Commutative Algebra
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Lecture – 07
Radicals, Extension and Contraction of Ideals

Let us recall.

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Radical of an ideal I is set of our x in A such that x power n belongs to I for some n in N. And then we proved that radical of I. What are the properties that we proved radical of radical of I? Is same as radical of I radical of I is the whole ring. If and only if I is the whole ring this is we did not prove this I guess, but we stated this then radical of I intersection J is same as radical of I J same as radical of I intersection radical of J

So, now, what can you say about radical of I plus J. So, this is certainly radical of I is contained here and radical of J is contained here. Therefore, their some is contained here. So, the natural question is whether they are equal. So, we need to check whether if I take an element here it is here. So, let us start with an element x in radical of I plus J.

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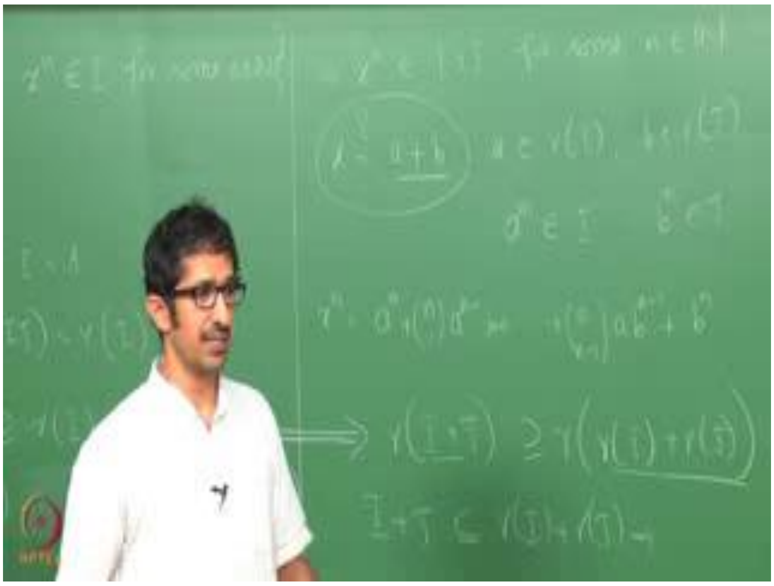

What is that mean? That means, x power n belongs to I plus J for some n in N. So, the question is whether we can say that you know this x power n is can be written as some a plus B where you know a belongs to some power of a belongs to I and a power of and whether this x can be written as, a question as x can be written as a plus B where a belongs to radical of I and B belongs to radical of J or in other words some a power m belongs to radical of sorry I and B power some n belongs to J. What we know is that x power n belongs to radical of x power n belongs to I plus J.

Suppose you write something like this what is x power n x power n is a power n choose one a power n minus 1 B again n choose n minus 1 a B power n minus 1 plus B power n. Now does not really seem to be any of in this form right. So, I mean it does not lead anywhere this. So, it may be you know this is it is not clear. We when have a direct approach.

So, let us try to modify, you know what are the other possibilities. So, radical of this is contained here. So, therefore, if I take one more radical on the left side on this side right hand side what do I get? If I take one more radical on this what do I get radical of radical of I plus radical of J radical of this one, and on this side, it is a same right; that means, this is still contained in this now do you see some obvious relation there.

So, what does this say let me write it here? So, this one implies that radical of I plus J is contained in radical of radical of I plus radical of J. Now do you see some obvious relation here? See you have 2, I mean you have an inequality here, you have 2 ideals on either side both of them are radicals right this is radical of an ideal this is radical of an ideal. Now do you see some relation between the ideals inside? Do you see relation between this and this?

Student: I plus J is contained in radical.

I plus J is contained in.

Student: Radical I plus radical J.

Radical I plus radical J and I is contained in radical I J is contained radical J. Therefore, this is contained here and what does that mean? This is contained here right I plus J is contained in radical I plus radical J.

