Commutative Algebra
Prof. A. V. Jayanthan
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 5
Properties of Prime Ideals

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So let us look at this map. A to A mod I 1 plus a mod I 2 so if I 1 plus I 2 is a then I 1 I 2 is same as I 1 intersection I 2 and we said phi is so, this is the natural map phi is surjective, if and only if they are co-prime. And then third point that we said phi is injective if and only if yeah is 0. Now this can be stated all these results are stated for n ideals so the proof is kind of similar I will not go into the proof, but the statements are let I 1 up to I n be ideals and we have this map phi from A to A mod I 1 cross a mod I n.

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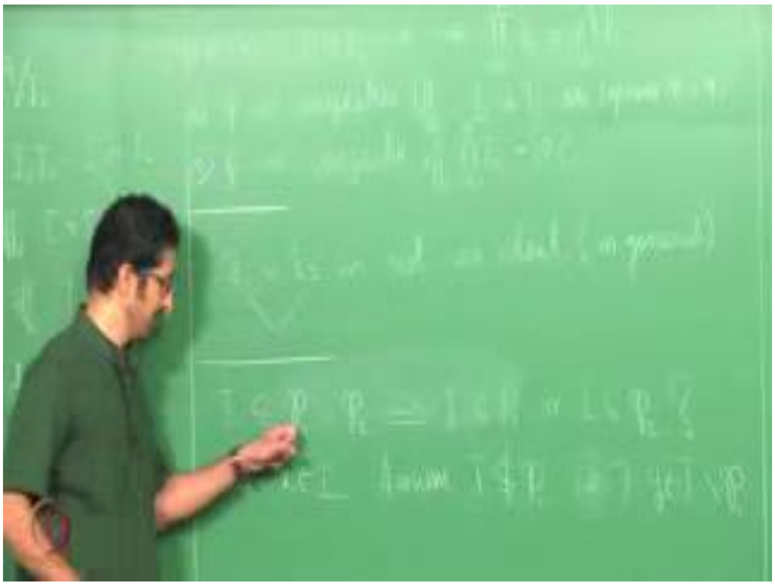

Then for all i not equal to j I i plus i j this A, this would imply that product i from 1 to n I i this is same as intersection I i i from 1 to n, c is surjective if and only if I i and I j are co-prime for all i not equal to j, and phi is injective if an only if this is intersection I i is this is 0.

The last point is pretty straightforward because it is the kernel is the intersection. The other ones are also and this is pretty similar to the proof that we gave. So I leave the proof as you know exercise to you. So we have seen that you know the ideals we have seen lot of basic properties of these ideals. Now suppose you have a 2 ideals I 1 and I 2, we have already seen that this is not an ideal in general. But suppose you have this is a prime ideal you know these 2 are prime ideals. Yeah so the question is if I have an ideal, which is contained in the union of true ideals. If it is there in the yeah can we say that it is indeed so in the integers as see as we saw now in the integers, if I take an ideal this is if it is contained in the union it is contained either here or, but is this always true.

So to start with let us look at an ideal I contained in union of 2 prime ideals. Suppose I is contained in p 1 union p 2. Can we say that I is of course, you know this is not an ideal can we say that I 1 is contained in p 1 or I is contained in p 2. Let us see if this is true so let us start with an element x yeah any doubt let x be an element in i. So we want to say that it is either we want to check whether it is here I mean whether this is true. Or this is true suppose this is not true, I want to say that then this has to be true I mean if for this to hold if I say this is not true then I have to say that this is true. Let so let us assume I is not that I is not in prime p 1. What does this mean? This means that there exists an element; there exists some y in i, but not in p 1 right. Now y is an i, but not in p 1 can you say something more about y.

