Commutative Algebra

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Lecture – 06
Colon and Radical of Ideals

Yesterday we were talking about; if an ideal is contained in union of prime ideals then it has to be in one of them. And similarly, if an ideal contains prime ideals contains an intersection then it should contain one of them.

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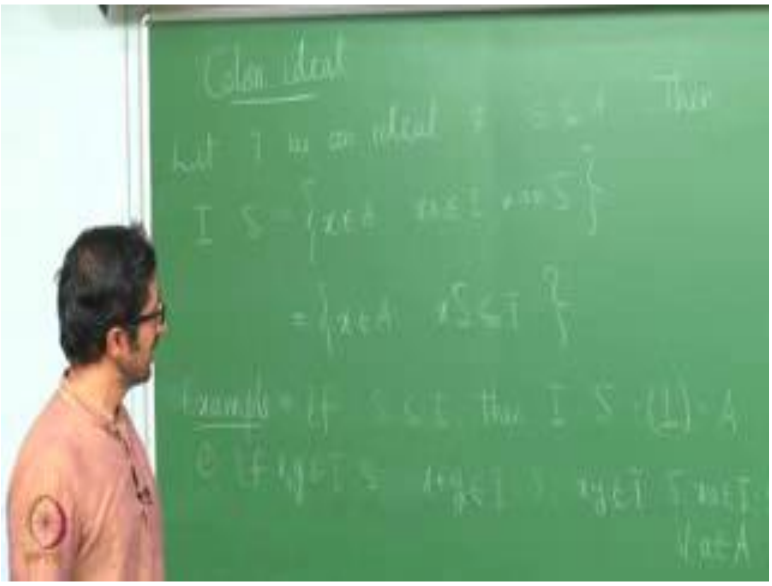

So, let us study more about ideals so today we will look at a specific kind of set. Suppose I have, I be an ideal and s be any set in A. Then I can say that I can define a new set which I will denote this by I colon s to be set of our elements x in A such that x s belong to I for every s in S.

Student: (Refer Time: 02:15).

Subset any set so I am looking at all elements which will multiply s inside I or in other words or in a you know in a general sense, where we are we are taking ideal I and dividing by all elements of S in some sense right. We are looking at all x such that x s is in I for a every S so we are taking all elements of the form x s and divided by s in S. That is, you know vague meaning of taking colon. We will look at this we will see so this is also you can also write this as x in a such that x s is in I.

Student: Since, s belongs to S and s is subset of A I know the s is subset of I.

So let us look at some examples first that will clarify few things. What happens if s is a subset of I itself then what happens to I colon S.

Student: I.

It is a whole ring right. Every element of a now suppose I take an element x in this one and y in this one can you say x plus y also belong to this set.

Student: What sir?

X s is in I y s is in I.

Student: X plus y.

X plus y s is also in I right.

Student: Yes.

So therefore, x is in I, I colon S so if x comma y is in I plus s, then x plus y is in I colon S what can you say about it is product x y belongs to I colon S.

Student: Yes.

Therefore, can you say something more?

Student: It is a sub ring.

It is a sub ring.

Student: Is it ideal?

So now the question, is this an ideal? So if I take x in I, I call an S. And I take an arbitrary element a in A can we say that x a belongs to I colon S.

Student: Yes.

X a s is same as a times x s x s is in.

Student: I.

I therefore, a times that will again be in I because I is an idea so therefore, x a belongs to I colon S for every a in A.

