Commutative Algebra
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Lecture - 35
Properties of Artinian Rings

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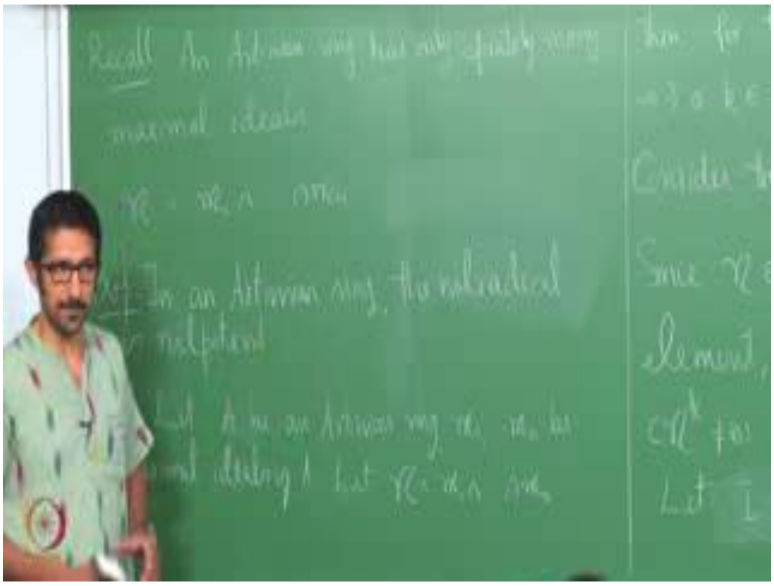

Last time we proved that an Artinian ring has only finitely many maximal ideals right. So, we showed that any prime ideal of an Artinian ring is maximal, and it has only finitely many maximal ideals. So, which means if you look at the nil radical, nil radical is sum intersection of m 1 up to m n.

Student: (Refer Time: 01:38).

Sorry this is Jacobson's radical as well as nil radical because in Artinian ring there are all prime ideals are maximal. So, nil radical is equal to the Jacobson's radical in a Artinian ring. So, now, the see if you look at the Artinian ring see if your nil radical is finitely generated if your nil radical is finitely generated. That means, I can write this as x 1 up to x n some x r, each 1 will have some each x i will is nilpotent. Therefore, x i power n i is 0 for some n i if I take n to be summation over all those n i s n power that n will be 0.

The max see x i power n i is 0 because they are all nilpotent, and if I take this to be summation n i, i from 1 to r then this is 0. Every element here is generated by some x 1 power alpha 1 x to power alpha 2 two x r power alpha r where summation alpha I is capital N, but summation alpha I is summation n i means at least one of the n i is will be bigger than equal to alpha I sorry at least 1 alpha I will be bigger than to n i; that means, that will be 0.

So therefore, this n power N is 0. So, nil radical is nilpotent if it is finitely generated, but is that true I mean can be without even assuming can we say that it is ridiculous nilpotent. So, let us try to see whether you know let n be A be an Artinian ring m 1 up to m n be maximal ideals of. We know that there are only finitely many maximal ideals. So, I want I know. So, let n be the nil radical of a, which is m 1 intersection m n because all prime ideals are maximal. So, a nil radical of a is precisely this, now there is a given any ideal we have we can you know form a descending chain of ideals, then for the chain n contained in n square contained in n cube and so on.