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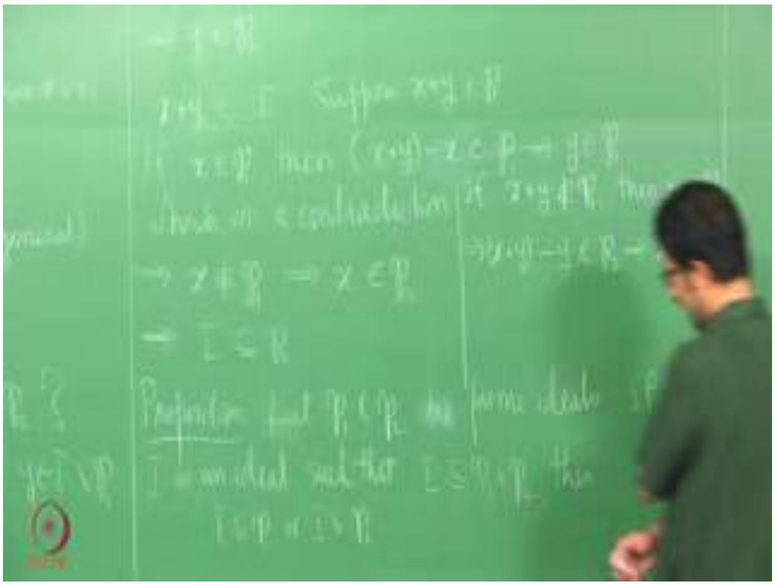

Then y is here, but not here. See to begin with this is there in there I is contained in the union. Therefore, y has to be in p 2. Our aim is to check whether x belongs to p 2.

So, let us let us look at the element x plus y. What can you say about x plus y this is in I right the element x plus y is in I therefore, it is either in p 1 or in p 2. Can it be in p 1, suppose it is in p 1 what does that say, can we get a contradiction x is in I x may be in p 1 or may not be in p 1.

Student: (Refer Time: 09:15).

So, if x is in p 1 that will imply that y is in p 1. So suppose if x is in p 1, then this would imply that x plus y. Suppose x plus y belongs to p 1 then x plus y is in p 1 and the x plus y belongs to p 1 so x plus y minus x belongs to p 1, which means y belongs to p 1 that is a contradiction. Therefore, x does not belong to p 1. X is in I and x does not belong to p 1 where can it be so this implies that x has to be in p 2 and that is exactly what we were trying to prove.

So we have started with an arbitrary element of I and proved that it is in p 2. So this implies that I am contained in p 2. So what did we prove now? So let p 1 and p 2 are prime ideals be prime ideals. If I is an ideal such that I is contained in p 1 union p 2, then I is contained in p 1 or I is contained in p 2.

Student: (Refer Time: 12:03).

Good so let us yeah so let us complete the proof. So suppose x plus y is not in x plus y is not in p 1, then where can it be? Then x plus y is in p 2, so therefore, x plus y minus y belongs to p 2 this implies x belongs to p 2. If so what is the assumption that we have made if I am not contained in p 1 it has to be in p 2.

Student: Property of prime number.

Because yeah so in in the case of 2 ideals we are not really using the property of prime are we, we are not. If this is I is contained in 2 ideals it has to be in one of them. That is what we are proving good observation. Now so therefore, it is natural to ask whether we can extend this for any finitely many ideals.

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So, the question is supposing I is contained in I 1 union I 2 then, can we say that it is contained in one of them.

Student: (Refer Time: 15:05).

No we are how do you when do you say ideal is prime.

Student: (Refer Time: 15:10).

Yes.

Student: (Refer Time: 15:15).

No because y is see we have started with y in i, but in general y is in p 2 does not mean that it is in I because it is in the I is contained in, so let us let us let us go ahead and you know try to prove in the same manner. Let see what happens start with a element.

So let us assume that let us assume this is true for the case of n minus 1 ideal we have proved for 2 so if I take if I is contained in a collection of n minus 1 ideal, then it is contained in one of them let us assume that. So assume that if I is contained in we did not use the property of prime in this. So let us not assume it for the time being and see whether we need it at some point of time if we do not need it will prove the general case if we need it we will impose the condition and you know prove it. So assume that if I is contained in I 1 union I n minus 1, so here the induction hypothesis is not just I 1 union I 2 I mean you should not confuse with the notation. If the induction hypothesis is if I is contained in any n minus 1 union, then it is contained in one of them.

