Commutative Algebra
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Lecture - 36
Structure Theorem of Artinian Rings

We were on course to prove structure theorem for Artinian rings, it says that an Artinian ring is uniquely up to isomorphism a finite product of Artinian local rings.

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Yesterday what we proved was that if we start with if A is an Artinian ring with this with the maximal ideals. So, we know that Artinian rings have only finitely many maximal ideals. So, if A is A Artinian ring with maximal ideals m 1 up to m k then some m n then we showed that a is isomorphic to a mod m 1 per k cross a mod m n power k for some integer k right, this is what we showed last time.

So, now what we want to show is that this is unique, in the sense that if I take any product if A is isomorphic to some product of Artinian local rings, then they are you know those A i s are uniquely determined.

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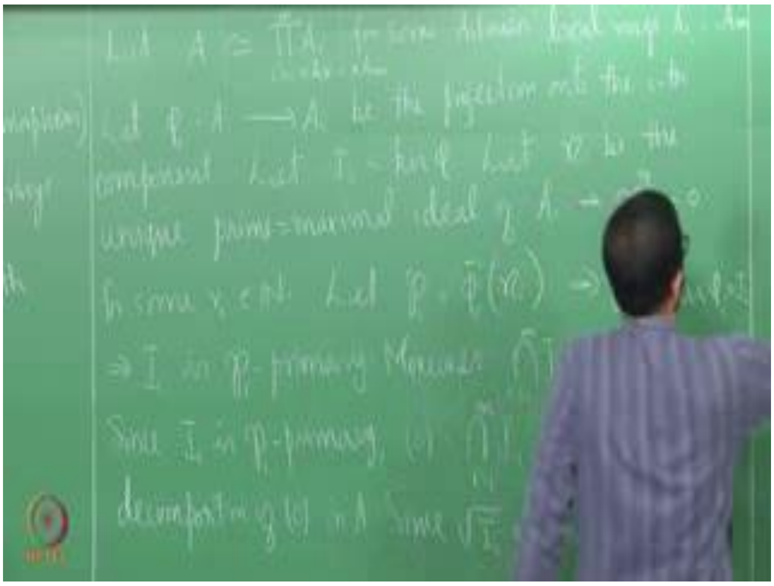

So, let A be isomorphic to product of A i, i from 1 to m for some Artinian local rings A 1 up to A m and for some Artinian local rings A 1 up to A m. Now what we want to show is that this A i s are uniquely determined, its it has an independent structure irrespective of you know the I mean this has an independent structure coming out of A. So, let us look at this map let P i be the map. So, this is A 1 cross A 2, this is A 1 cross up to A m I can. So, this is an isomorphism and I can compose the isomorphism with projection onto A i and called it phi i. If A to A i be the projection onto the i-th component and let I i be equal to kernel phi i.

Now, this A i s are Artinian local rings, which means A i has only 1 prime ideal which is also maximal and this has only one maximal ideal, but any prime ideal of an Artinian ring is maximal therefore, A i has only 1 prime ideal. So, let n i be the unique prime equal to maximal ideal of A i. So, therefore, this is n i itself is nil radical of A i, A i is Artinian local, therefore it is Noetherian, therefore nil radicals nil potent; that means, n i or some r i is 0 for some r i in n.

Now suppose I take let P i be the inverse of n i in A. So, therefore, this is prime P i will be prime ideal and hence maximal ideal because our A is also Artinian ring, P i is a maximal ideal. Now see n i power r i is 0 that means, if I take P i power r i an element in P i power r it will be mapped inside by phi i it will be mapped inside n i power r i, but that is 0; that means, P i power r i is contained in the.

Student: (Refer Time: 06:40).

Kernel of phi i that is P i power r i is contained in I i, now p i is a maximal ideal whose radical is whose power some power is contained in I i therefore, I i is primary to its a primary ideal primary to the.

Student: (Refer Time: 07:03).

Ideal p i; so this implies that P i power r i is contained in kernel phi i which is I i, this implies that I i is P i primary again because P i is maximal. So, this implies I i is P i primary and 1 more thing what is the intersection of I i? Intersection of I 1 I 2 to I m what is intersection this is an isomorphism. So, if I take the map.

Student: (Refer Time: 08:09).