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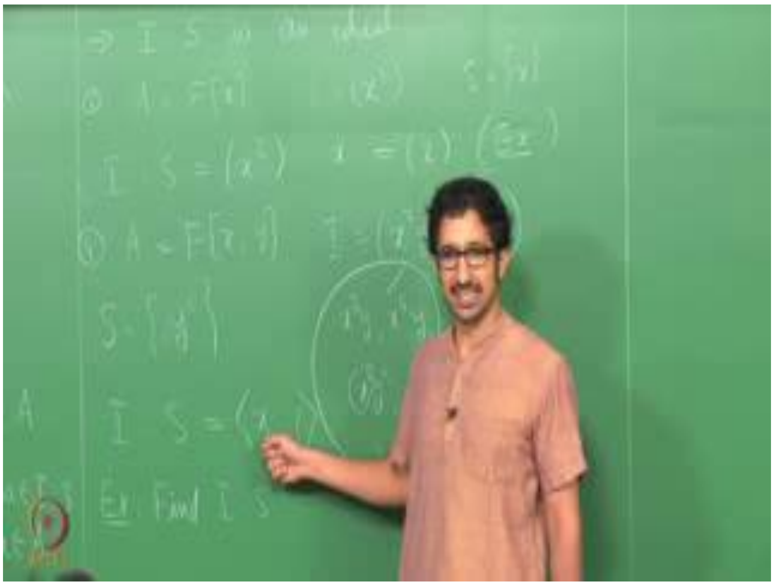

So, this these properties say that I colon S is an ideal. Now let us look at some simple examples. Let me take I so you are my ring is let us say any field any pid any field f and polynomial ring over f, take I to be equal to x square and s is x.

Student: First term x should be 1.

Sorry one, I mean ideal generated by one that is it, so what would be I colon S. I am looking at all elements in.

Student: Constant all polynomials in; constant.

Will 1 plus x be there?

Student: Constant term will be 0.

So this is so this is x square. So in some sense now you are saying all polynomials with constant term 0. And that is generated by that is ideal generated by x. So what have we done we have just.

Student: Divide.

Divided that is exactly what I said here we are simply dividing this. In the normal cases in the usual you know first examples what we see is that we are basically dividing them. You know dividing by elements of s so this is the ideal generated by x. Let us look at one more example. I am sorry a is f x comma y, so this see, this is you know you have said your intuition and I have accepted it, but you should write down a complete proof of this. When you have to write down an ideal is this how do you do that.

Student: Proofs.

You have to prove both the inclusions which inclusion trivial.

Student: X.

This is contained here is straightforward.

Now, you have to prove this inclusion. For which you can simply use the property of variables you know. And when are when do you say 2 polynomials are equal.

Student: Compare x with.

So just comparing the degree and coefficients you get so, but this proof is you know important in writing down, so let us look at another example, let me write this as and take my s to be y square, so what would be I colon S? Can you tell me what would be I colon S there is there is an obvious element there?

Student: X, x is there.

X is there is there right so let us let us let us see if we can find the generators of this. X is there do we have something else can we think of some other element; see here you have to look at maybe I will. See you have to look see we have to divide all the elements of I by y square, and see what are the elements. Now from here, can you think of elements that you can obtain from here?

X cube y x power 4 y and so on. Right, you multiply this element by x square x cube and so on you get this.

Another possibility is x square y square x square y cube x square y power 4 and so on right. These are another set of elements in this ideal another set is in a combination of these 2, x cube y square right and then again x power 4 y cube and so on. Similarly, for this one; so when you divide by y square this none of them is divisible by y square right, this has only y term there. So you cannot really divide this by y square now dividing this by y square what do you get? X square x square y x square y square, but when we are considering the ideal and we are trying to you know specify the generators of the ideal do we need to mention x square, x square, y x square, y square and so on. Because if x square is there x square y is there, and if that is there x square y square is there and so on.

So we only need to make sure that this element is there, but now from the other one we already know x is there. So x square is also superfluous. Because that you can obtain from x. So the question is, is there anything more here. So exercise. Find I colon S in this example. And find means you have to prove whatever the ideal if you are trying to prove that it is generated by x and you have to if you are saying that, it is indeed x then you have to prove that this is equal.

