can you think of if I take any, if you take any ideal so given a ring what are the possible ways of constructing new rings right. One is sub ring then adjoining a variable, another way is taking quotient. So if you take A mod I this has a natural. This is certainly an abelian group and it has a natural multiplication coming from A right, this is an A module so how do you define scalar multiplication here a times x plus I is.

Student: A x plus I.

A x plus i. With this scalar multiplication A mod I is an A mod I.

Student: A to the power n.

A to the power n, so if you I will come to the direct sums little late, another important example which is if you do structure theorem of finite abelian groups. There is a very nice example that comes up there. Well this is one way to look at it. So take a vector space. Let v be a vector space over a field F. Then there exists you know linear operators. Let T, T from V to V be a linear operator.

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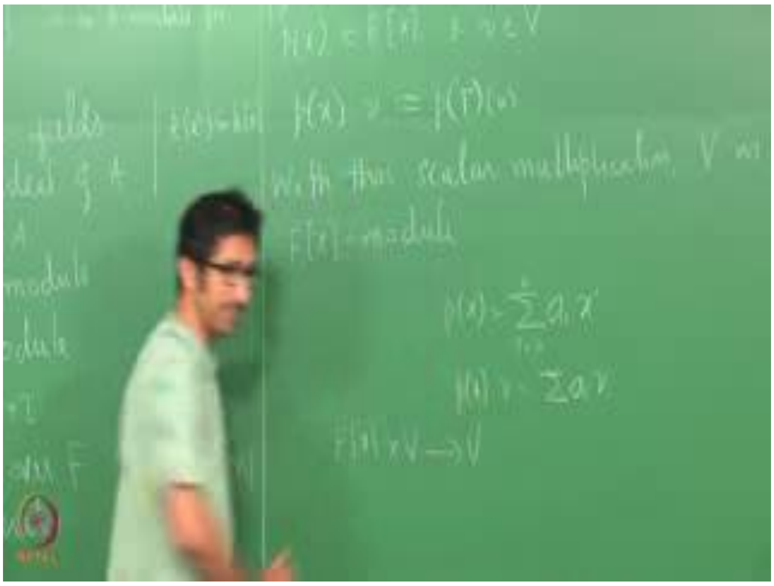

Then you can make V a module over k x. So take a polynomial p of x in k x.

Student: In previous case the f is there.

Sorry, f?

Student: But we already said that a vector space is module over a field.

Module over a field; now what we are going to do is we are going to define a module structure for V over this polynomial ring I have to define a multiplication. Can you think of a multiplication here? So I want for an v in V can you think of a definition for this.

Student: Coefficients multiplied by v.

Coefficients so what?

Student: Some equation a I v etcetera.

So, how does the degree or you know how do you say that it is linear can you say that it is linear.

Student: (Refer Time: 11:34).

This is p of t acting on v. So with this, we can say that with this, scalar multiplication v is a k x module a F x module. Now F x is a when you want to study this diagonalization and the Jordan canonical forms and spectral theorem etcetera, one can you know there is an approach through this, for example, you can view v as an F x module use structure theorem of modules or pid to get a specific structure for v and use this multiplication to get so to say what T on a specific decomposition with respect to T and so on.

Student: Sir what is the problem if we multiply we with the f coefficient of that.

So, you are saying if p x is equal to summation a I x power I, I from 0 to n. Then p x v equal to you are saying a I v I is that what you have a I v.

Student: A I v x.

No you are you have to define a map from F x cross V to V. So we will see more examples as we go along first let us look at some properties of modules; so as in the case of rings.