If I take the map x going to x 1 bar x 2 bar up to x m bar, or a you know x 1 x 2 to x m here the kernel is precisely the intersection of all I i is, but that is 0 because this is an isomorphism right. So, therefore, moreover intersection of I i is 0, because this is an isomorphism and this is by the and precisely the composition with the projection. So, if phi sends A to x 1 x 2 up to x n this is phi i is A I mean. So, phi since x 2 x 1 up to x n then phi i of x is x 1 phi 1 of x is x 1, phi 2 of x is x 2 and so on. So, the kernel is precisely the intersection of all this and that is 0. Now, 0 is equal to this intersection and each I i is.

Student: P i primary.

P i primary; that means this is a primary decomposition of 0 in a. So, this since P i is an I i is I i is P i primary, the 0 equal to I i intersection of I i is a primary decomposition of 0 in A. See here see if you look at this phi i is A composite with I mean the isomorphism composite with the projection, then A i is isomorphic to A mod I i, but that does not say that A i is are uniquely determined; but if we say that this I i s are uniquely determined not I mean if we can describe I i without the help of this isomorphism; here we are A i is isomorphic to A mod kernel of this map, that does not give any description of I i without mentioning about this relation. If I can describe I i without mentioning about this relation; that means, it is uniquely determined. Without any other information this without using this information, if I can describe I i that will imply that A i s are uniquely determined.

Now, what we have concluded is that all these I i s are nothing, but I mean the I i s using this I i s I can give a primary decomposition of 0, but again a primary decomposition is not unique right as we saw in k x y the ideal x square comma x y, it has infinitely many different decompositions. So, that does not uniquely determine the components, but later we proved one kind of uniqueness. The first uniqueness theorem which said sorry the second uniqueness theorem which said some primary components are uniquely determined, the primary components associated with the isolated primes.

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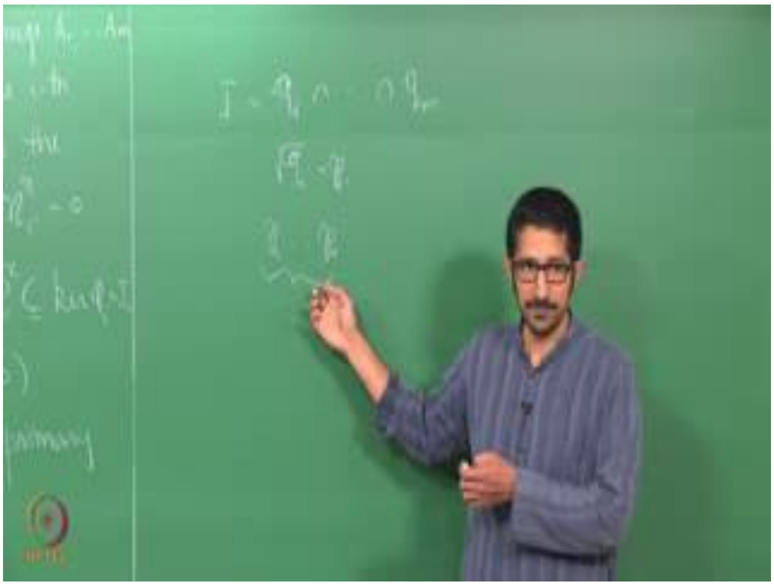

See in the in the primary decomposition, so I have A i have an ideal I 1 intersection. So, let q 1 intersection q m they are all prime primary ideals, the radical of q i let us say P i.

Now, if p 1 up to p r if these are the isolated primes that is the minimal primes; in this there are minimal prime ideals and embedded isolated prime ideals and embedded prime ideals, if these are the minimal prime ideals isolated prime ideals then q 1 up to q r are uniquely determined. They are unique in any decomposition in any primary decomposition that q 1 up to q r have to be there this is what we proved some time back.

So, now coming back to this case, let us look at this primary decomposition; do we see some isolated or embedded components here, what are the primary I mean what are the radicals of each I i?

Student: (Refer Time: 13:51).

It is.

Student: (Refer Time: 13:52).

P I, now what can we say about p i?

Student: (Refer Time: 13:58).

P i s are all maximal. So, there cannot be an embedded there cannot be a prime ideal that contains P i right there cannot be a prime ideal that contains p i. So, all these P i s are.