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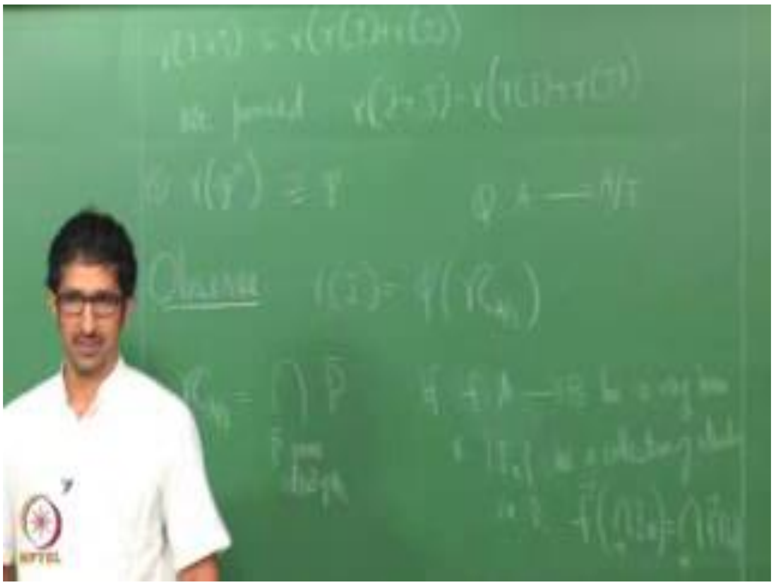

And that would imply that radical of I plus J is contained in so, what did we prove therefore, radical of I plus J is radical of radical of I plus radical of J.

Student: We could have just used that one.

Which one?

Student: The I s radical of I and J radical of J.

So, what are you saying? So, we have we have only used those 2 properties right this.

Student: No I am saying that only using this property we do we can use this again.

Ok.

Student: substituting I is radical of I J is radical of J.

So, from here you can say radical of I mean radical of I plus radical of J. And this is; obviously, contained here. These 2 are I mean both of them followed from the basic properties of radicals itself.

So, that is another property. Now what can you say about. So, we have you know if I take a prime ideal and look at radical of this what would be. This can we say something about this. First of all, can we say that p is can we see the p is contained here. If I take any element in p say x, x power n belongs to p power n therefore, x belongs to radical of p power n. Therefore, this contains p for sure, is this equal to p.

Did we prove something about radicals?

Student: Radical of I J.

We need to prove. So, let us let us prove this let us keep this aside make an observation. This is I think I made this observation sometime back, but let us do that again. Radically of an ideal is what?

Student: Minus.

This is phi inverse, phi is the natural map from a to A mod I, phi inverse of nil radical of A mod I?

Now, what is nil radical of A mod I this is intersection of all let us call this p bar; p bar prime ideal of?

Student: A mod I.

A mod I; what is the relation between, so, if I have a collection of ideals? Suppose I have a ring homomorphism. F from A to B, B A ring homomorphism and I alpha be a collection of ideals in B what is I mean can you say something about.

Student: Intersection.

This is. So, over alpha this is same as f inverse I alpha. This is in fact, basic set theory. Now, so, let us apply this here what do you get? What is f inverse of the nil radical, I mean phi inverse of the nil radical? What is the phi for what are the inverse images?

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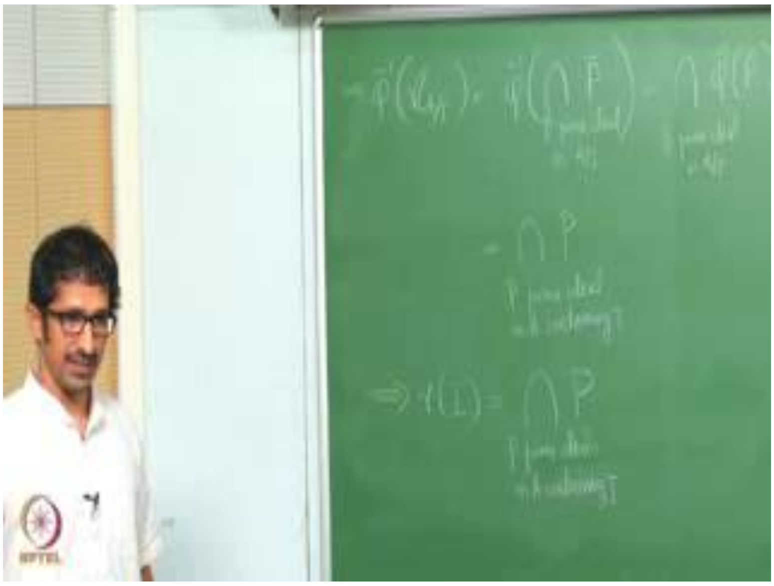