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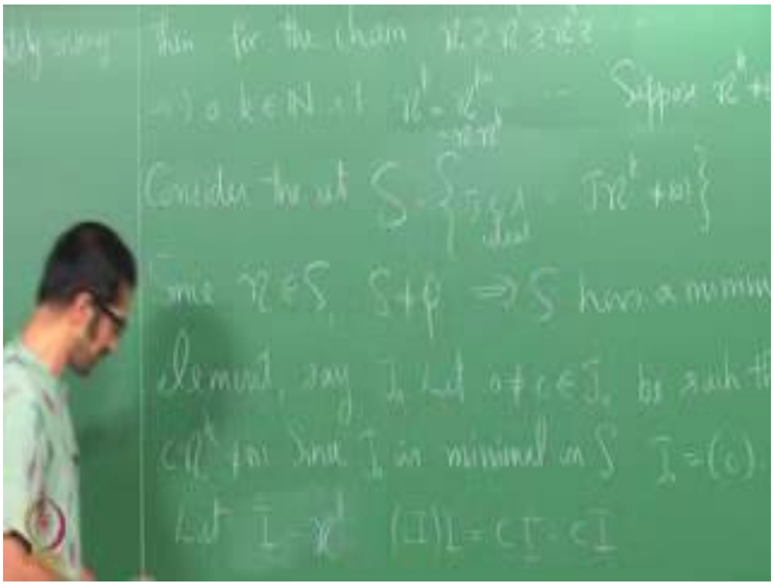

See this is we are in inside an Artinian ring. So therefore, if we look at this chain I can always find a k as such that n k is equal to n k plus 1 and so on right. This implies there exist there exists a k in N such that n power k to be equal to n power k plus 1 so on. See here the question is whether this is 0 or not.

Again you know this is nil ridiculous Jacobson's radical, if this were you know if your ideal n where finitely generated again from here one can conclude that n power k is 0; because I can write this as n times n power k. So, n power k is n times n power k, n is an ideal contained in the Jacobson's radical. So, by nakayama lemma this n k has to be 0, but again at the moment we are not in a position to apply nakayama lemma because we do not know whether the nil radical is indeed finitely generated or not.

So therefore, let us continue let us suppose this is nonzero. So, there is. So, what we are saying is that n k is equal to n k plus 1, now n k is nonzero means n k plus 1 is nonzero and so on so forth. So, consider the set call this S set of all ideals of a such that J n power k is nonzero. I will look at all ideals of A with the property that J n k is non 0, this is a non empty collection I will write it as proper ideal then you know this is a non empty collection why is this a non empty collection?

Student: (Refer Time: 08:21).

N belongs to S right this n, n k this is n times n k and this is nonzero therefore, n belongs to this one right n is in S, S is in non empty. Now we are in an Artinian ring every collection of ideals has a.

Student: Maximal.

Minimal element in the Noetherian ring case, every collection of ideals has a maximal element, in the Artinian case every collection has minimal element. So, S has a minimal element, say call that J naught. So, J naught is a minimal element such that J naught n power k is nonzero then therefore, there x is at least some nonzero element in J naught. So, let 0 not equal to some C in J naught be such that c n power k is nonzero, but then; that means, the ideal generated by c n k is nonzero or in other words J naught is minimal therefore, J naught has to be equal to the ideal generated by c, since J naught is minimal in S, J naught equal to the ideal generated by c.

Let I be equal to n power k then what we now is that c, c I times I this is C I square, but C I square is same as c I itself this is see I look at the ideal c n power k. If I multiply by n power k what I get is c n power 2 k, but c n power 2 k is same I mean I power n power in I power 2 k or sorry n power 2 k is same as n power k therefore, this c I itself.

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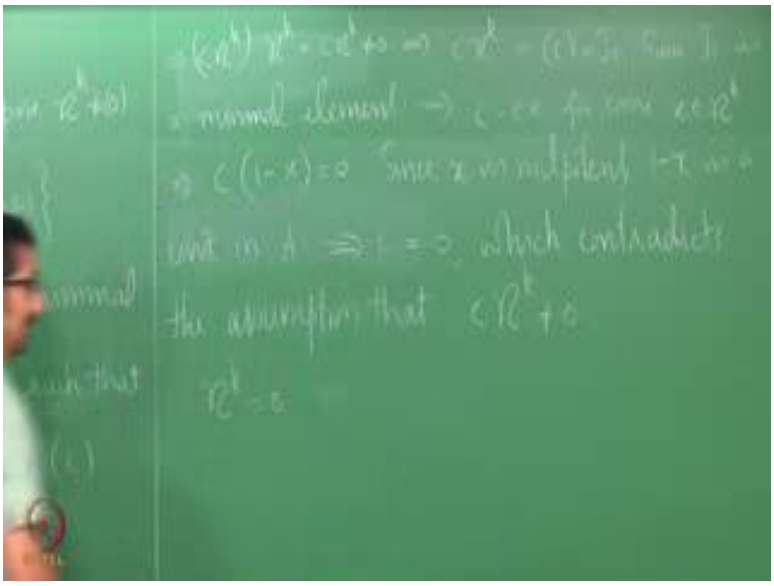