So I am just denoting it by I 1 up to I n minus 1. If I take a collection of n ideals this should be the induction hypothesis should be true for any n minus 1 collection, I 2 up to I n I 1 I 3 up to I n. So then I is contained in so now, let us assume let I be contained in I 1 I n. Again by induction hypothesis if I is contained in any n minus 1 ideals, union of n minus 1 ideals then we are through. If I is contained in so I will just use the notation like, this I i1 union I in minus 1 then by induction, I is contained in in one of them. So therefore, we can assume that I is not contained in any n minus 1 collection.

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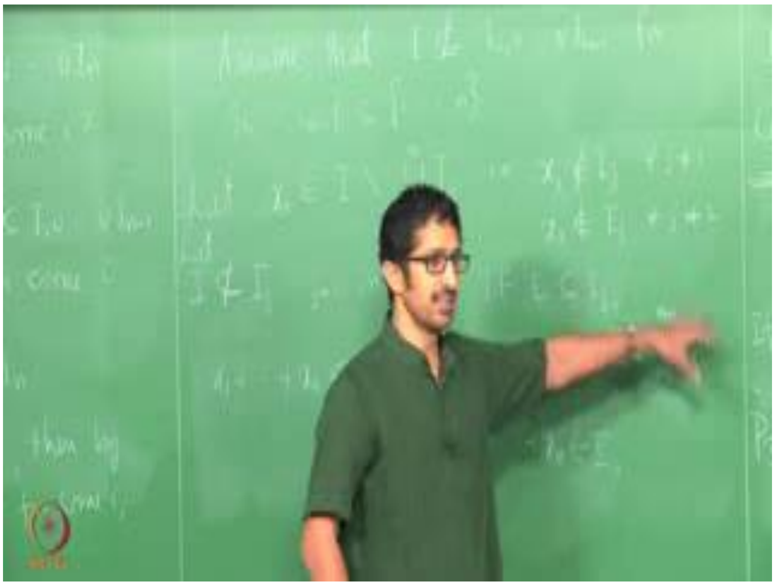

So therefore, assume that I is not a subset of I i1, I in minus 1 for this I 1 up to I n minus 1 contained in, you take any n minus 1 of them I cannot be contained there. Because if I is contained there then we are through by induction hypothesis. So therefore, we assume that this is not true. So now, let us choose an element let x i be an element in i, but not in the union of so I j, j from 1 to n j not equal to i; that means, x 1 is in i, but not in I 2 I 3 up to I n, I is not contained in this union right. For any I this is no I is not contained; that means, there exists an element here, but not in this one.

So x 1, x 1 is in not in j for all j not equal to 1 x 2 is not in j for all j not equal to 2 and so on. I am choosing elements like this. So let us assume that you know let us assume that I is not contained in I j, j from 1 to n minus 1. See my aim is to show that I i is contained in one of them. So let us assume it is not in the in the first n minus 1 ideals. And I want to show that I want to see whether I is contained in I n. Now here we want to again you know use this trick of creating an element in i, but it is not in other you know ideals. So if I take x 1 up to x 1 plus x 2 up to x n, what can you say about this element? This belongs to I certainly, but then we are starting see we want to say that I is contained in I n. So let us let us start with let us start with this we will try to prove by contradiction. Suppose this is not in any one of them. Let us see if we get a contradiction.

So x 1 up to x n, this is in i, so what? So it has to be in one of them right I is contained in the union it has to be in one of them; that means, x 1 plus etcetera up to x n is in one of the I j's.

Now x j belongs here, so I can remove that, x 1 x 2 up to x j minus 1 plus x j plus 1 up to x n belongs to I j from there. We cannot proceed see there were only 2, so therefore, when we remove one there was only one remaining and we could conclude, but here we more than 2.

So therefore, say if this is for example, if this is 1, that would imply that x 2 up to x n belongs to I 1. So what so we have to modify the proof the same proof does not work. So therefore, here comes a small trick.

