Commutative Algebra
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Lecture - 4
Operations on Ideals

So last time we talked about nil radicals.

Student: Sir, can we find non-divisors of infinite field?

In a field we have non 0 divisors.

Student: Sorry an infinite ring.

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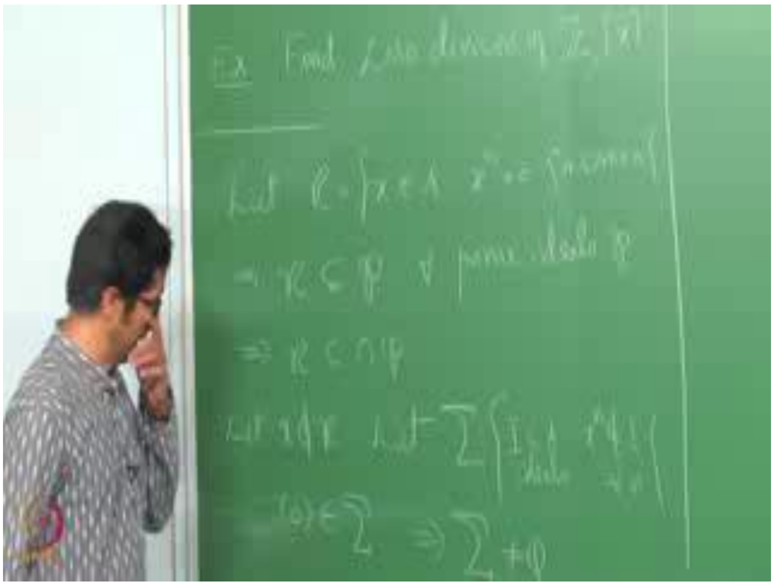

Infinitely so this is from a given ring let us say you know a finite ring for example, a finite ring example of a finite ring Z n right, from this can you construct a ring which is which has infinitely many elements using this ring.

Student: (Refer Time: 01:00).

Sorry?

Student: (Refer Time: 01:06).

Infinite Cartesian product, which has or there is another.

Student: Z n x.

Z n x, if you put this right this is polynomials with coefficients coming from Z n x. Do you now this Z n is contained here this has many 0 divisors.

Student: Sir actually my doubt was to find the number of.

Number is no I mean here this will have so this is exercise. Find 0 divisors of Z let us put 6, once you know Z 6 you can probably you know do it for Z n. It is a I can give you, but that would not you know it is better that you try understand that and then we will discuss this so we were talking about the nil radical that is set of all nil potent elements so last time we first we proved that if so let n base the collection of our x in a x power n is 0 for some n then. This is contained in p for our prime ideals.

So, this implies that n is contained in the intersection of all prime ideals. So therefore, natural question is whether this is equal and we try to answer that. So we started by looking at an element x which is not a nil potent. Then we looked at this collection of all ideals such that x power n is not in I not in I for all n. This collection is non-empty because the ideal 0 belongs to sigma. Therefore, this is non-empty.

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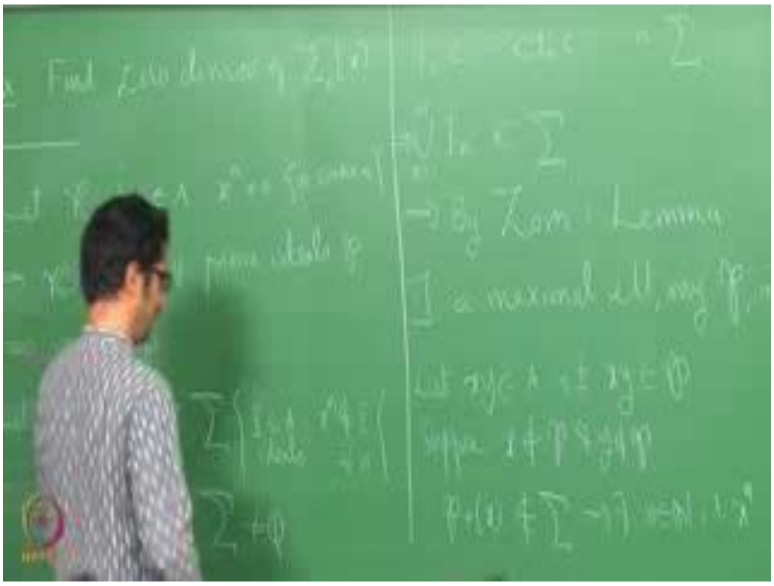

Then we looked at any chain of ideals in sigma. Then we proved that this chain has a maximal element, which is the union of union of I n, n from 1 to infinity. This belongs to sigma. Which means that every chain in sigma has a maximal has an upper bound in sigma. Therefore, by zooms lemma there exists a maximal element.