So that means, c I. So, what we have shown is that c n power k the ideal c n power k acting on and multiplied with c n power k is same as c n power k which is nonzero. So, we have reduces ideal further, see we have got a minimal element J in J I have a nonzero element c I mean minimal element J naught, I have a nonzero element c in J naught then what we showed is that J naught has to be equal to the ideal generated by c.

Now, for that I reduce it further here, I look at the ideal generate by c n power k what we are showing is that c n power k multiplied with n power k is again nonzero. C n power k is contained in c therefore, by again by the minimality c n power k has to be equal to J naught or which is equal to c there is contained in. So, this implies that this is equal to ideal generated by c which is equal to J naught, since J naught is a minimal element so. So, therefore, c is equal to c times x for some x in n power k. C times 1 minus x is 0, but now what is x? X is in n power k.

Student: (Refer Time: 13:32).

It is in Jacobson's radical, now we it is a nilpotent element right x is nilpotent n power k is again contained in nil radical therefore, any element here is nilpotent x is if x is nilpotent what can you say about 1 minus x.

Student: (Refer Time: 13:54).

It is a unit right since x is nilpotent, 1 minus x is a unit in A and that would imply that c times a unit to 0 will imply that.

Student: (Refer Time: 14:16).

C is 0, but c is 0 means that our J naught is 0 and that would imply that that is a contradiction to this assumption, which contradicts the assumption that c n power k is nonzero. So, therefore, n power k is 0 that is. So, ultimately what we have shown is that in an Artinian ring the nil radical is nilpotent is the proof clear. So, in an Artinian ring the nil ridiculous nilpotent. Now if you recall we had proved sometime back a result that if you take a ring in which the 0 ideal can be written as product of maximal ideals some finite product of maximal ideals.

Student: (Refer Time: 16:06).

Then the Artinian is of a and the Noetherian is of a r equivalent. Now look at the situation where we are in A is in an Artinian ring the nil radical is nilpotent. So, if we start with an Artinian ring every prime ideal is maximal. So, nil radical and there are only finitely many.

Student: Maximal ideals.

Maximal ideals, so nil radical is a finite intersection of.

Student: Maximal ideals.

Maximal ideals now look at this says that I mean, if you look at the intersection the product is contained in the intersection.

Student: (Refer Time: 16:58).

Right the product of ideals I times J is contained in I intersection J. Now m 1 up to m n the product will be contained in m 1 intersection up to m n. Now look at the product the n power k, n power k is 0 which means m 1 power k the product m 1 power k up to m n power k that will be contained in n power k but that is 0, therefore this is 0.

So now 0 is product of.

Student: Maximal ideals.

Maximal ideals what is that mean.

Student: Ideal (Refer Time: 17:38).

So we are starting with an Artinian ring, which means the ring is.

Student: Noetherian.

Noetherian; so what we have shown what we have seen now is that, a Noetherian ring an Artinian ring is.

Student: (Refer Time: 17:54).

Indeed a Noetherian ring; we have seen this is not true in the case of modules right, we have seen Artinian module which is not Noetherian. We looked at this q mod z and subgroups of some subgroups of q mod z which has every descending chain terminates, but you always have a ascending chain which never terminates. So, a Noetherian an Artinian a module need not be Noetherian a module, but now in this case we are seeing that an Artinian ring is always a Noetherian ring. So, let us write this term. So, now, you I mean see once we say that an art an Artinian ring is Noetherian, this makes perfect sense. Because nil radical will be finitely generated, in Noetherian ring nil radical is nilpotent; once it is finitely generated this is when one can prove that way or if Artinian ring is Noetherian ring then nil radical is nilpotent again.