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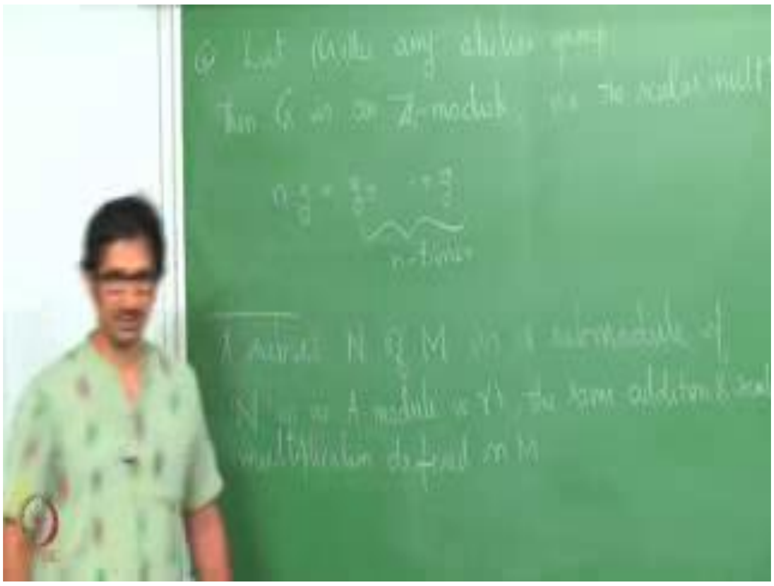

First we will look at now sub modules. What would be a sub module? Sorry I forgot to include one another important example here let me.

Student: (Refer Time: 14:38).

Suppose for example, if you take. P one x p 2 x sorry.

Student: Through the identity.

This is a special case of what we discussed here. This is that an arbitrary T and this is for identity operator this works correct. If you take this is another important example; if you take any abelian group. So for a set to be module you need it to be an abelian group. So start with an abelian group then you can have a module structure on G over Z then G is an Z module, via the scalar multiplication. What should be the scalar multiplication? How do I define that n times g?

Student: (Refer Time: 16:19).

What are the operation here is so, let us let me write G plus then g plus g n times. So any abelian group is also a an Z module. Now what should be sub module a sub module should be a subset with?

Student: (Refer Time: 16:54).

Same as a operations same as in the case of module M.

Student: And.

And it itself; that itself should be a module. So a subset N of M is a sub module, if N is a module is an a module with respect to the same addition and scalar multiplication defined on M. And you have a subset with the same structure. Structure remains the same. So some examples if you take any subspace over a vector space it is a sub module. Similarly, take any ideal, if it is an ideal then any sub ideal is also a sub module and so on.

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We will see a lot of examples a map f from, so let us start by saying let M and N be a module. A map f from M to N this is an A module homomorphism. If what should be the condition, it should be a abelian group homomorphism to start with or in other words f of m 1 plus m 2 should be same as f of m 1 plus f of m 2 then.

Student: (Refer Time: 19:43).

F of A m should be a of f of m for all A in A M, M 1 M 2.

Student: (Refer Time: 20:03).

Do we have to specify that?

Student: No.

We do not have to specify that because it is already there in m there is no f of 1 there and both of them are modules over a some examples. Now, what should be a zeroth example? If you take any 2 modules M and N, if M and N are any 2 modules, then the 0 map is a homomorphism. So I will henceforth only say homomorphism it means a module homomorphism or in the context of rings it should be a ring homomorphism. If needs to be specified separately I will do that. Otherwise it is a default in homomorphism in this default set up. Some homomorphisms from Z to Z can you give me some homomorphism from Z to Z identity yes, but more.

Student: F of n.

F of n equal to?

Student: Minus n.

Minus n yes, n equal to.

Student: M times n.

M times n fixed. Will this be a module homomorphism? F of n 1 plus n 2 is m times n 1 plus n 2, which is m n 1 plus m n 2 and f of some k n is m k n which is k times m n which is k times f of n.

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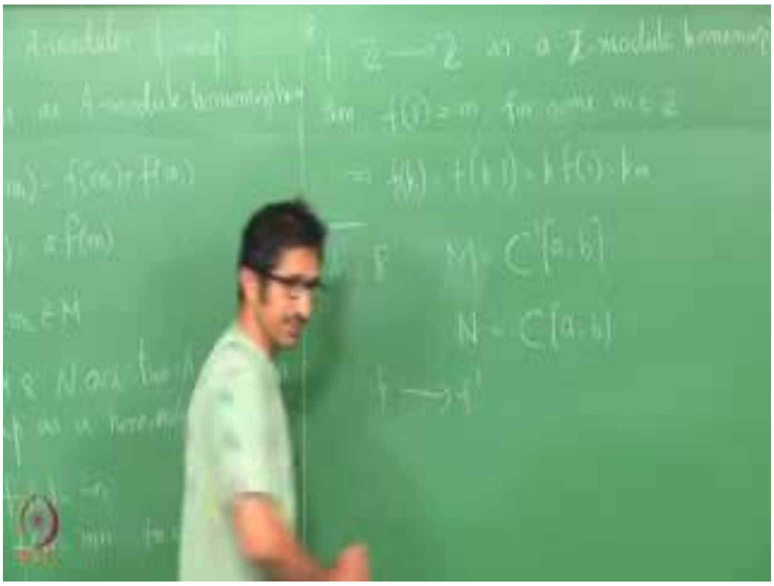