So, let us call that say p in sigma. And we wanted to prove that p is a prime ideal. So if suppose I have x y in such that the product belongs to p. We want to say that either x belongs to p or y belongs to p. Suppose, both so suppose x is not in p and y is not in p. That would imply that this p plus x is an ideal which is bigger than p. Therefore, it is not in sigma; that means, there exists this is not in sigma, which means there exists some power n such that x power well I am using the same this is this is x. So let us call this a and b a b belongs to this suppose a is not here b is not here then a such that this is p plus a.

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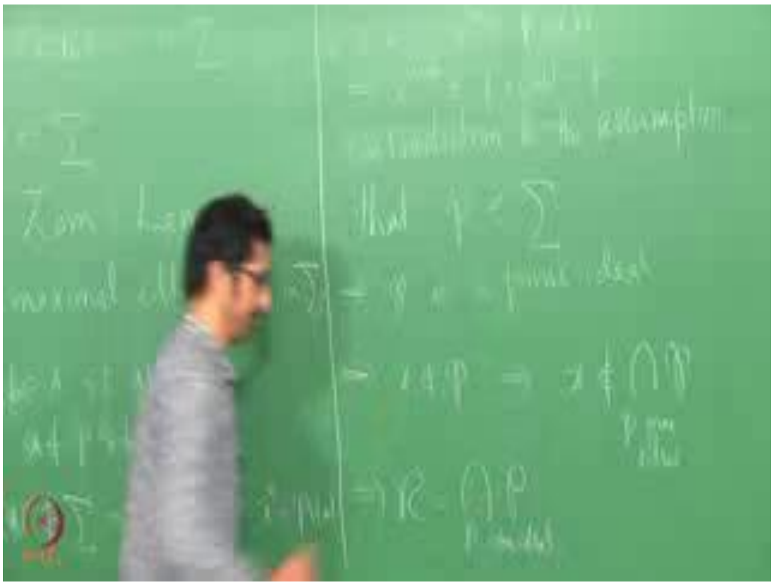

And there exists m such that x bar m belongs to p plus b. So this would imply that x power n plus m belongs to p plus a b which is p because a b belongs to p. Therefore, that is a contradiction, because p is chosen contradiction because p belongs to p sigma that p belongs to sigma.

So therefore, this says that contradiction says that either a belongs to p or b belongs to p. This implies p as a prime ideal. So p is a prime ideal in sigma implies that p does not contain x. So what we have done is if x is not an nil potent we have come up with a prime ideal, which do not contain the given element. So therefore, this implies that x is not in p and that implies that x is not in the intersection of prime ideals. So if x is not an nil potent, then it is not in the intersection of prime ideals. So this implies that the nil radical is indeed the intersection of all prime ideals. Is the proof clear? So now, what we have seen is the intersection of our prime ideals.

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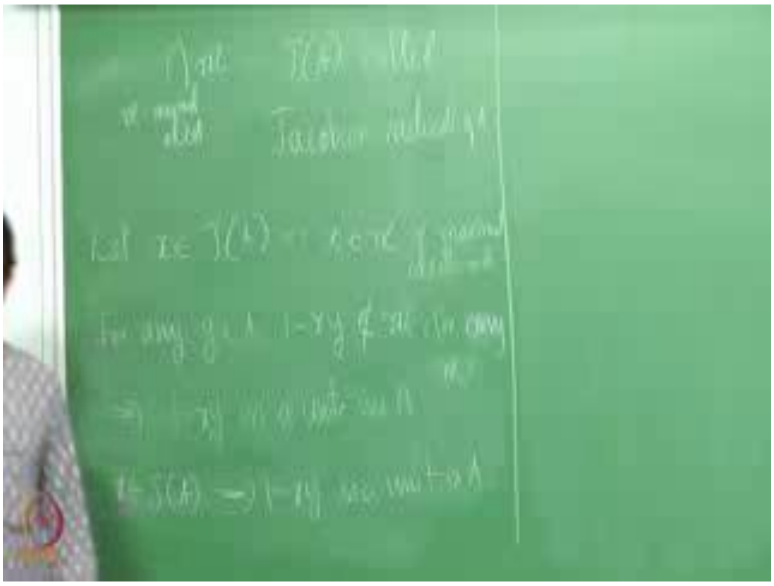