Phi inverse of nil radical of A mod I, this is equal to phi inverse of intersection p bar p bar prime ideal in A mod I. This is same as?

Student: Intersection of phi.

Intersection of phi inverse p bar; now what can you say about phi inverse of p bar?

Student: It will be a prime ideal in a.

It will be a prime ideal in a containing I. So, this is equal to and they are in one to one correspondence right. So therefore, this is this intersection is equal to I will write p, p primes ideal in a containing I. So, where did we start with? We started with radical of I, it is the inverse image of the nil radical, and we have proved that the inverse image of the nil radical is nothing, but intersection of all prime ideals containing I. So, therefore, what we have proved is radical of I is intersection of p, intersection of all prime ideals in a containing I, is this clear it? Is the intersection of all; so, radical of an ideal is intersection of all prime ideals containing that ideal?

So, now let us get back to this property. Radical of p power n contains p. Now by this property what is radically of p power n? It is the intersection of all prime ideals.

Student: Containing p.

Containing p power n, can you think of some prime ideal containing p power n?

Student: P.

P right p is a prime ideal containing p power n. Therefore, this is intersection of all prime ideals containing p power n, and p is one such prime therefore, this is contained in the in all the elements in the intersection in particular p. Therefore, this is contained in p. We have already proved it is it contains p.

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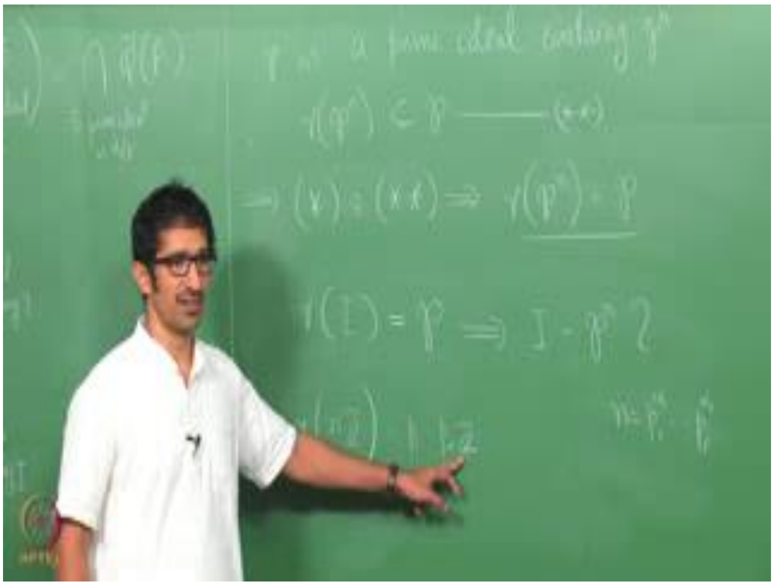

So, radical of p is a prime ideal, containing p power n. Therefore, radical of p power n is contained in p. So, these 2 together this if you called this star and if you call this star, then implies that radical of p power n equal to p.

Student: Radical of I J is.

Which one?

Student: There was the property written there.

Radical of I J is equal to radical of.

Student: I intersection radical of J if you use the effect here. So, radical of p to the power n is radical of p.

Radical of?

Student: P and radical of p is intersection of all prime ideals containing p so.