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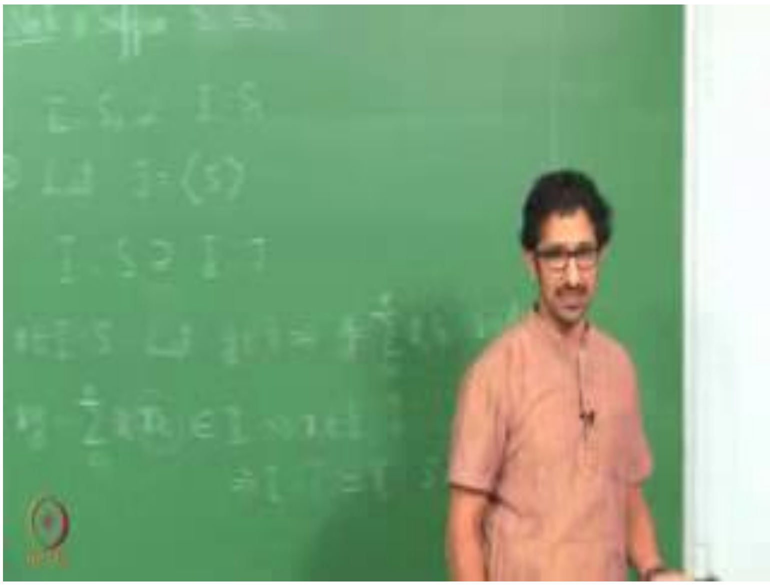

Fine now let us look at one observation. Suppose J is the ideal generated by the set S. Before that let me suppose you have S 1 is contained in S 2. What can you how do you compare I colon S 1 and I colon S 2.

Student: I colon S 2.

I colon S 2 is contained in I colon S 1. Right because this ideal takes everything in S 2 inside I, in particular it will take S 1 inside I. So therefore, if x belongs to this set it has to be here. So therefore, we have the reverse enclosure. Now let J be the ideal generated by S. Do you know the notion ideal generated by a set? Set of all polynomial combinations of you know or a linear combinations of elements of S, that is basically the ideal generated by S.

Now, can you compare I colon S and I colon J. Of course, the first remark say is that this inclusion is true. Is the other inclusion also true? From here this is contained here. Because J contains S J contains S. Therefore, I colon is contains I colon J now what about this one is this true. Let us start with an element here. X n I colon S. Now we want to check whether it is in I colon J. Now take any element in J, this means this element y is of the form some a I x I a I s I I from some finite summation a I belong to a and s I belong to S. Any element in J is of that form. Now do you see the result? What can you say about x y this is summation a I x s I I from 1 to n? What can you say about x s I?

Student: X s I.

We have started with x in I colon S, which means x multiply every element of s inside I which means all these are in I therefore, what can you say about this linear combination. That is again I because a I is in I a I is in A. X s I is in I, I is an ideal therefore, this is an ideal this is in the ideal I, there for the whole summation is in the ideal I. Therefore, this belongs to I. So what have we proved? For any x in I colon S and any y in J x y belongs to I which means x belongs to I colon J this implies x belongs to I colon J or in other words I colon J is equal to I colon S.

So when we are taking colon, whether we take a set or the ideal generated by that set they are the same.