Now, any prime ideal is a maximal any maximal ideal is a prime ideal so what can we say about can we say something about the intersection of all maximal ideals. Of course, this is intersection of ideals, this is again an ideal. And this is called Jacobson radical. It is the intersection of all maximal ideals. Now is there a way to look at the elements or you know find the elements of this ideal. See here to check whether an element is in the intersection of all prime ideals. If it is nil potent or not that is a characterization it is nil potent if and only if it is in the intersection of all prime ideals. So in this case, do we have something nice you know nice characterization? So let us look at let us start with an element in the Jacobson radical.

Now, this is there in all I maximal ideals. Now if I look at so this is says that x belongs to m for all maximal ideals. So let us look at 1 minus x y, see if this is so for any y in the element 1 minus x y, can it belong to m for some m maximal ideal, because x y is.

Student: (Refer Time: 11:38).

X y belongs to m. Therefore, this cannot belong to m for any m. So this implies that 1 minus x y is a unit in A. Why is this because every non-unit is in some maximal ideal? So here this is an element which cannot be in any of the maximal ideal. Therefore, it has to be a unit. So what we have proved now is that x belongs to Jacobson radical of a implies 1 minus x y is a unit in A.

Student: (Refer Time: 12:37).

Sorry does not belong for any m.

So, there is this natural question whether the converse is true. If this is a unit in a can we say that for every y in A for every y in A this is a unit. Can we say that x belongs to maximal ideal all maximal ideals?

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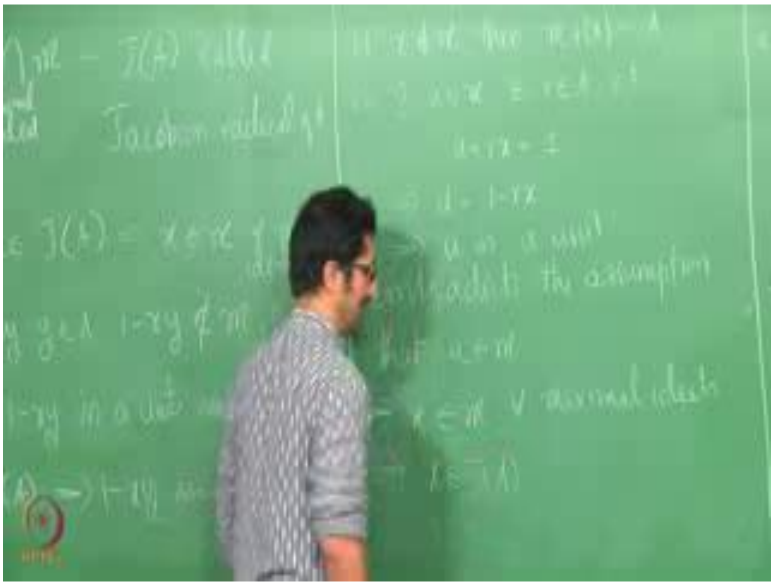

Suppose x does not belong to a maximal ideal then, then this A. Right or in other words that is there exists some element u in m and r in a such that u plus r x is equal to 1. Right which implies that u is equal to 1 minus r x, but now our assumption is that 1 minus x y is a unit for every y in A. Which means u is a unit by the assumption that is a contradiction; u belongs to the maximal ideal m. So this implies that x is there in all maximal ideal. So that implies x is in the Jacobson radical.

So, what have we proved now we proved that Jacobson radical is the set of all elements x such that 1 minus x y is a unit for every y in A.

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So, Jacobson radical is equal to set of all x in a such that 1 minus x y is a unit. An immediate corollary of this is that if 1 plus x is a unit for every x in m, so let us let me write, let m be a maximal ideal, and 1 plus x is a unit for every x in m. Let us see what could be the conclusion.