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So an Artinian ring is Noetherian. So, let A be an Artinian ring, let m 1 up to m n be the only prime equal to maximal ideals of A then n power. So, let n be equal to m 1 intersection m n, then the product m 1 up to m n is contained in n, this implies that 0 is equal to m 1 power k. So, I will write this as m 1 power k up to m n power k, this is contained in n power k which is 0, this implies that m 1 power k to m n power k this is equal to 0. Therefore, the 0 ideal is a finite product of maximal ideals of a, and a is Artinian therefore, that implies a is Noetherian. What about the converse is a Noetherian ring Artinian is a Noetherian ring if a is Noetherian can we say is Artinian.

Student: K x.

K x or z.

Student: (Refer Time: 21:40).

Any PID, f I look at an ideal having a nonzero element in any PID, then look at a I mean a to be nonzero element in PID then we have this descending chain a contained in a square contained in a cube contained in a power four and so on that never terminates, it is a domain. So, it will never terminate I mean it will be strictly decreasing chain which will never terminate. So therefore, an Artinian ring sorry, Noetherian ring need not necessarily be an Artinian ring, but let us look at the you know let us look at this proof what have we used to ultimately to conclude that a is Noetherian.

To start with we are starting with an Artinian ring, to conclude a is Noetherian we showed that 0 is a product of.

Student: (Refer Time: 22:48).

Maximal ideals; now suppose we start with a be a Noetherian ring, in the Noetherian ring nil radical is nilpotent right. Now what condition can you think will ensure that 0 is a product of maximal ideals?

Student: (Refer Time: 23:20).

I mean that is a see here, what are the conditions that we used in Artinian ring. These are the only prime ideals. If this were true in the case of the ring that we are starting with then the whole thing goes through, we can just replace this by Artinian here you replace this by Noetherian replace this by Artinian or in other words if A is a Noetherian ring such that every prime ideal is maximal.

Student: (Refer Time: 24:00).

Then it is Artinian that is what the proof says right. If your ring in your ring every prime ideal is maximal then the nil ridiculous same as intersection am Jacobson's ridiculous which is same as intersection of Jacobson's radical is intersection of maximal ideals. Now if your ring is Noetherian nil radical will be nilpotent; that means, the intersection of maximal ideal whole power k will be 0 which means this will again hold true. Once you reach this stage then Noetherian (Refer Time: 24:38). Artinians are equivalent.

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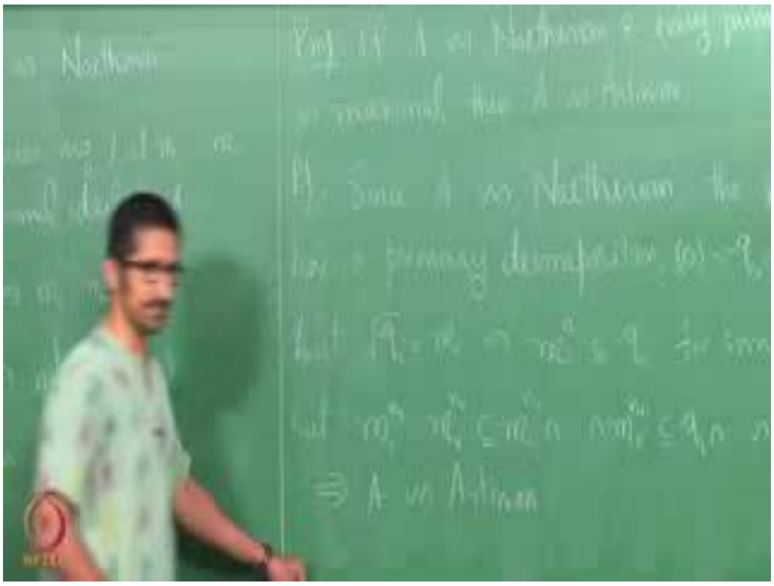

So therefore, the converse of this is if A is Noetherian and every prime ideal of is maximal then A is Artinian. See A is Noetherian, the 0 ideal has a primary decomposition right the 0 ideal has a primary decomposition. So, let us write 0 equal to q 1 intersection q r; now each radical of q i. So, let radical of q i be equal to m i. Radical of q i is equal to.