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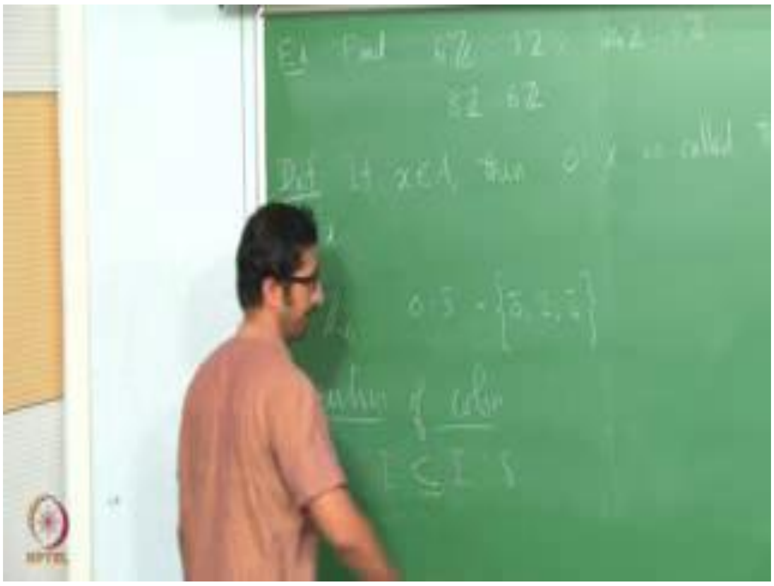

So, let me give you one or 2 exercises. Find 6 Z colon 3 Z 20 4 Z colon 5 Z. What would be 20 4 Z colon 5 Z.

Student: (Refer Time: 19:32).

Think about this. It is find this one. There is one important colon or you know if x is in a then, this so this is the, I will write this as simply 0, instead of 0 with bracket. 0 is an ideal the 0 ideal I will write it as just simply 0. Then this is called the annihilator of x.

So, if you look at Z 6 what is the annihilator of 3 bar? So, what would be this it will be an ideal right if you simply say 2 bar that is not enough. It will there will be certainly then. 2 bar anything more?

Student: 4.

4 bar right. That will also be there. Let us look at some more properties of first observation can you relate I and I colon S.

Student: Yes.

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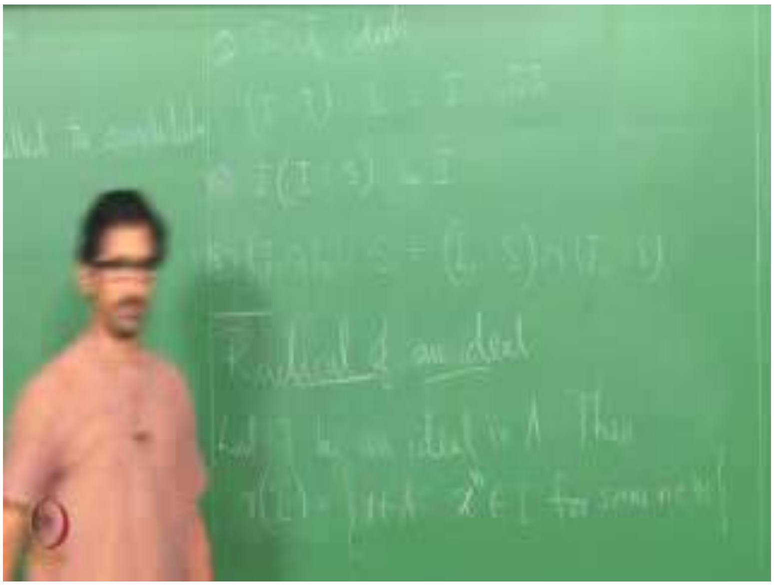

I every element of I will multiply at S inside I right so this is always the case. Now if you take 2 ideals, I colon J 1 colon J 2. So think of the division that I said. I am first dividing by all elements of J 1, and then dividing by all elements of J 2, that should be same as. I take the product of J 1 and J 2 and divide in some sense. Say if you look at the integers. And in the case of ideals this is true, the colon operation indeed behave like that. So this is some something one can you know the simple ideal theoretic verification, which I will leave it to you, to check from this first one it follows that, if I look at I colon S multiplied with I this is contained in I right.