So, 1 plus x is a unit for every x in m. So I start with some non-unit, so let y be some element of some non-unit in A. This will imply that 1 plus so x y belongs to m, see if I multiply with any unit, if I multiply if this is a unit and if I multiply with any unit this will remain a unit right. Now if y is a non-unit then 1 plus x y is in sorry x y is in m and that implies that 1 plus x y is a unit.

That means what does that mean x y is in the sorry x is in the Jacobson radical, but we are starting with x in m. If I take in Jacobson radical is intersection of all maximal ideals. Therefore, Jacobson radical is contained in each maximal ideal, but here we are saying that m is contained in the Jacobson radical which means, both are equal this implies x is in is in Jacobson radical and that implies m is equal to J of A and that implies that A is a local ring.

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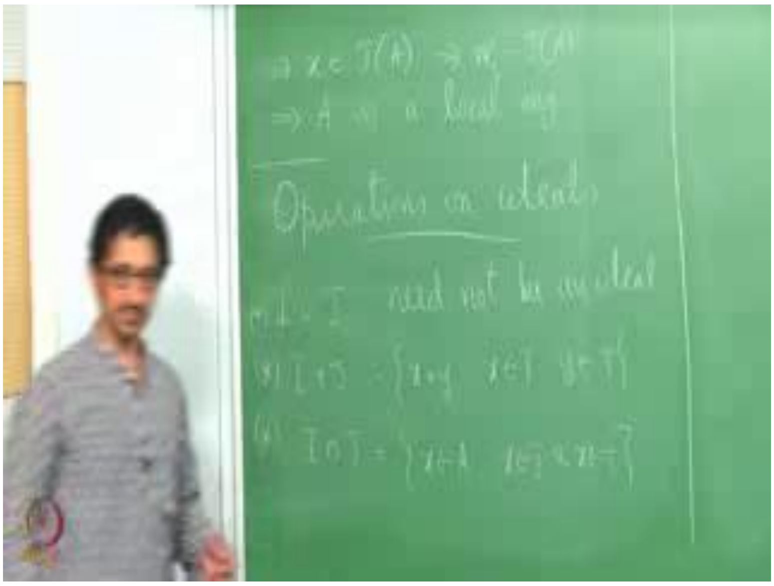

It has precisely one maximal.

We will come back to these properties as and when we go along. Let us look at some of the properties on ideals. You have seen that the operation taking the union, this need not be an ideal. When is this ideal I union J if one is contained in the other right. It is if and only if I union J is an ideal if and only if I is contained in J or J is contained in i. This is something that you have seen for the case of vector spaces group theory and at least in these 2 cases you must have seen earlier. This need not be an ideal and this is an ideal if and only one is contained in the other.

Another interesting operations operation on I and J is I plus J. So by definition this is set of all x plus y where x is in I and y is in J. Then this is an ideal in A. Third one is a I intersection J this is the set of all x in a such that x is in I and x is in J. Again this is always an ideal intersection of 2 ideals is again an ideal.

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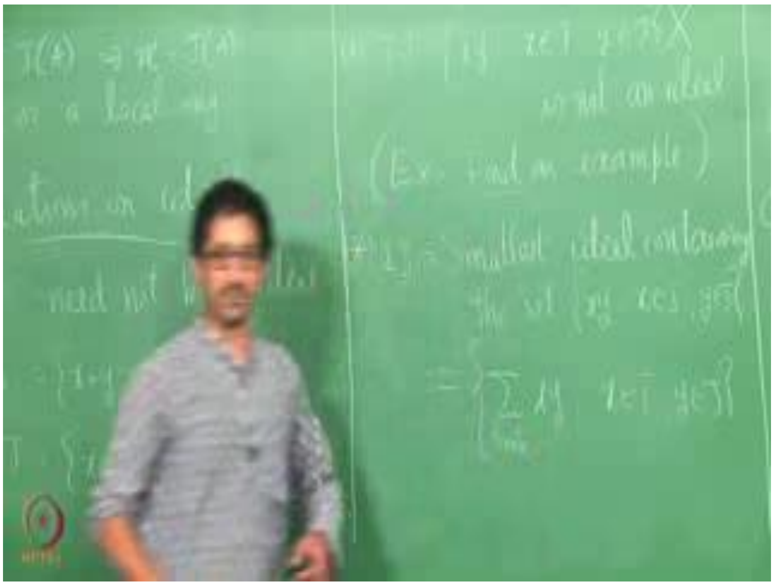

Another operation that you have probably not seen so far in the vector space theory or group theory, is I J the product ideal I J. So what could this be or what should this be x y such that x in I y in j.