So?

Student: And p is itself a prime ideal containing p the radical of p is actually equal to p.

So, what we see again what you get is a containment right. Radical of p is contained radical of p will contain p right. And you are taking a product there see there is a product as well as intersection right you are saying let us see radical of p power n this is equal to radical of p. All of them are p itself that this is equal to radical of p now.

Student: Now if we use the fact that you see that actually equal to the intersection of all prime ideals containing p.

We have to use the fact that. So, then what do you get. This is contained in p.

Student: Yeah.

Right and then this is containing. Why should this p is?

Student: P is all containing.

P is contained and we have proved exactly Just that it we did not use this property we simply observed it from here we have to ultimately we have to use this.

Student: Yeah.

This fact that is there in the middle the other side other 2 sides you can use any of the properties, but yes this is another way to observe this property.

So, looking at this do you think of a natural question.

Student: Maximal idea.

Yes, radical of what do you want to ask.

Student: Radical of.

What would be radical of maximal ideal by the way?

Student: Has to be.

It has to be this itself right because it will contain m and we have proved that it is the whole ring if and only if this is the whole ring. So, therefore, this has to be m itself.

Now so, let me see radical of an ideal is intersection of all prime ideals containing that given ideal. So, suppose this is prime ideal itself, can we say something about I or you know what would be natural question to ask about I.

Student: Why is that prime?

It need not be a prime ideal right. So, that does this implied I is some power of a prime? See for example, suppose you take the case in Z when can, suppose I have n Z what would be radical of this n Z. Suppose your n is p 1 power alpha 1 up to p n p r power alpha r. This is the prime factorization of n what would be radical of this?

Student: P 1.

P 1 up to p r Z. And when can this be a prime ideal?

If r is 1 that is the only way radical of n Z can be a prime ideal if r is 1 then.

Student: Power.

N is a power of a prime. So, is that generally true? This is something think about it we will address this question sometime later, but it is nice to experiment with you know this question try to understand.

See one of the ideas in commutative ring theory is you know trying to see how far properties of you know Z goes through or how does these properties generalize into commutative ring theory. So, this is one such that we can see in Z, but is it true in general in ring theory when we take radicals. So, let us think about this question we will come back to this question at a later stage.

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Now, let us move on to study, you know if I have a ring homomorphism f from A to B be a ring homomorphism. Suppose I take an ideal let us take an ideal I in A, then f of I need not be an ideal right. This we have already seen we have you can take lot of examples and whenever and most often if you take, Z 2. One quick example even if you take any of the you take any ring homomorphism which is not on to we can find this you know you do not really need to do only with Z n Q take you have seen lot of examples of ring homomorphism now. So, take one of them which is not on to and that will give you this one.

But then this set is relevant and we say let I e be the ideal of B generated by f of I. What does that mean that is I e is set of all x i or you know a i f x i i from 1 to n a i in B x i in a sorry x i in I and n in N. Take all finite linear combinations of these elements that is basically the ideal generated by this set. It is like vector space generated by this set it is the same thing in the case of ideals.

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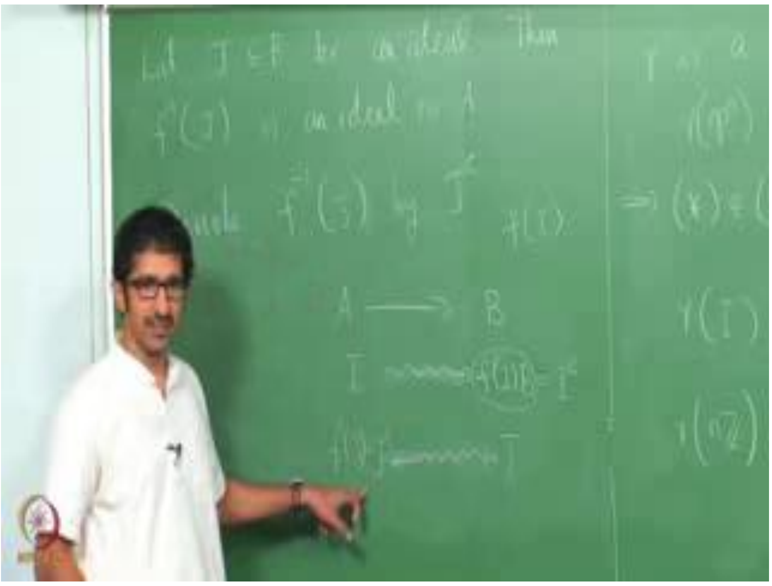