Student: M i.

I mean this is a prime ideal, but we our assumption is that every prime ideal is.

Student: Maximal.

Maximal; so radical of q i is m i, now we proved another property of ideals in an Noetherian ring. Every ideal in the Noetherian ring contains a power of its radical right; that means, m i power n i belongs to q i, is contained in q i for some n i. Now if I take n to be maximum of n 1 up to n r, what can you say about m 1 power I do not really I do not even require that, I can simply say m 1 power n 1 product m r power n r this is contained in m 1 power n 1 intersection up to m r power n r. Now each one is contained in q i this is contained in q 1 intersection q r, but this is 0.

Now, we are through m 1 up to m r is a I mean product of maximal ideals is contained in 0, which means that they are equal; that means, 0 is a product of finitely many maximal ideals, and that implies that a is Artinian because a is Noetherian. So, this every prime ideal of A is maximal implies that it is it has only finitely many. So, to say that see if you if a ring has only one prime ideal need not necessarily imply that it is Noetherian or you know.

Student: Artinian.

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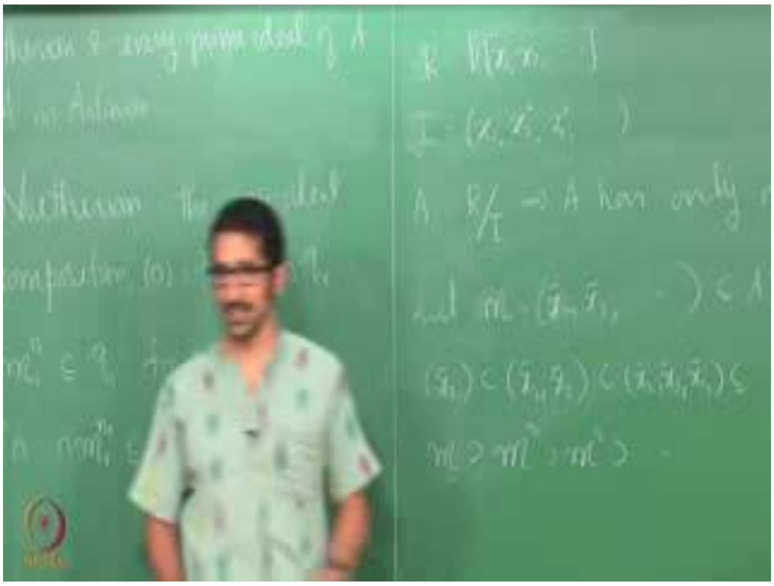

Artinian say for example, if I look at look at the ring R equal to k x 1 x 2 I mean infinitely many variables, and look at the ideal I equal to x 1 x 2 square x 3 cube and so on. and take A to be equal to R mod I, then a has only 1 prime ideal which is the ideal generated by see any prime ideal here will be a prime ideal containing this ideal. If a prime ideal contains I it has to contain all the variables, but that is a maximal ideal. So, the only prime ideal containing I is the maximal ideal generated by all the variables. So, A has only 1 prime ideal right. Now this ring is neither Noetherian nor Artinian right.

If you look at the ideal generate by. So, x 1 bar 0, x 2 bar, x 3 bar and so on the ideal generated by all the variables in A, then you know this is an infinite chain x 2 bar contained in x 2 bar x 3 bar contained in x 2 bar x 3 bar, x 4 bar and so on, this is a chain here. Now can you think of a descending chain which never becomes 0?

Student: (Refer Time: 31:58).