So, this is a module homomorphism, can we say something more in this, suppose f from Z to Z is any homomorphism. F from Z to Z, is Z module homomorphism. Then can we say that it should be of this form? How do you say that? If you take f of one equal to n so let us use m for some m, then what should be f of any k? F of k I can write it as f of k times 1 which is k times f of one which is k n.

Student: M.

K m; so therefore, this is precisely of this form. Which means any homomorphism from Z to Z is of the form f n equal to m n. Some more examples; A is R and m is C 1 a b and n is C a b and f going to f prime. Is that what you have in mind? This is a so more generally any vector space homomorphism should be a.

Student: Modulo.

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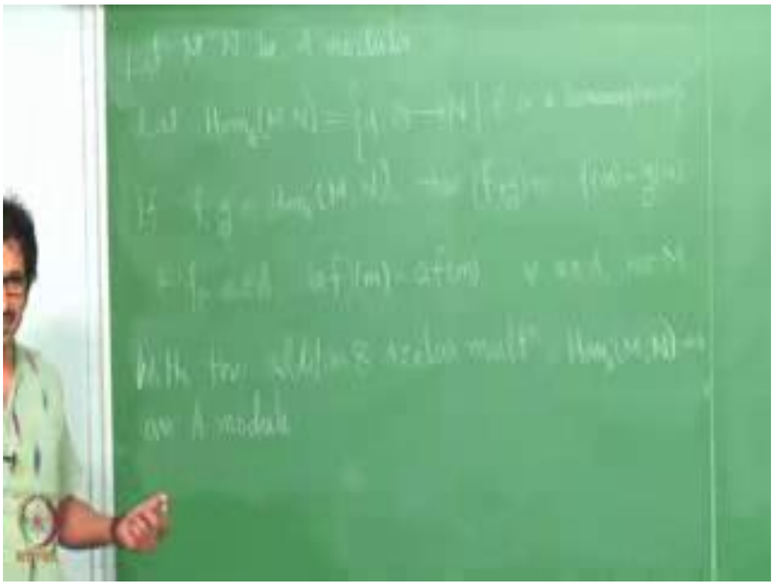

Modular homomorphism over the same field; now, there is suppose you fix 2 modules M and N. Let M N be a modules and look at all the homomorphisms from M to N. Collection of f from M to N. F is a homomorphism. F is a an a module homomorphism.

What can you say about this set? Is this familiar for example, in the case of fields, if you take a to be a field and m and n be vector spaces then this is denoted by l m n right set of all linear maps from m to n, that is also a module over or in the case of fields that is also a vector space over the ground field right so here what do we expect.

Student: It is a module.

That this is also an a module so what should be the addition and scalar multiplication. We have to first say that this is an abelian group. And then say that there is a scalar multiplication. Which makes it an a module. So if I have f comma g in Hom A M comma N, then how do I define f plus g. F plus g should be a homomorphism which takes any m to f m plus g m. It should be you know see if you restrict your ring to a field, this should be set of all linear maps. So you can take the inspiration from there right. And similarly how do you define alpha of f in a in A a of f is the homomorphism that acting on any m should give you a of f m. With this addition and scalar multiplication Hom A M N is an A module.

So, therefore, let us come back to what we showed here. What is Hom of if I take Hom Z Z, what did we see here it is.

Student: Any.

Any f has a unique integer. And also any n given any n I have a homomorphism or in other words, there is a one to one correspondence between Hom Z to Z.

Student: Z.

Now what we need to check is that whenever you define a homomorphism you have whenever you define a map the first 2 concepts involved related the concepts involved are whether the map is injective surjective or both. When they are both it is called an isomorphism right. So in this in the case of module homomorphisms also the module isomorphism is a module homomorphism which is both injective and surjective.