I one intersection I 2 colon S. So if I take an element x which multiplies everything inside the intersection. This will so if x s is contained in I 1 intersection I 2 which means x S is contained in I 1 and x S is contained in I 2 therefore, it is in the intersection. Now suppose x s is in the intersection which means x s is I 1 in I 1 xs is in I 2, therefore it is in the intersection x s is in I 1 intersection I 2 therefore, it is here. So these 2 are equal. We will see some more uses of this colon and so on, later in the course. Let us move on to study another type of ideals called radical ideal. So radical of an ideal. So let I be an ideal in a, then I look at the set r of I to be set of all x in a such that x power n belongs to I for some N. I am looking at all the elements in I all the elements in A whose some power belongs to I.

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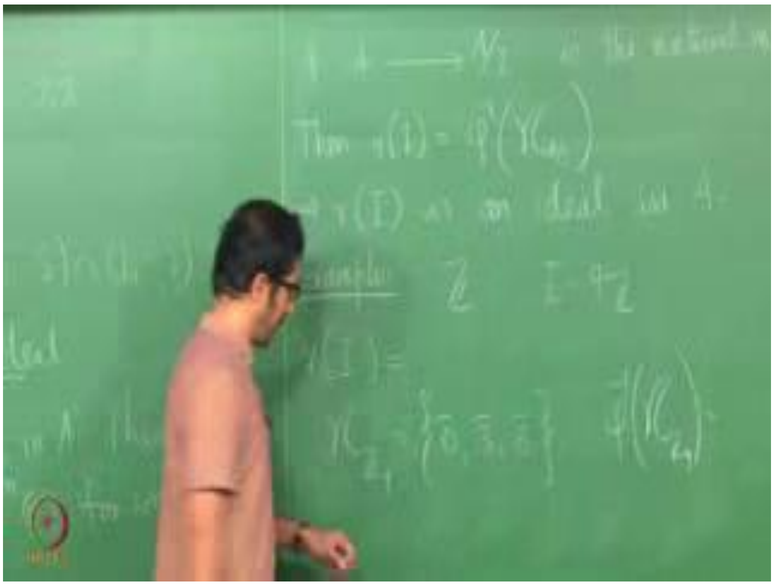

Or in other words if you look at the ring, A mod I can you identify this set. In A mod I, I am looking at all the elements. Whose some power is in I, but if an element is in I what would be it is image here, it would be 0.

Student: 0.

Right or in other words we are looking at.

Student: All nil potent.

All nil potent elements or in other words this is the.

Student: Nil.

Nil radical there; or in other words we are looking at suppose this is our natural map, in radical of I there is nothing, but the inverse image of the nil radical of A mod I right. I am looking at all elements, so if I take an element x plus I whose nil potent in A mod I. What is that mean x plus I whole power n is.

Student: I.

0 plus I or in other words x plus n belongs to?

Student: I.

I which is similarly if x plus x power n belongs to I, then x plus is a nil potent element here in A mod I. Therefore, the radical of ideal there is nothing, but the inverse image of nil radical in.

Student: A mod I.

A mod I; so therefore, this immediately tells you that I is an ideal. See if you look at this definition a priori there is no reason why this should be an ideal to start with right, but now what we are seeing is that this is inverse image of an ideal.

Student: Yeah.

Inverse image of a an ideal in a ring homomorphism is.

Student: Ideal.

Always an ideal therefore, this r of I is an ideal in, let us look at some simple examples. Let us take Z itself. And I look at let us say my ideal I is 9 Z. What is the radical of I? Can you identify some elements which are there in radical of I?

Student: We can use that thing.

Yes.

Student: (Refer Time: 30:36).

So what are the nil potent elements in Z mod 9 Z. What are the nil potent elements in Z mode 9 Z, what is the nil radical of Z mod 9 Z 0?

Student: 6 0 3 6.

0 3 6 so nil radical of Z 9 this tells 0 3 6. 6 times 6 is 0 right so 6 square is 0 3 square is 0 0 is there always anyway. So what is it is inverse image? Phi inverse of what is it is inverse image in Z what is that ideal.

What is the inverse image of this ideal? What is this ideal in Z 9 is Z 9? A principal ideal ring; so what is the relation between an id ideals of A mod I and ideals of A.