Student: X y.

Why that this should be the natural definition right. Why are you saying that this is not?

Student: it is not ideal.

So if you define it like this then this is this set is not an ideal. I will leave it you to exercise find an example. Where will you look for? If you look at if you look for an example in in Z.

Student: it is a PID.

Sorry it is a PID. I will leave it to you know experiment try to get an example and show me that this is not an ideal. So therefore, this is not the way to define what we do is I J, I define it to be set of all or the ideal generated by this. You know or in other words the smallest ideal that contains all these products. The product ideal is as we saw or you know as you should be seeing if you take individual products it does not form an ideal. So it is natural to talk about ideal which is the smallest and containing this one. I will define in I define to be that as my product ideal I J is that clear.

So, I J, I define it to be so I can make a definition in many ways, smallest ideal containing the set or this is equivalent to writing out the sets in the element like this, writing out the elements in the set like this x y where y belongs to J, finite summation this is we are we are in the ring theory and whenever I write sigma for summation, it will always be a finite summation because we do not deal with infinite summation. So there is a ring called power series ring, which has elements are power series, but the operation in that ring is summation which we take only a finite summation. There is a difference, we will I take a power series as an example at some point of time and I will probably deal with it we will see, but whenever I write summation in in this course it means finite summation.

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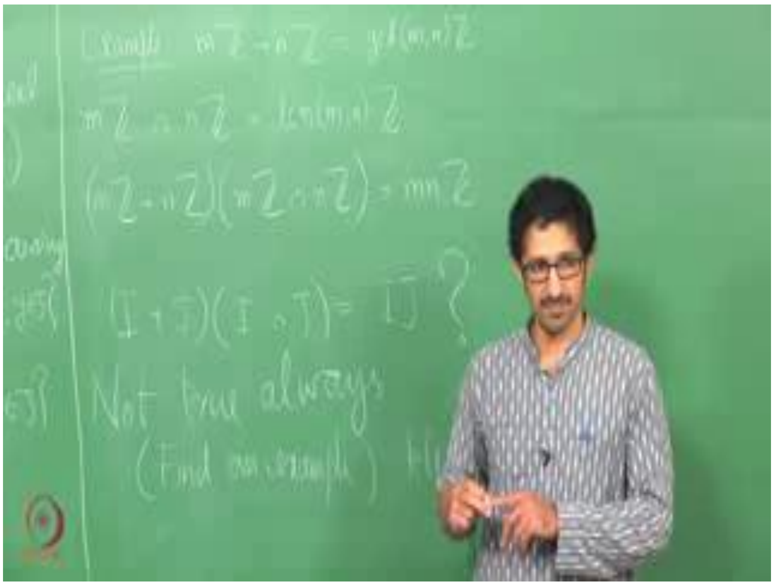

This is the product ideal. Just to look at some quick examples. What is m Z plus n Z this is GCD m comma n Z and m Z intersection n Z this is what would be this ideal.

Student: LCM.

LCM of m comma n Z, so therefore, m Z plus n Z times m Z intersection n Z this is m n Z. So in general should we say I plus J times I intersection J is I J. Right this is the analog should be, Z is a very beautiful ring. You know so we cannot expect everything that happens in Z to be true in general. So this is whenever taking an example and trying to pose a general theory Z is not a nice ring for that because Z. In Z lot of nice things happen which do not happen in other rings.

So, this is not true always. So again if you look for examples in k x k x and Z are kind of very close, ring property wise both are PIDs and you know their properties wise they are very close. So therefore, if you look for examples in polynomial ring in one variable again you would not be able to find.

So, try to think of an example, so when you are trying for an example it should not be in Z it should not be in k x, so that itself should give you a indication maybe I can even tell you look for an example in k x comma y, polynomial ring in 2 variables. Now this is this holds true if I plus J becomes the whole ring.