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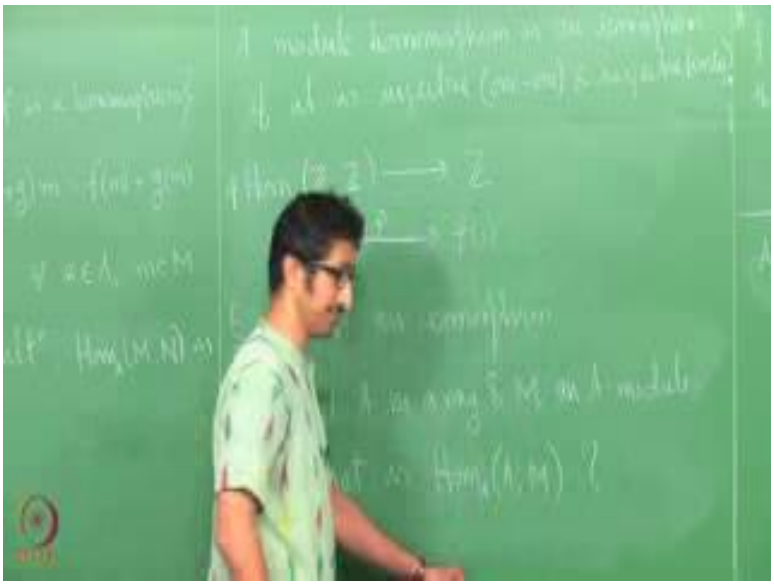

So, first let us define those terms. A module homomorphism is an isomorphism, if it is injective or one to one, and surjective onto. So one can see that Hom A Z, Z to Z there is a map f going to can you exactly defines what the map should be.

Student: F 1.

F of 1 right; so exercise, that this is so let me call this map phi, phi is an isomorphism. You have to show that this is a homomorphism first.

Student: Z module.

Z module homomorphism so we were A is Z. First you have to show that it is a homomorphism f plus g act will be mapped to f plus g acting on which is acting on 1 which is f acting on 1 plus g acting on 1 which is same as phi of f plus phi of g. Similarly, I mean some n times f will be n times f of 1 which is same as n times phi of you know f.

So therefore, this is homomorphism. And verify that it is both injective and surjective. So this is an isomorphism. Now can you concept can you see this concept in a more general setting. So let I will put another exercise. Let A be a ring and m an A module what is think about it try to arrive at an answer.

So, when I say what is if you have a very specific? Answer: find a specific map homomorphism isomorphism whatever possible think about this.

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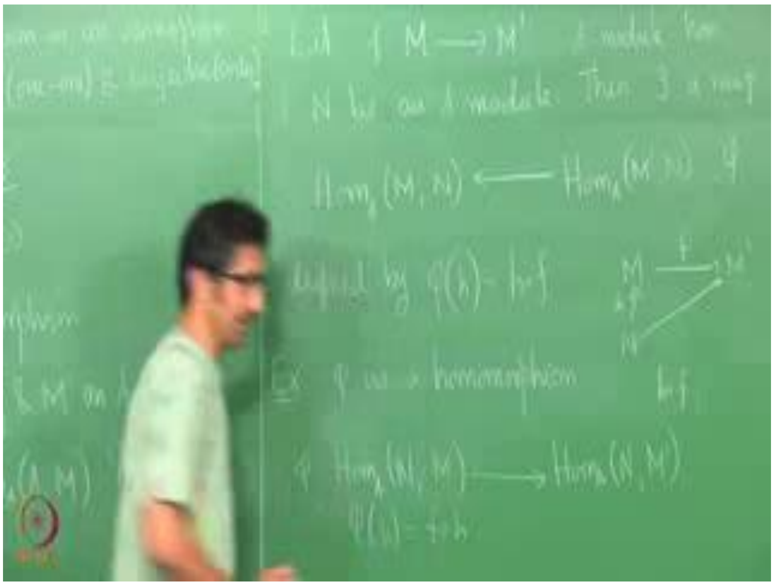

Now, suppose I have a map f from M to M prime a module homomorphism. Then we have 2 modules right, Hom A so and n be an a module. Then I have 2 modules Hom A M N and Hom A M prime N. Can you see some interaction between? Them I mean can you see some maps between these 2 modules.