Student: (Refer Time: 14:21).

Maximal I mean isolated, they are all isolated means each I i is.

Student: Uniquely determined.

Uniquely determined; that means, A i s are uniquely determined because what we need to do is I mean. So, what we are proving here is that if you take a primary decomposition of an Artinian ring, if you take a if you take an Artinian ring and look at a product then this product cannot be too different from you know I mean this product has to be of this form where you look at the 0 ideal in A look at its primary decomposition. In the primary decomposition you will have only isolated primes because you take any primary decomposition each primary ideal will be primary to some prime ideal, but every every prime ideal of a is maximal. So, therefore, you cannot have an embedded component there.

So, look at every I mean look at the primary decomposition, look at each component that is nothing, but and they are therefore, they are each primary component associated with the unique associated with the isolated primes are unique. So, take those unique take that unique primary decomposition, each A i has to be A mod I i for some I j I mean I i, that is precisely what we are proving here. So, let me complete this since radical of I i equal to P i is maximal P i s are isolated primes associated to 0.

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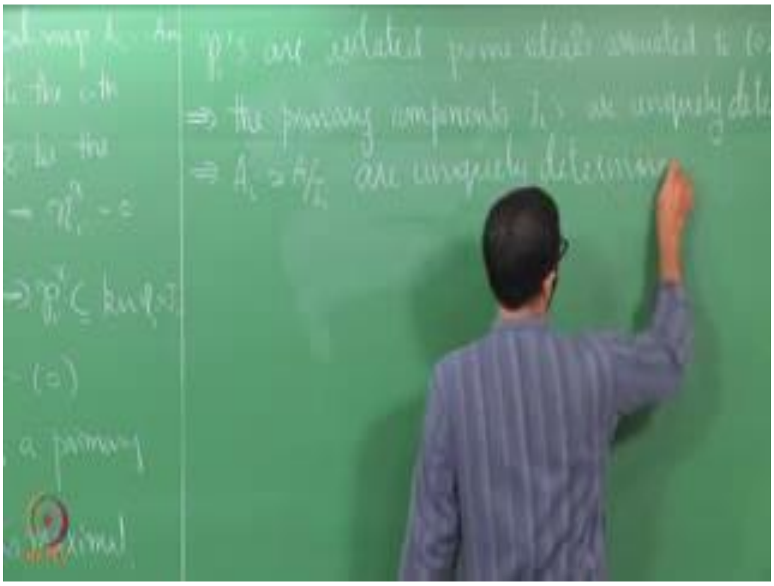

Therefore the primary components I i s are uniquely determined; that means, A i isomorphic to A mod I are uniquely determined. So, that proves the structure theorem. So, up to isomorphism, so in an Artinian ring what you need to do is look at 0 ideal, look at its primary decomposition, and look at I mean look at any primary decomposition because any primary decomposition will be the same, because all the components are all the primary components are.

Student: (Refer Time: 18:15).

Primary to maximal ideal therefore, they are all isolated; there are no embedded components there. So, therefore, they the primary decomposition is unique, in that primary decomposition look at take the i 1 up to i m a mod i m i mean a is isomorphic to a mod i 1 cross up to a mod i m that is precisely an Artinian, how an Artinian ring looks like?

So, we were talking about see Artinian ring is same as Noetherian ring with all maximal ideals or prime ideals maximal. So, this property that all prime ideals are maximal has another name called dimension 0.

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So, what is dimension? Dimension is this is called cruelled dimension, if you take a chain. So, this is a let A be a ring, the length. So, then dimension of A is by definition maximum of n such that there exists a chain of prime ideals, P naught prime ideals of A if there is A i mean look at look at all n for which there is a chain like this and look at maximum.

Student: (Refer Time: 21:02).

0 need not be a prime ideal right for example, what is dimension of k field.

Student: (Refer Time: 21:21).

It will be.

Student: 0.

Zero right there is because you do not have there is only one prime ideal which is 0 itself. If R is if A is a PID then dimension of a is see.

Student: (Refer Time: 21:52).

If you take any integral domain you can always start with.

Student: 0.

0 is a prime ideal.

Now, what will be the next prime ideal? It should be a nonzero prime ideal, in a PID a nonzero prime ideal is.