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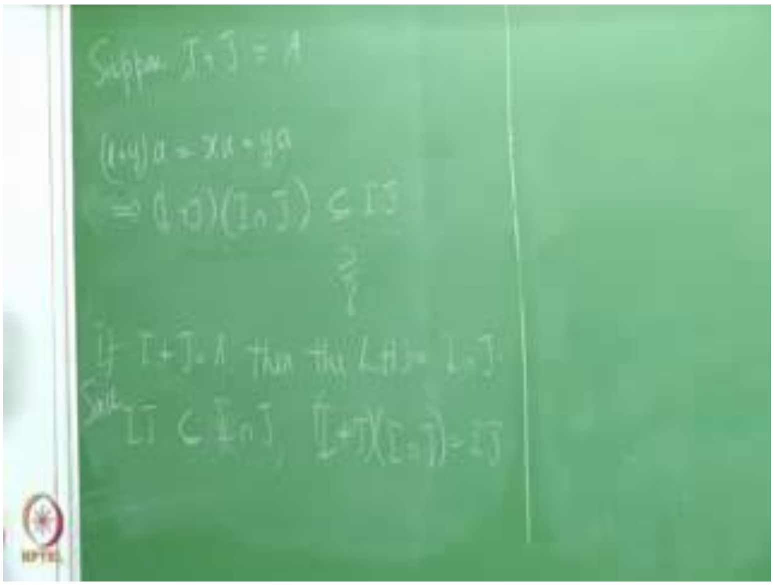

So, let us see suppose I plus J is the whole ring. In that case see when we write such a can you see one inclusion, can you say that this is contained here. This is contained here; which way is true see this is again a this is a product ideal this is a product ideal. So if I can say that every generator of this belongs to this one, then we are through now. How does any generator look like, x plus y times let us say a where a is in the; so I look at x plus y times a then this is x a plus y a x a is in I J y a is in I J therefore, this is in I J therefore, the left hand side is contained in the right hand side?

So, I plus J times I intersection J this always contained in the product ideal I J. The question is can we say that this is true. So suppose now this becomes clear right if I plus J is the whole ring what does I intersection J, sorry what is this product this product would be simply I intersection j, but this if you take any generator of I J it is of the form some product x y, where x belongs to I and y belongs to J that product belongs to I as well as j.

J or in other words it is in the intersection which means it is a, is that clear so if I intersection sorry I plus J is this the whole ring then the left hand side is equal to I intersection j, but at the same time I J is this is always true let us what we saw just now. Since I J is always contained in I intersection J we have I plus J I intersection J is equal to I J. So therefore, if I plus J is a then we have this equality.

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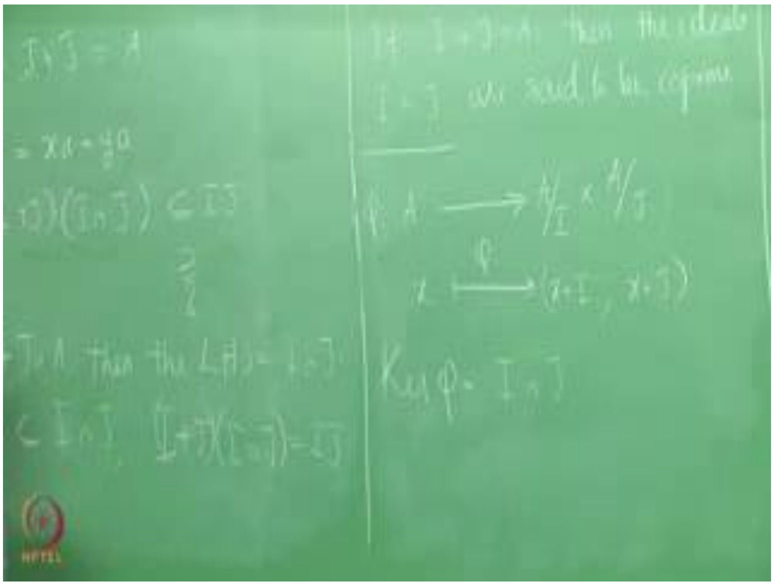

Now, if I and J sum up to a then we say that the ideals are co-prime. If I plus J is equal to a then the ideals, I and J are said to be co-prime. So the terminology again came from the ring of integers. When you say 2 integers are co-prime when you know there are no common prime divisors or there you can you know J their GCD is 1. Or in other words if you look at the ideal generated by I so take 2 integers to be m and n ideal generated by m plus ideal generated by n is the whole ring and that is exactly what we have.