Student: (Refer Time: 34:23).

So I have a map from M to M prime. Suppose I have a map from M to N and I have map from M to N. So if I have a map from M prime to N, then composing with f I can get a map from M to N. That is, given an element here composing with this I have. Or in other words I have a map from here to here.

Student: (Refer Time: 35:13).

So, if I have a map if I take a homomorphism let us say h. Here h. Then look at f composite h this will be a map from it will be a map homomorphism from it is a composition of 2 homomorphisms is again a homomorphism. This will be a homomorphism from M to N.

Student: Sir h will take m dash to m.

Yes.

Student: So the domain of.

I am sorry I wrote the wrong direction h composite f. Look at the h composite f this will be a map from M to N; that means, given a homomorphism here, I get a homomorphism. Or in other words I have a map from here to here. So then there exists a map phi from here to here defined by phi of h equal to h composite f. Prove that phi is a homomorphism, straightforward verification. I will leave it you to do that we will study more properties of this little later.

So, one you know there are these questions like depending on the properties of f how we can derive properties of phi. For example, if m f is injective can we say something about phi. If f is surjective, can we say something about phi if f is an isomorphism can we say something about this and so on? So therefore, we will come back to that little later. In the same manner so this is one way another Hom module associated with this would be Hom A N M and Hom A N M, M prime again. Can you see a map between them?

So, now I have a map from N to M and N to M prime. So I have map from N to M, N to M prime. Can you see some interaction? Given a map from N to M I can compose with f and get a map from N to M prime. Or in other words I have a map from here to here, phi of, phi of h equal to f composite h right. So in both cases you should verify that this is a homomorphism. Now suppose I have is this clear are the maps clear.

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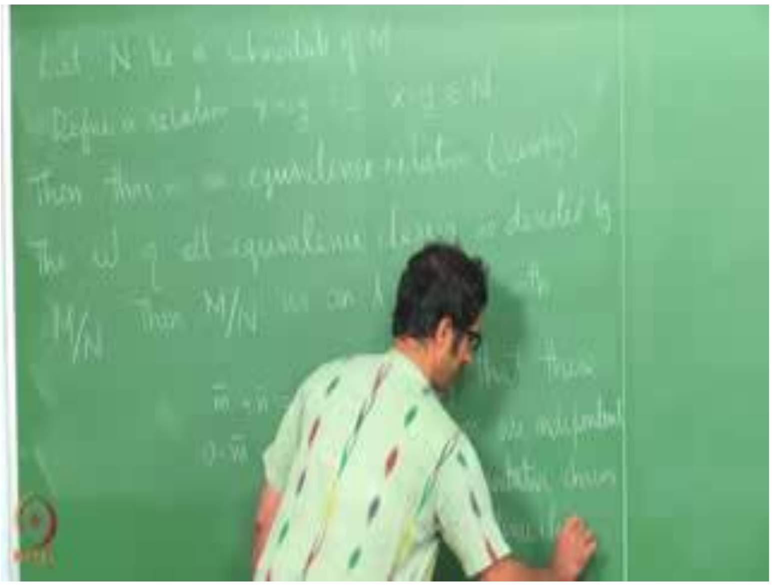

Now, suppose I have a submodule, submodule of M. Then as you must have done for the case of ideals for a groups ideals vector spaces etcetera, you can define an equivalence relation x define a relation x equivalent to y by if and only if x minus y belongs to N. Then this is an equivalence relation. You must have done that for groups as well as ideals. And this equivalence relation gives me a set of equivalence classes. So the set of all equivalence classes is denoted by.

Student: M mod n.

M mod N; so this one has a natural module structure on this then M mod N is an A module with m 1 plus N. So this is slightly different this plus, this is a notation for an equivalence class. So let me just write m bar.

So, I am looking at equivalence class of m plus equivalence class of n I defined the sum to be equal to equivalence class of m plus n. One has to see that this is well defined. The well definedness etcetera the procedure is exactly same as in the case of groups and ideals that you have done. So I will not be going through that again. A dot m bar is same as a m bar verify that these definitions so I should do like, this these definitions are independent of the representative of chosen for equivalence classes. Basically these are well defined the addition and scalar multiplication are well defined.