Student: (Refer Time: 22:10).

Maximal; so you cannot have another. So, therefore, dimension of A is 1. Can you give me another example where dimension of dimension is 0?

Student: (Refer Time: 22:34).

If you take an Artinian ring or example for example, Z n, if you take Z n it is going to be an Artinian ring right. If n is how do you say that this has dimensions 0 if n is prime then it is a field, if it is not what are the prime ideals of Z n? See this is nothing, but Z mod n Z what are the prime ideals of Z mod n Z?

Student: (Refer Time: 23:19) prime.

P Z n, where p divides?

Student: N.

N right now they are all maximal; they are all maximal in p z n. So, therefore, the only prime ideal here you can start with this p naught there is nothing more. So, therefore, these are all; can you generalize this looking at I mean can you from a take A and bring it to an Artinian ring, what is that we are doing here z is a PID right. So, what is that we are doing here?

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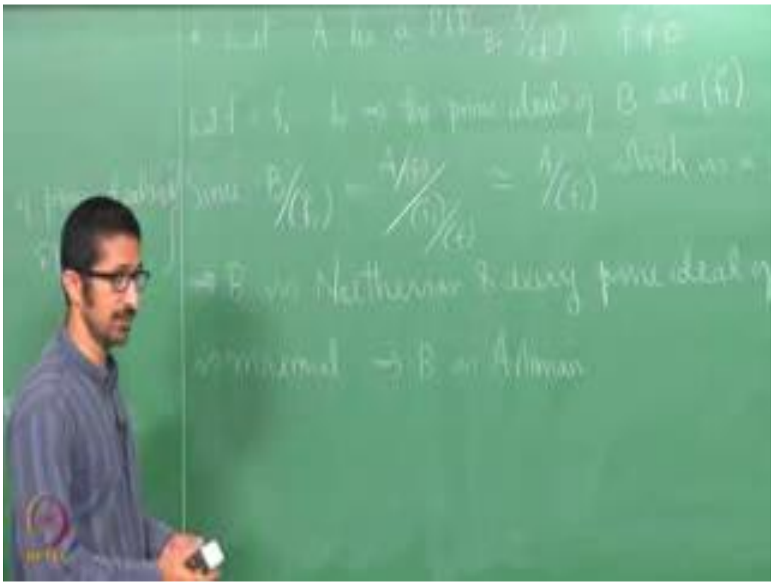

If I take A be a PID, can you construct an Artinian ring from A, how do you get an Artinian ring from A, it should be PID is Noetherian.

Student: (Refer Time: 24:46).

So, every prime ideal should be maximal, now what are the you know prime ideals of A? Nonzero prime ideals are all maximal, everything is generate every ideal is generated by 1 element.

Student: (Refer Time: 25:13) p power a mod p power.

A mod.

Student: P power n for.

P power n for some n p is.

Student: (Refer Time: 25:26).

Prime element in a.

Student: (Refer Time: 25:27).

No when you say prime element it is already nonzero. This is see the only prime ideal that contains. So, only prime ideal that contains p power n is the ideal generated by p. So, you have only 1 prime ideal in such a chain here right there is only 1 prime ideal here, but do we can we say more is it necessarily be of this form.

See for example, Z 6 is this Artinian.

Student: (Refer Time: 26:20).

So more generally you do not you cannot you do not need to save and powers f nonzero.

Student: (Refer Time: 26:24).

Right, this is going to be Artinian, why is this Artinian? What are the prime ideals of this? The prime ideals of this ring are prime p which contains f, now what are the prime ideals that contain this f.

Student: (Refer Time: 26:53).

F is in A nonzero, f is in a PID therefore, it is in a u f d f has a primary I mean prime factorization, I will write this as f 1 up to f r then the prime ideals of I call this B, prime ideals of b are precisely.

Student: (Refer Time: 27:20).