Now, if I take an ideal J be an ideal J in B what can you say about f inverse of J. This is an ideal in a right. So, this is denoted by contracted I J is contracted. So, this is see e is for the sim for extended. We are extending I to B in some sense or here we are contracting J to a this is denote f inverse J by J c.

So, imagine the situation I have A to B, I have I here that is f of I B we will write it this is basically, I extended. Now I have J here that comes back to a by f inverse of J which is also denoted by J contracted. This is Just to show that this is the set of all linear combinations of elements from here and here. If you look at this, I e let us it is of the form be f I.

Student: Will they contain the map f?

Yes.

Student: Equal to I goes to f of I.

So, I am looking at I goes to f of I, but here I am talking about the correspondence between ideals in A and you know through this extension and contraction. If I look at only f of I this is not an ideal. So, we can relate I with I e, I extended. We are only I am not saying that this is not I should properly do this I look at this connection between ideals here and ideals. There I goes to I extended and J goes to J contracted. That is the natural question right we have 2 ways of moving right, one I start something here go there and come back, what do we get similarly I start with an ideal here come back here and then go back again extended do you get the same ideal. First let us look at one or two examples, again you know them.

Student: Due to the contraction question then extending that gives the same thing. That it is just f of J inverse.

Will it give the same thing?

Student: No sir.

F of f inverse of J.

Student: It is one-one f of J.

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So let us look probably, you have this example in mind. If you have this natural inclusion map what do you get. So, I have I start with any n Z, I go here what is the extension of this. It will be a whole Q right if this is I then I e is Q. So, if I contract it back I get the whole of Z. They are not equal.

Now, suppose I take Z to Z x. The natural inclusion map, n is map to n itself. What do I get I start with an n Z here, what does it is extension here what does it is extension here?

Student: Z x.

N Z x and what if I contract it here what do I get. So, this is my I, this is my I extended. So, I send it back, I extended and I contracted. I n Z x come back to this one, what do I get? Do I get n Z or do I get anything more? So, the question is what are the integers in n Z x what are the integers available in n Z x.

So, before going into this one let us make simple observation that will you know simplify this question.

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Suppose I have, you know if A is a sub ring of B, if A is a sub ring of B, and this is the natural inclusion map. And if I have an ideal I here then what does I e is nothing, but I B right I look at all B linear combinations of elements of I. And what is if I have J here what is J contraction? Now, forget about I here I am starting with an ideal arbitrary ideal B arbitrary ideal J in B and looking at J contraction.

Student: J intersection A.

This is nothing but J intersection with A right. When I am taking f inverse if J contains an element of B, but not in A it is inverse images empty set there is nothing there, and if it is an element in a let us precisely that element itself. So, therefore, this is nothing but J intersection A.

So, let us come back to this one, what would be it is contraction here? It is set of all integers in n Z x, what are the possible integers in n Z x? It is basic that is that requires a proof, but this is intuitively it is clear that it is n Z all integers in n Z x, what is n Z x it is collection of all polynomials whose all coefficients are divisible by n.

Now, it should contain n Z naturally. Now can it contain any other natural number, I mean any other integer. See it is by definition it is, it has to be if an element is there n should divide that. So, therefore, any element there is a multiple of n. So, therefore, it is all integers or n Z x intersection Z is precisely n Z you want to do ask something no.