So, this is if you take any module in and a sub module you have this quotient module M mod N.

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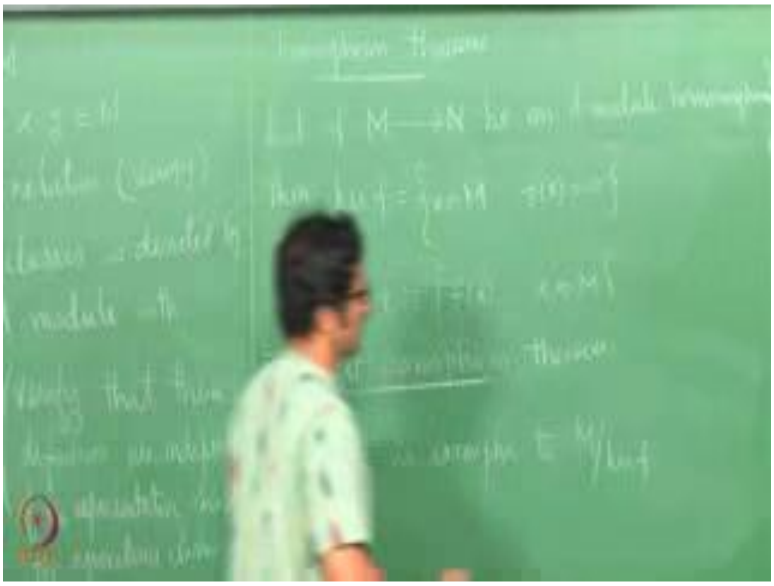

Now, first isomorphism theorem as in the case of groups and groups rings and vector spaces and so on, we have, if I have a, a map f from M to N be an a module homomorphism. Then I have definition for kernel, kernel of f is defined as all x in M such that f of x is 0. And another notation is image of f this is set of all f x, x belongs to M or you can say set of all y such there exists x in M with f of x equal to y. The first isomorphism theorem says what should be the first isomorphism theorem.

Student: (Refer Time: 45:36).

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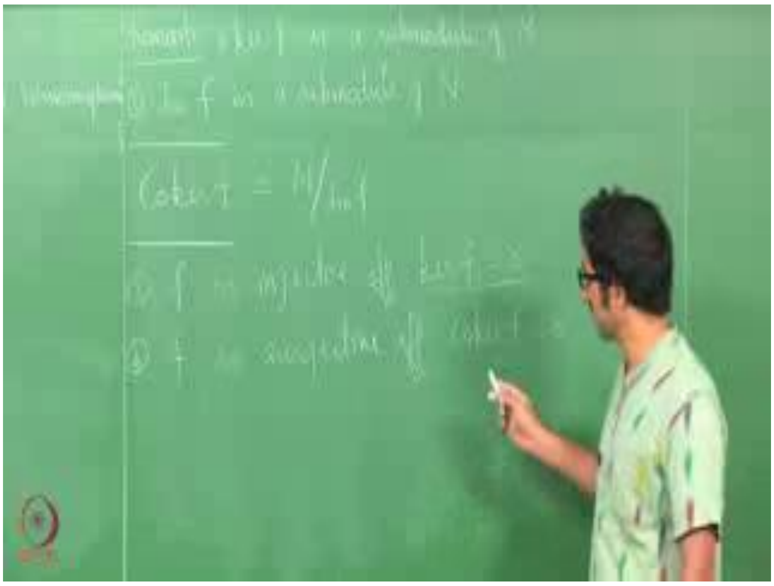

Image of f is isomorphic to quotient kernel of f. Before this one more remark, that kernel of f is a submodule of M and what can you say about image of f? Image of f is a submodule of N. There is one more definition co kernel of f is defined to be N mod image of f.

So, now f is injective if and only if kernel f is 0, and f is surjective if and only if co kernel of f is 0. So these are you know this kernel f is 0 if and only if f is injective. This you must have seen in the case of rings in groups rings vector spaces and so on. This is a very strong property that you know that holds for homomorphism. In general, if you want to check injectivity you have to check take 2 arbitrary points, if their images are same then you have to prove that these 2 points are same.

But in the case of these structured maps that is maps which preserve the structure you only have to check whether an arbitrary element is map to 0 or not. So we will continue later.