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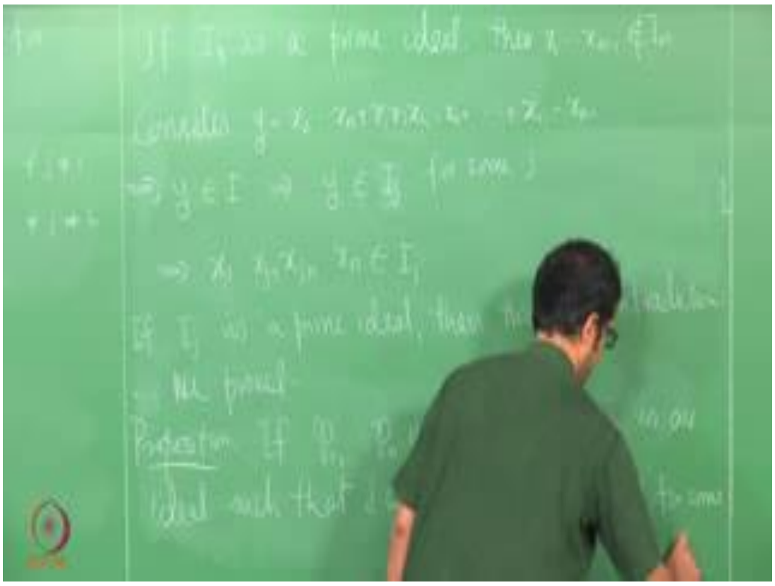

Suppose I look at this element x 1 up to x n minus 1. Look at this product. Can we say something about this product?

Student: (Refer Time: 24:45).

This is there in I 1, this is there in, and there in p 1 it is there I mean I 1 it is there in I 2 it is there in I n minus 1 can we say it is there in I n or not.

Student: (Refer Time: 25:11).

X 1 x 1 belongs to I 1, but not in any anything else, x 2 belongs to I 2, but not in any of them. So this product we cannot say anything about whether it belongs to I n or not. Now suppose my I all these I j's are prime. Then can we say something about whether this element belongs to I n or not. Suppose you are I n is prime. Does this belong to I n, because none of them belongs to I n. Therefore, this cannot belong to I n. So if I is a prime ideal, sorry if I j, I n is a prime ideal then x 1 to x n minus 1 does not belong to I n. Similarly, if I 1 is a prime ideal, x 2 up to x n will not belong to I 1. If I 2 is a prime ideal x 1 x 3 up to x n will not belong to I n I i2. So therefore, I look at this product.

Now, consider the element y equal to x 2 up to x n plus x 1 x 3 x 4 up to x n, x 1 up to x n minus 1. So from each product I remove the corresponding I th element. Is this element in i? It is there in I right this will say that y is in I i is contained in the union. Now in which ideal this will belong to. This implies y is in I j for some j. Now let us come back to this product. Just for understanding purpose, suppose this is in I 1. Now is this portion belonging to I 1, assuming I 1 is prime. Will this component belong to I 1, what about this component? All the rest will be in I 1. So therefore, if y is in I 1 y minus this portion will be in I 1 that will imply this is in I 1, but that is where we need the assumption on primes.

So, then so this imply y is an I j for some I j. So this will imply that x 1 up to x j minus 1 x j plus 1 up to x n belongs to I j. If I j is prime is a prime ideal, then this is a contradiction. So what did we prove? If I is contained in a union of prime ideals, then it has to be in one of them. Right is that clear now. If all I 1 up to I n all of them are prime ideals, then I has to be in one of them. So therefore, we proved if p 1 up to p n are prime ideals, and I is an ideal such that I is contained in p 1, p n. Then I is contained in p i.

So how did we prove this? We use induction for n equal to 2 we have proved it. Suppose n is bigger than 2, then if I is contained in any of the n minus 1 primes, then by induction we are through. So therefore, we can assume that I is not contained in any n minus 1 collection. We want to say that I is contained in one of them. Suppose not suppose I is not contained in any one of them then we choose an element which is in one of the primes, but not in any other n minus 1 we can do this by again the assumption that I is not contained in any n minus 1 collection union of any n minus 1 collection. So I have x 1 x 2 up to x n then I form a an element in i, by taking a product avoiding one of them. Take an n minus 1 product all the n minus 1 products of x 1 up to x n minus sum them up, this cannot be in any of the prime ideals because the ideal is prime all the ideals that we are considering a prime that is the idea of the proof.