F 1 mod f f 2 mod f and f r mod f, these are the only prime ideals of, but they are all maximal because this is this is generated by an irreducible element they are all maximal. So, therefore, the only prime ideals of B are generated by f 1 up to f r and they are all maximal, hence b is Artinian then the prime ideals of B are f 1 bar f r bar, since b mod f 1 bar this is isomorphic to this is equal to A modulo f mod f 1 mod f which is isomorphic to A mod f 1, which is a field because f 1 is an irreducible element nonzero reducible I mean it is an irreducible element therefore, this ideal generated by irreducible element in A is maximal, therefore this is a field. Therefore, these are all maximal ideals; that means, every prime ideal in B are maximal, B is Noetherian because A is Noetherian this implies B is Noetherian and every prime ideal of B is maximal this implies B is Artinian ok.

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Now, let us look at one more nice property of Artinian rings, let A be an Artinian local ring with the unique maximal ideal, then the following are equivalent. Every ideal of A is principal, m is a principal ideal dimension as an a mod m vector space the module m mod m square here what we are saying is in an Artinian local ring if the maximal ideal is principal then every ideal is principal an Artinian local ring is a principal ideal ring, if and only if the maximal ideal is principal. So, 1 implies 2 implies 3 is straightforward; 1 implies to every ideal is principal implies m is principal. Now if m is generated by 1 element.

Student: (Refer Time: 32:00).

Then this is by Nakayama's lemma 2 implies 3 is straightforward application of Nakayama's lemma. So, 1 implies 2 trivial, 2 implies 3 follows from Nakayama lemma. So, let us prove 3 implies 1 that is the vector space dimension of m mod m square is less than or equal to 1 implies every ideal of A is principal. So, let first of all if dimension a mod m of m mod m square suppose this is 0 what does that mean? If this is 0 means that m has to be equal to m square, a vector space if it contains 1 nonzero element it will have dimension at least one. So, this is 0 means m has to be equal to m square, but again by Nakayama lemma if we are in a Noetherian local ring m is finitely generated. So, therefore, by Nakayama lemma m is equal to m times m which means m has to be 0, and m is 0 means what?

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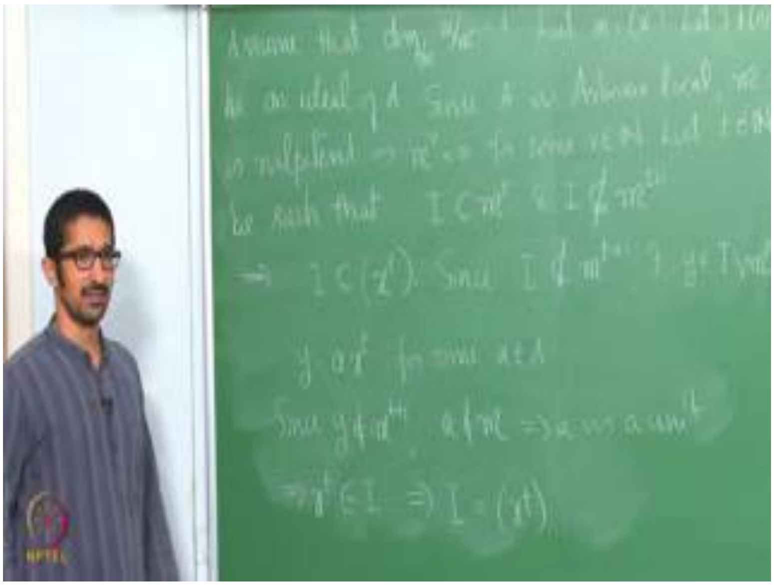

Student: (Refer Time: 33:33).

K is a field right that implies I mean A is a field. So, therefore, we can assume that now let us assume that dimension of dimension of m mod m mod m square is 1. Now we are in a Artinian local ring m is the.

Student: (Refer Time: 34:09).

Sorry.

Student: (Refer Time: 34:10).

M is m is 0 a mod m is always a field right for any maximal ideal a mod m is a field, now m is 0 means a mod m is same as a or you know these are characterization that we have discussed earlier; a field is nothing, but maximal ideal being 0. So, suppose let m be generated by some element x, a is an Artinian local ring and m is the maximal ideal there well sorry now we need to prove that every ideal of a is principal. So, let us start with a nonzero idea let I be a nonzero ideal an ideal of a, we want to say that I is generated by 1 element.

Now, m is the unique maximal ideal in a therefore, the nil radical of a is m, nil radical of is nil potent therefore, m power r is 0 for some r right what does that mean? Since A is Artinian local, m is nil potent; that means, m power r is 0 for some r.