M, M square, M cube; when you do this each time one variable drops, but you know it will never terminate, it will never become 0. If there were only finitely many variables and if you look at something like this m power n becomes 0, but here in this case none of them will become 0. So, this ring is neither Noetherian nor Artinian.

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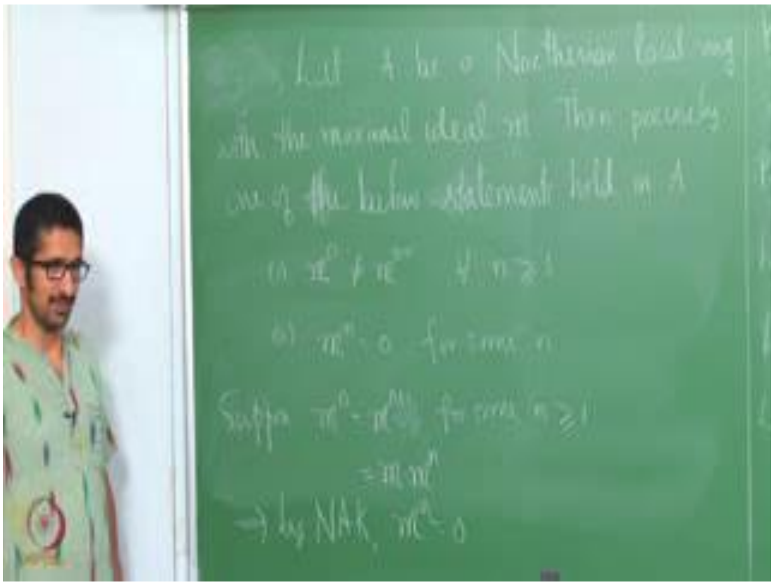

Now, in a Noetherian ring, if your ring is Noetherian something like that cannot happen in the sense that suppose A is; so let me let A be a Noetherian ring, I want a local condition here A be a Noetherian local ring. In fact, that is see this is indeed a local ring right there is only 1 maximal ideal there is only 1 prime ideal which is the maximal ideal generated by the variables. Now let A be a Noetherian in local ring with the maximal ideal m.

So, see here the situation is that we have an increasing chain which never terminates, and a decreasing chain which never terminates. In the Noetherian ring there are only 2 possibilities: one is that m power n is not equal to m power n plus 1 for all n bigger than or equal to 1, and another option is m power n is 0 for some n. So, let us assume and this is fairly straightforward conclusion, suppose this is see it can be that you know there is a I mean see in this case what we say is that the ring is not atinian right. So, a Noetherian ring can be simply a Noetherian ring which is not Artinian or an Artinian ring, but in the Artinian ring case this is always true. So, that is exactly what we are saying.

So in the Noetherian I mean in this case if suppose m power n is equal to m power n plus 1 for some n. suppose this is you know this does not hold. So, let me then precisely one of the below conditions hold below statements hold in A. Either this or this both cannot happen together; if this holds true well and good, if this is not true that is there exist some m power n which is c equal to m power n plus 1, we are not saying that it is Artinian, we are the simply saying that for this maximal ideal m power n is equal to m power n plus 1 for some n, but then we are in the Noetherian ring case m power n is a finitely generated module, I can write this as m times m power n up in a nakayama lemma by nakayama lemma m power n is 0.

So once this is true that says that your ring is Artinian. So, in a Noetherian ring either it is not Noetherian or not Artinian. In that case we will have this true or in other case it will be Artinian and in that case this is true that is precisely what we have proved. Now Artinian rings are more special than Noetherian rings, now we have already proved that in the sense that Artinian rings are Noetherian rings with some additional assumptions.