Student: (Refer Time: 32:51).

So this is I j is not contained in j yeah, where did we use it.

Student: (Refer Time: 33:05).

No y is in i, sorry we are I n we are using this. We are only using to say that x we can choose elements which are not in n minus 1 of them, are we using only that much. So let us go through the proof again I start with x i which is here, but not in any one of them therefore, y is in I therefore, y is in I j for some j. Therefore, x 1 up to x j minus 1 x j plus 1 up to x n belongs to I j if I j is prime this is a contradiction.

Student: (Refer Time: 33:57).

No see we have to no see what is our assumption and this is we are getting contradiction to this assumption ultimately see if we are through if I is contained in one of them. That is what we are trying to prove I mean the proof we do not really have to do anything if I is contained in any one of them.

Student: (Refer Time: 35:19).

We cannot choose we cannot choose x j right x j minus 1. For example, if I is contained in I j for some j we cannot choose x j, not x j any sorry anything other than x j anything other than x j or is it, sorry if I is contained in I j you look at this union yeah I from 1 to n I not equal to j, except j. So this is so I i no I i want to say I equal to j is always there. So take for example, let us take j equal to 1 and if you take I from 1 to n minus 1 I cannot choose an element here because when I take I without this it is already empty. I 1 I is contained in I 1 so I cannot choose an element here. I mean this is I mean comes from the assumption that it I is not contained in any n minus 1 union. It is a byproduct of that assumption.

Student: (Refer Time: 37:24).

Sorry.

Student: (Refer Time: 37:30).

That is that is from the initial hypothesis, that I is that is not in any of the assumptions. This is the hypothesis that I is contained in the union of n ideals.

Student: (Refer Time: 37:45).

See this is the hypothesis. What is our hypothesis. I is contained in union of n ideals. So if I take any element in this, it is in this, which means it is in one of them. That is the hypothesis later once that we are this is an assumption in the proof. This is also an assumption in the proof, but that follows from here that is a byproduct of this one. So is this clear? Now let us look at you know now what happens to another possibility of these union intersection.

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Suppose a prime ideal is contained in the intersection. You have a prime ideal contained in the intersection. Can we say it is in one of them? Does that imply p is contained in again it is the yeah this is contained I 1 or p is contained I 2. Again prime ideal and this is true for any idea so let us look at the other possibility, p is p contains intersection.

So I have 2 ideals their intersection. It is contained in the prime ideal. Can we say I 1 and I 2, one of them, is in p so this is not obvious. It does not have a obvious answer looks like yeah so let us see at least in the case of when we think about this, in the case of integers this is right integers, this is same as a product, sorry this same as the taking the LCM. LCM is contained in p means LCM is divisible by a prime means one of them has to be divisible by the prime ideal therefore, it is the prime integer.

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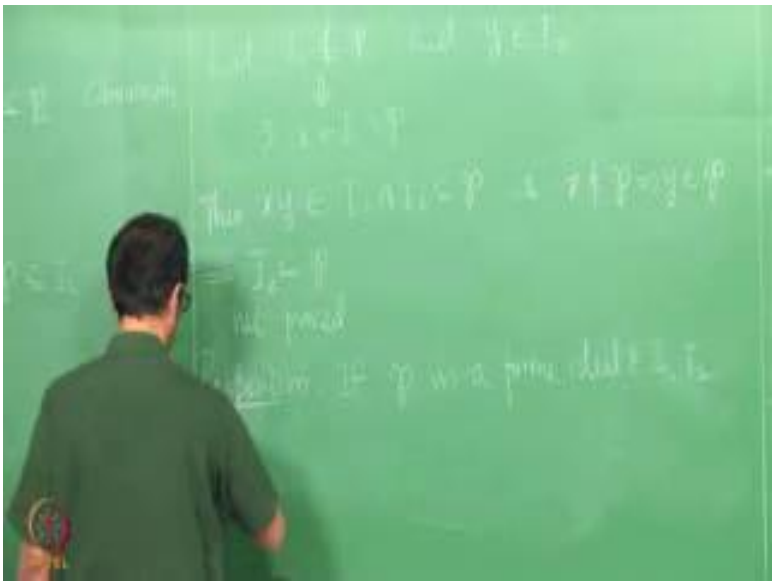