Student: (Refer Time: 36:21).

Sorry.

Student: (Refer Time: 36:26).

This 1 Nakayama lemma if m if dimension of m mod m square. So, we proved this, if as a vector space over a mod m, if m mod m square is minimally generated by x 1 bar x 2 bar up to x r bar then m is generated by x 1 up to x r, this is something that we proved immediately after Nakayama lemma. So, this implies in fact, these 2 are equivalent to 3 are equivalent is non long back. The main contribution here is that they are equivalent to every ideal being principal. So, m power r is 0; that means, see I is not contained in m power r. So, therefore, look at the least see I is contained in m, m is the unique maximal ideal. So, every ideal is contained in m.

Now, look at the least n let t be in N be such that I is contained in m power t, and I is not contained in m power t plus 1. See I is contained in m and m power r is 0 and I is nonzero; that means, there exist some positive integer such that I is contained in I is not contained in m power t and I is contained in m. So, I look at whether I is contained in m square, if I is contained in m square look at check whether I is contained in m cube if it is not that 2 is 2 is equal to t.

Similarly, we look at the least integer for which I is contained in m power t you are not (Refer Time: 38:53) the largest integer for which I is contained in m power t. So, therefore, this implies that I is contain what is, but what is m power t m power t is.

Student: (Refer Time: 39:09).

Ideal generated by x power t now. So, I want to say that I is contained in x power t, x power. So, I is not here; since I is not an m power t plus 1, there exists y in I, but y is not in m power t plus 1; that means, now y is equal to a y is equal to a x power t for some a in A; because y is in I and I is contained in x power t. Now can a be in the maximal ideal maximal ideal is generated by x, can a be in maximal ideal? See if a is in maximal will be equal to some a prime x which means y is prime x power t plus 1 that will imply y is in m power t plus 1, but we our assumption that y is not there.

Therefore, since y is not in m power t plus 1, a is not in m, but what are the elements which are not in m? In a this is the unique maximal ideal. So, if it is not here then a is a unit this implies a is a unit and that would imply that x power t belongs to I. I is contained in x power t, x power t belongs to I that will imply that I is generated by one element which is precisely x power t. So, y is equal to A times.

Student: X power t.

X power t. So, a times x power t is here I mean is in I y is in i.

Student: (Refer Time: 41:50).

So, a power x a times x power t is here, a is a unit any multiple of this is here. So, multiplied by inverse; see this is somewhat clear what are we doing here we are looking at we are taking an arbitrary ideal, we are starting with an arbitrary ideal that ideal is contained in m. Now we look at the largest integer such that I is contained in m power t. So, I is contained in x power t, now I every element of I is A i is not contained in x power t plus 1, I has contained in x power t ideal generated by x power t and I is not contained in ideal generated by x power t plus 1 every element of I is some a power t I am sorry a into x power t, but then a cannot be in m because if a is in m that will imply that y is in.

Student: (Refer Time: 43:00).

X power t plus one. So therefore, a has to be unit which means I has to be generated by this x power t.

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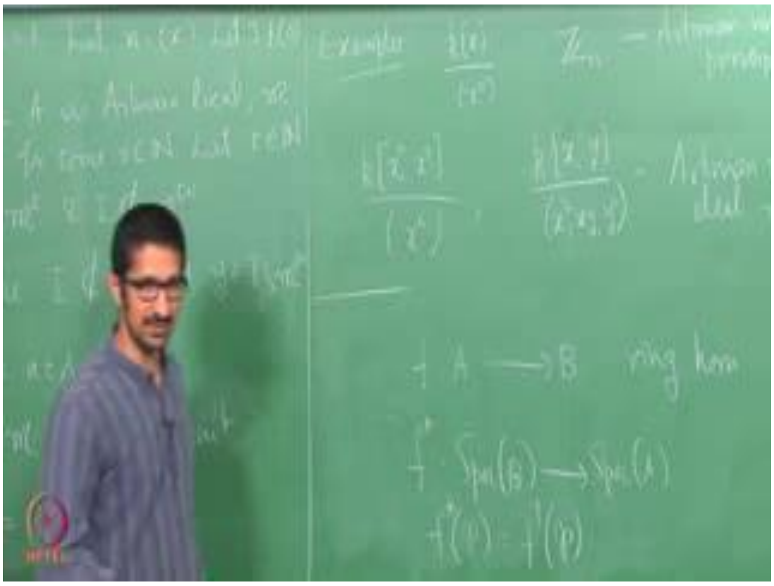