Student: One to one correspondence.

One to one correspondence with what?

Student: Ideals of.

Ideals of?

Student: A containing.

Ideals of?

Student: Ideals of a containing I.

Ideals of a containing I are in one to one correspondence with ideals of.

Student: A mod I.

A mod I this is an ideal in A mod I so what is the corresponding ideal in Z is my question. It should be an ideal containing.

Student: Z, I.

Which is Z, what is I here.

Student: 9 Z.

9 Z it should be an ideal containing 9 Z.

Student: 9 Z.

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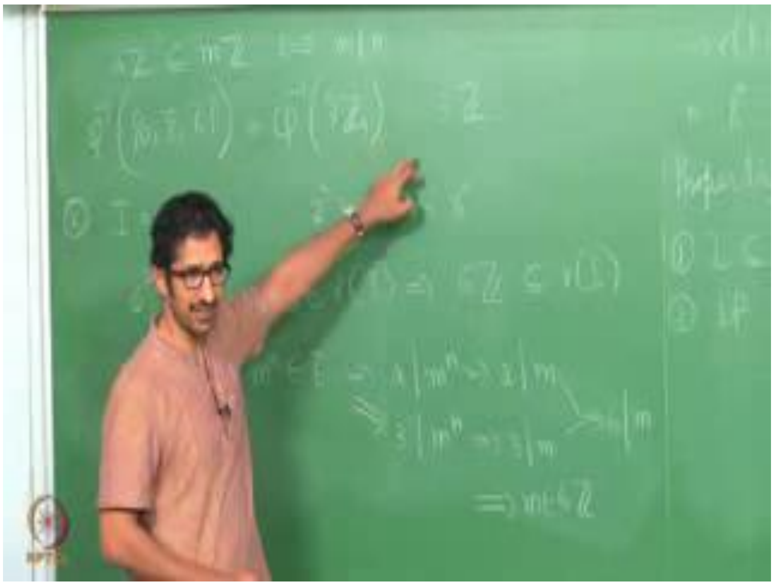

What are the ideals containing n Z, if you take any n what are the ideals containing n Z? When does an m Z contain n Z when does n Z when does this happen.

Student: M divides n.

This if an only m divides n so what are the ideals containing 9 Z.

Student: Divisors of 3.

What are the devices is my question?

Student: 1 Z.

There are only.

Student: 1.

1 3 9 which means what are the possible ideals.

Student: 3.

1, which is the whole ring then 3 Z then?

Student: 9 Z.

9 Z so what does the inverse image of this ideal.

Student: 3 Z 3.

3 Z right so phi inverse of 0 bar 3 bar 6 bar is nothing but this see this is the ideal generated by 3 in Z 9 right. So therefore, this is 3 Z. That is the radical of 9 Z.

Is this clear let us look at another example, I S 12 Z. What is the radical of 12 Z? So let us write 12, 2 square times 3. See it will be an ideal generated by one element. So if you can identify one element probably the least, whose some power belongs to this ideal your through. Will 2 Z will any power of 2 belong to this ideal? Will any pure power of 2 belong to this ideal 2 4 8 16 32 64 will any of them belong to this ideal? For any element to be in this ideal 3 should divide that is one necessary condition.

Student: Minimum power of divisor.

2 4 also should divide. I mean 12 should divide that integer. So therefore, we cannot think of if you take say for example, ideal generated by 2 that will not be the radical. Can you think of one element whose some power belongs to this ideal?

Student: 6.

6 what is 6 square.

Student: 36.

36 does it belongs to this one yes write this times 3 this is 2 square 3 square which is 6 square. 6 square belongs to this one. Therefore, 6 belong to radical of I, now can we say it is equal. For any if I have an element if I have an integer m in Z, such that m power n belongs to I, can you say something about this m? What are the possible prime factors of m? You should you should be able to say at least these are there can you say something of that sort?