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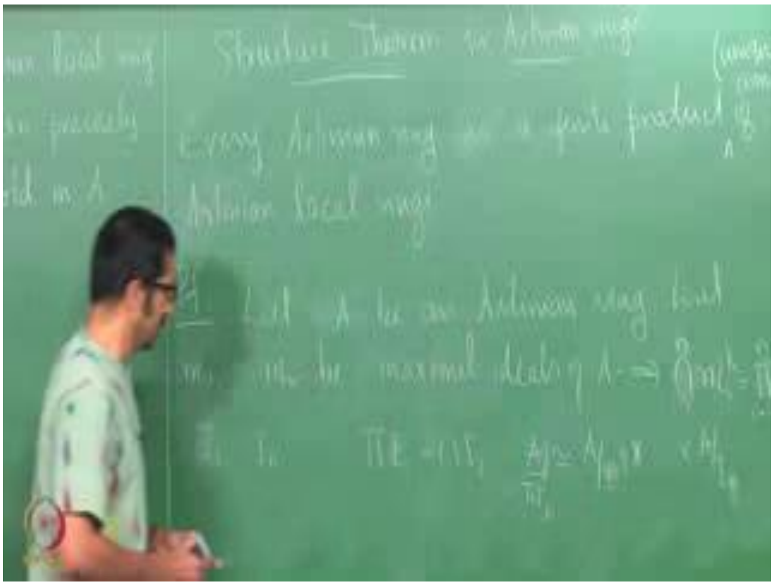

But it is you know there are its more generally there are more stronger results which describe Artinian rings, which one can in fact, give a very nice structure theorem, every Artinian ring is isomorphic to a finite product of we should say unique. In fact, I should say every Artinian ring is uniquely is a finite product of Artinian local rings, finite product of Artinian local rings finite product up to unique up to isomorphism.

What does that mean? what we are saying is that if I take an Artinian ring A then I can write this A as A 1 cross up to A n where each A i s is an Artinian local ring and each A i is uniquely determined up to isomorphism; that means, if I have some A 1 prime cross A m prime this m is equal to n.

Student: (Refer Time: 40:36).

A i is isomorphic to some you know ideals. So, how do you prove this? So, first let us let m 1 let A be an Artinian ring. So, we know that there are I mean all the prime ideals in A are maximal and there are only finitely many maximal ideals. So, let us write m 1 up to m n be maximal ideals of A, now this implies intersection of the product m i power k this is all in our intersection 1 to n, this is equal to the product m i power k i from 1 to n this is 0; see this is nil radical. So, therefore, this nil ridiculous nilpotent we just use the fact that nil ridiculous nilpotent to say that this will be 0.

Now let us recall a theorem that we proved some time back, if I have you know look at this I 1 up to I n, they are all co prime; this is co prime if and only if the product is I I is intersection I j and in that case what we know is that A mod A is isomorphic to a mod I 1 power k cross up to sorry I 1 cross up to a mod I k and I n right when they are co prime the intersection hold. So if you send an (Refer Time: 43:18) if you send a map from A to this product, this will a going to x I mean A plus I 1 comma up to a plus I n the kernel will be.

Student: Intersection.

The intersection; so this will be isomorphic to the intersection which is also same as the product (Refer Time: 43:44) this we have already proved this is a isomorphic.

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Now, let us bring it in our situation what do we get, I look at A mod m 1 power k cross up to A mod m n power k, this is isomorphic to A mod intersection of m 1 power k or the product which is 0. So, A is what we have shown is that A is a product of. Now what can you say about this ring each if A i is equal to A mod m i power k, then this is Artinian ring because A is Artinian, and this is a local ring, see this m i power k is a primary ideal which is primary to the maximal ideal. The only prime ideal containing this would be m i itself there are no other prime ideal contain. So therefore, this is a local ring then A i is an Artinian local ring. So, what we have shown is that an Artinian local ring is a sorry an Artinian ring is product of Artinian local rings.

Now, we show that this product is unique up to isomorphism, in the sense that you start with I mean this is one isomorphism that we have exhibited. Now suppose a is isomorphic to product of some Artinian local rings, then there those components cannot be different from these guys. That is what we would like to show that will imply that it is the I mean the structure theorem that would imply that it is unique up to isomorphism, that we will do in the next class.