Now, there is an interesting natural map if I take ideals in a, if I look at them see there is a quotient ring a mod I, if you have 2 rings a one and a 2 their Cartesian product is also a ring with you know addition and multiplication compound twice. So therefore, there is you know this is a ring, commutative ring and what is it is identity element what is the identity element of this ring 1 comma 1 right or 1 bar comma 1 bar. Bar denotes the corresponding equivalence class.

So, therefore, this is a commutative ring with identity and there is a natural map from A to this ring. A to A mod I there is a natural map. And A to A mod J there is a natural map. Now there is a natural map from A to A mod I cross A mod J, where any element x is mapped to x plus I x plus J.

Now, what can you say once you have, once you have such a map you start asking questions right. What can you say about this is a natural map this is a natural ring homomorphism? Can you say anything about the nice properties of what would be the kernel of this map? Kernel of this map is so let us call this map phi, kernel phi is I intersection J. Now this is surjective, sorry A to A mod I is surjective. A to A mod J is surjective. Now can we say that phi is surjective?

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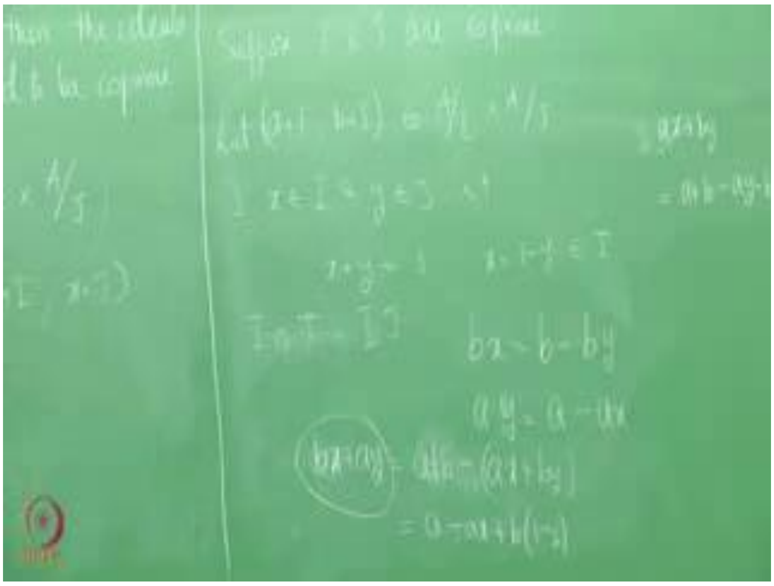

So, the question here is given an element a plus I b plus J can we find x in a such that x is congruent to a mod I and x is congruent to b mod J. See think about this in Z, in Z instead of I think of let us say primes. If they are distinct primes the Chinese remainder theorem says that you can do it. But so therefore, what should be the equivalent, if I and J are co-prime so let us let us try to see suppose I and J are co-prime.

Now, can we say that the map is surjective? So how do I check whether the map is surjective? So what should be the natural, so I have a let a plus I and b plus J be in A mod I cross A mod J. I want to say that there exists an x which is there exists an x in A mod I in A such that x is congruent to A mod I and x is congruent to b mod J, how do I look for that see there is already an information that x I and J are co-prime, what does that mean it means I plus J is equal to the whole ring which means there exists x in I and y in J such that x plus y is equal to 1. B x is b minus b y and a y is a minus a x. So b x plus a y this is b minus b y a plus b minus a x minus a x plus b y.

Now, modulo I what is this? Modulo I this would be a is in it may be so this is a plus a minus a x plus b into 1 minus y. Modulo I see this is in i, this is in i, so modulo I it would be a. Similarly, modulo b modulo J I can write this as b minus b y plus a times 1 minus x. So modulo J it will be b.

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So, therefore, this is precisely the a y maps to a plus I and b plus J so this implies phi is surjective.

Now, I will leave this as an exercise to say that phi is so this is the converse is also true. That is if c is if phi is surjective then I and J are co-prime, and phi is injective or one-one if and only if that is trivial, I intersection J is 0. And you can do this as we have done this for 2 ideals. This can be done for many I mean n ideals finitely many ideal. We will continue tomorrow.