Student: 2 anti-power n minus 1.

Sorry.

Student: 2 anti-power n minus 1.

No I am asking can you tell me some prime factors of m.

Student: M by 2.

Prime factors of m. Prime divisors of m can you think of one.

Student: 10 by m.

No, that is not a prime factor.

Student: A.

Can it be 5 for example, m equal to 5 can it be that.

Can m be equal to 5? If I take m equal to 5 can any power of 5 belong to I? No. I is 12 Z remember. So what should be some minimal you know some you have to say some.

Student: (Refer Time: 38:53).

See that is what I am asking give me some prime factors of m.

Student: 2.

2 has to divide m right. You cannot get a number which is not divisible by 2 and whose power belongs to this, because any element here is divisible by 2. Right m power n belongs to I means it is here which means 2 has to divide m power n which means 2 has to divide m similarly.

Student: 3 as to.

3 also divides m. 3 divides m power n, which means 3 divides m. Which means 6 divides m what are we already say said that 6 belongs to radical of I, since this is an ideal 6 Z is contained in radical of I.

What we have observed here is that, if any element belongs to radical of I, 6 should divide that element, or in other words it is here. So this implies 2 has to divide m power n 2, has to divide m. And this also implies 3 has to divide m power n this implies 3 divides m so these 2 together implies that.

Student: 6.

6 divides m or in other words this implies that m belongs to 6 Z. So what we have proved is radical of I is 6 Z.

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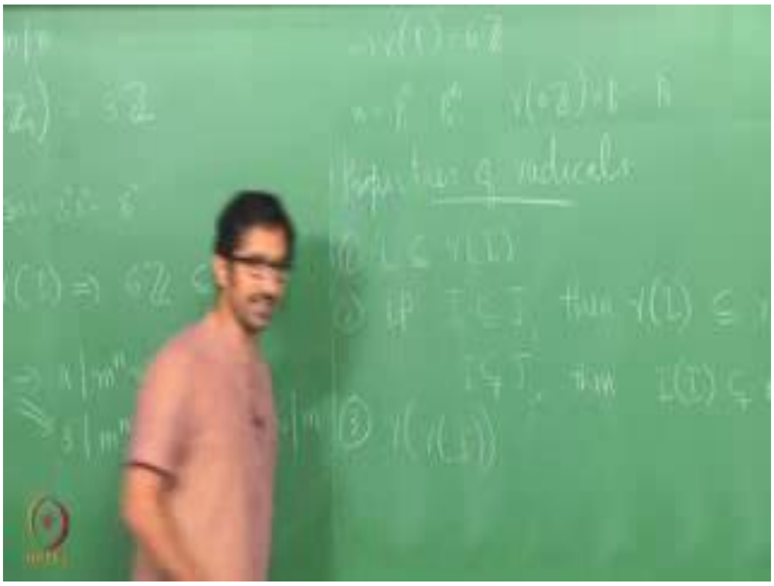

So now, do you see the technique at least in the case of Z? If I have if your n is p 1 power alpha 1 your n is p 1 power alpha on up to p r power alpha r, then what is the radical of n Z?

Student: P 1 to p r.

P 1 to p r; so now, let us look at some basic properties of radical of can you compare I and radical of I. What is the definition radical of I is nothing, but all elements in a whose some power belongs to I? So taking n equal to one itself or in other words if you look at this definition, this is phi inverse of, this is an ideal 0 belongs to this. What will be phi inverse of 0 fee inverse of 0 is contained in phi inverse of the nil radical which means. I is contained in.

Student: Radical.

Radical of I, if is contained in J then can you compare the radicals radical of I is contained in radical of J.

If I is strictly contained in J then can you say something here, this is true will it be true?

Student: Yes.

Look at.

Student: Yes.

We had 2 examples 6 Z and.

Student: 6 Z.