So, every principal every ideal is generated by some power of the maximal ideal. So, examples like well not all Artinian local rings satisfy this property, but for example, if you take k x mod x power n, I mean of course, z, z n you take any z n that will satisfy. If I look at an example where this is not satisfied, an Artinian ring whose maximal ideal is not a principle ideal that x square, x cube, x power 6 this is this is an Artinian. So, this is Artinian rings with maximal ideal principal, these are now if you look at k x y mod x square x y y square, these are Artinian rings with maximal ideal not a principal not principal here the maximal ideal is generated by x and y. Here the maximal ideal is generated by x square and x cube; we will prove what are one of the most important results in commutative algebra that you know has large implications in important applications in algebraic geometry called Noether normalization and Hilbert's nils Talenza.

So, before this let me just recall some basic facts, suppose I have a ring homomorphism from a to b this is a ring homomorphism then correspondingly I have a map f star from spec B to spec A what is spec B.

Student: (Refer Time: 46:38).

Set of all.

Student: Prime ideals.

Prime ideals of B and this is set of all prime ideals of A. So, what is f star of p?

Student: (Refer Time: 46:50).

F inverse of p right that will be a prime ideal here; so therefore, this is now see we have we were talking about integral dependents integral in I mean integral extension and so on.

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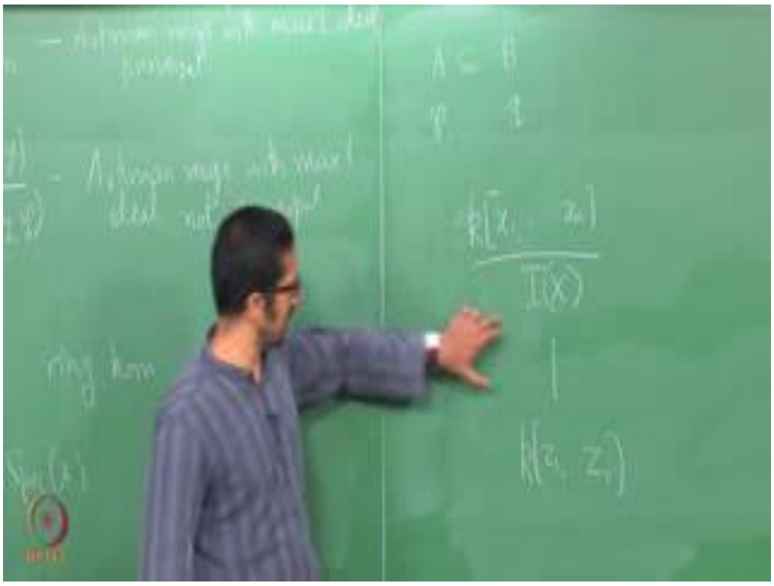

So, what we showed some time back is that if I have an integral extension, and if you start with the prime ideal here then I have a prime ideal here, which you know intersects with A to give P, this is called lying over theorem right the first one we proved given any prime ideal that exists a prime ideal lying over this and then using this we prove the going up theorem.

So, now suppose you take. So, the see geometry one thing is that when you are studying when you want to study curves or you know geometric objects, one way is to reduce them to a slightly simpler situation and then study; there are many ways, one of them is. So, one thing is suppose I have a suppose given. So, I have an algebraic set for which I have an ideal I. So, I have this polynomial ring k x 1 up to x n this is called the coordinate ring of the variety v. So, this is corresponding to verity or you know algebraic set x.

Now, if you look at this ring one of the questions that are relevant in geometry is, whether this is a finitely generated module over some polynomial ring. So, this is indeed true is what the result Noetherian normalization say we indeed have A i mean we have any finitely generated. So, any finitely generated k algebra looks like this k x 1 up to x n modulo some id. So, the Noether normalization theorem say is that this is indeed true and we will precisely say what can be you know in some nice cases what are these Z 1 up to Z r, one can take this as you know some linear combination of x I s. So, I will do this in the next class.