So therefore, let us try. So let us start with we want to check whether I 1 is contained in p or I 2 is contained in p. So let us assume I 1 is not contained in p. And I want to say that I 2 is I want to check whether I 2 is contained in. So let us start with an element in sorry yeah let us start with an element in I 2. Now how do we think about a contradiction in this case, or you know how do we think about getting I 1 is not in p. This implies that there exists some x in I 1, but not in p. So now, I have 2 elements x and y what do we do with them.

Student: (Refer Time: 41:33).

So the first possibility is looking at x plus y. What can you say about x plus y, x is in I 2 I 1 a y is an I 2. So this is in I 1 plus I 2. That is not in our hands right it is little far from, so what other possibility x y what can you say about x y it is in I 1 as well as in I 2 I 2. So that; that means, this is in the intersection, which means it is indeed in p, but if this is in p 1 of them has to be in p, but you know there is one element which is not in p, x, x is not in p therefore, y belongs to b and that is exactly we want to say right.

So therefore, then x y is in I 1 intersection I 2 which is contained in p, and x is not in p this implies y is in p. So this implies that I 2 is contained p. So therefore, we proved if p is a prime ideal, and I 1 comma I 2 are ideals, such that I 1 p is contained in I 1 intersection I 2, sorry p contains the I 1 intersection I 2 then, p contains I 1 or p contains I 2. And the same is so looking at the proof what if p is equal to I 1 intersection I 2. Can we say something more, if this were in equality?

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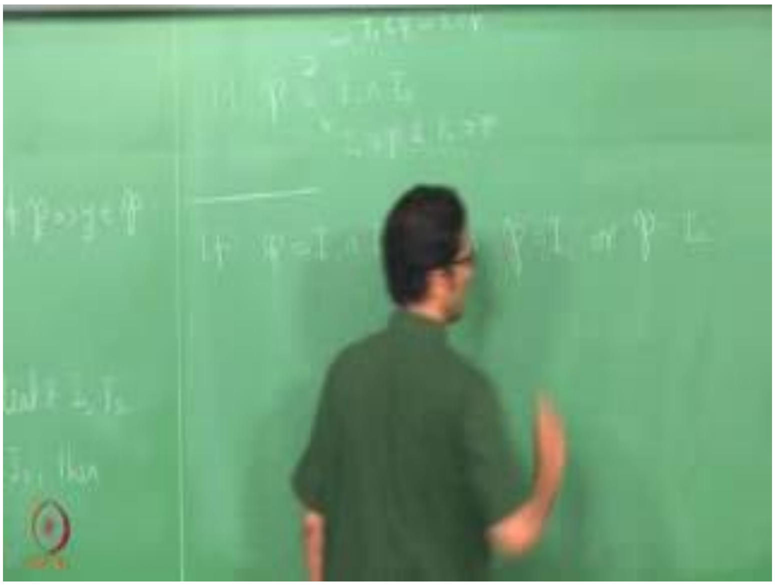

If p 1 intersection p 2, p is equal to I 1 intersection I 2, so this see this will already say that this right equality means these both the inequalities are true. This inequality will imply that I 1 is in p 2 or I 2 is in p 2.

This inequality implies that I 1 contain p and I 2 contain p. This is p, so if I 1 is contained in p that would imply that I 1 is equal to p. If I 2 is contained in p that would imply that I 2 is equal to p. So if p is equal to I 1 intersection I 2, then p is equal to I 1 or p is equal to I 2. These are some of the very important properties of prime ideals that we will you know one uses throughout you know study in commutative algebra. We will continue with more operations on ideals tomorrow.