So, now do you see in general do you see some relation between these extensions contractions and so on. Can you say something about I extend this and contract it back. Can you say some see I have an ideal I, I extended to B. So, this is in general. So, let f from A to B be a ring homomorphism. Let I be an ideal in A. So, I extended to be and then contract it.

We have already seen that it is; it need not necessarily B equal to i. It will contain I right this is if I start with an element in I then x by, you know x times one see B is again commutative ring with identity. So, f inverse sorry f of x is in I extended therefore, f inverse of.

Student: F of x.

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F of x that contains x therefore, x is there in this one. So, I is contained here now. And J be an ideal in B, therefore what similar?

Student: J contraction extension.

J contraction extension?

Student: Is contained in J.

Is contained in J; if I start with element x in J contraction extension; that means, there exist some y. So, how do you show this?

Student: J contraction.

Y is in J contraction extension.

Now, I want to say that y belongs to J. By definition what is this ideal this is set of all finite linear combinations of the form you know some Z J is Z I, where Z I belong to f inverse of I mean J contracted which is f inverse of J right. And a i belong to a B sorry f of.

Student: F of.

F of J contracted and this is finite linear combinations. See if I start with this if I start with an arbitrary element and take something of this form. One can of course, you know proceed, but this is nothing, but ideal generated by all elements in f of J contracted. So, if I prove that every element in this Z I in f of J contracted. Then all the linear combinations will be in the J contracted extended.

See if I show Z I belong to J contracted, this implies Z I belong to J. If I show this that will naturally imply that, this will imply J contracted extended is contained in J. Is that clear? I do not have to take all elements of this form, if or in other words this is contained J will imply the ideal generated by this will be in J, or in other words J contracted extended will be in J. That is what I am saying. So, we only have to prove that this containment is true this will automatically imply this is true.

So, let us try to prove this is true. That becomes pretty obvious now. Right f of J contracted is contained in J.

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If I take an element in y is in J contracted what does that mean? This implies that f of y is indeed in J, and that implies that f of J contracted is contained J. And that implies that f of J contracted and the ideal generated by this or in other words J contracted extended is contained in J. Is this clear?

Student: Yes, sir.

So, here what did we do? To prove that an ideal let us called is call I 1 is contained in an ideal I 2. We only have to prove that generators of I 1 is contained in I 2. If all the generators of I 1 is contained in an ideal I 2. Then the whole I 1 has to be in I 2. So, that is the idea that we have used here the generators of J contracted extended is nothing, but f of J contract. So, to prove that this is here we only have to prove that the generators are in J. And that is exactly what we do. And this is a trick that we will keep using in future as well, whenever we do module theory as well. In modules, if again same thing if a module is contained in other module can be proved using the same trick.

So now, can we go one step further what do we get? I start with an I, I extended to B come back to A, I get I extended contracted. What if I again go back to B? What I have is I extended contracted and then next. So, let me remove these brackets I will Just simply write one after, I extended then contracted and then extended. So, by this property see I extended contracted is it contains I.

Now, if I extend this. So, I can think of this as I extended this contains this is contained in I extended right, but I have an ideal say this containment is there if I extend this it will automatically see if I have I 1 contained in I 2 that will automatically imply I 1 extended will contain I will be contained in I 2 extents.

So, therefore, this equation implies I extended is contained in I e c e, while thinking of this as taking this ideal and contracting and extending using property 2, we have this is contained in I e I extended. So, therefore, these 2 together says that I extended contracted and extended again what we get is I extended.

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Similarly, if I take; so this is property 3. So, this is equal to I extended property 4 is I take J contract it extended and contract again what should we get.

Student: J contraction.

J contraction that is again natural try to prove this, this way. And there is one more property. So, if c is the set of or contracted ideals of A, that is all I mean some f inverse J for some J in A and B is the set of all extended ideals of B, well I am using same notation. So, E is, and then do you see some relation between these 2? Then see the set of all I such that I e c is I, and e is set of all ideals J such that J contracted extended is equal to J. So, we will start module theory in the next class.