If you take 6 Z and 12 Z both their radicals are 6 Z or if you look at this example whatever the alpha 1 up to alpha r b radically is going to be this. So this is not true. If I take I have already, you know see a radical of I in some sense you can think of this as you know you take an element in I and taking the possible nth root; all possible nth root of elements in I whatever n b. So once I have done that your elements of r of I you cannot really take anymore nth roots. So if you apply this again. It should possibly not change. This is a simple verification. One containment is clear from the property 1. Right the right hand side is contained in the left hand side. This is here now can you say that this is here.

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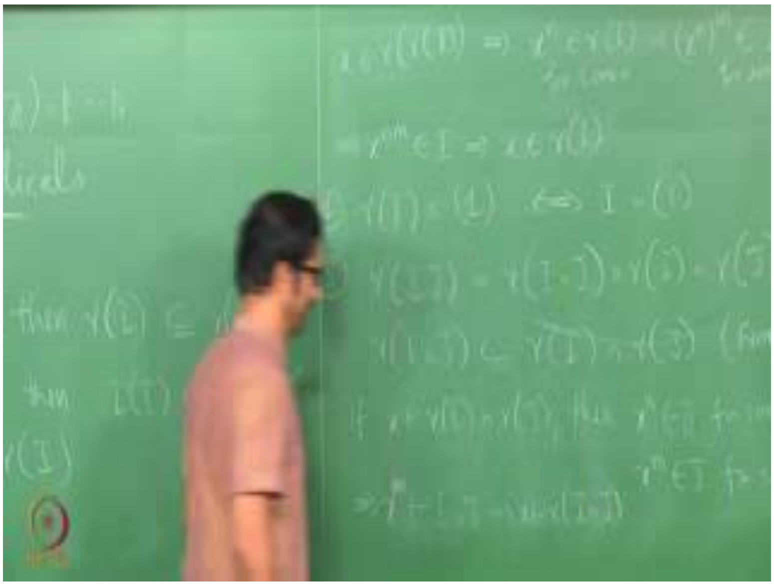

If I take x is in radical of radical of I; that means, x power n belongs to radical of I, but that would imply that x power n whole power m belongs to I. So this is for sum n and this is for sum m and that would imply that x power n m belongs to I by definition this implies that x belongs to radical of I.

Radical of I, is the whole ring if and only if I is one this is a simple verification I will leave it to you to do this. I am I will you know do one another problem. How do you compare a radical of the product? This is same as the radical of I intersection J which is same as radical of I intersection radical of J. So let us first look at these 2 comparisons. Can you see one of them contained in the other? I intersection J is contained in I therefore, radical of I intersection J is contained here similarly.

Student: Contained here.

It is contained here. Therefore, this is radical of I intersection J is contained in radical of I intersection radial of J from the first property. Now, if x belongs to I intersection radical f J some power x power n belongs to I for some n x power m belongs to J for some m now can you think of a power which is in both I and J. There are many candidates for that x power n m x power n plus m even x power maximum of n and m. All of them will belong to, so it is implying x power n plus m or m n belongs to I intersection J this implies x belongs to radical of I intersection J. So therefore, this equality is true proves the other equality and stop. Again can you see some equality there? I mean inequalities containment this is contained here I J is contained in I J is contained in J.

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Therefore, I J is contained in the intersection therefore, radical of I J is contained in this contained in radical of I intersection J by property 1.

Now, suppose I have x belongs to a radical of I intersection J. This implies x power n belongs to I for some n and x power m belongs to J, I intersection J. So x power n belongs to I as well as x power n belongs to J.

Student: J.

J now can you tell me and x power n belongs to J. Now can you give me an element in the product from these 2 information x power 2 n these times this belongs to.

Student: (Refer Time: 50:50).

The product ideal this belongs to I J. This implies x belongs to radical of I J. So this implies and that we have already proved that this is contained. So we will see more properties of ideal on Wednesday.