Commutative Algebra
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Lecture - 34
Second Uniqueness Theorem, Artinian Rings

Let us begin.

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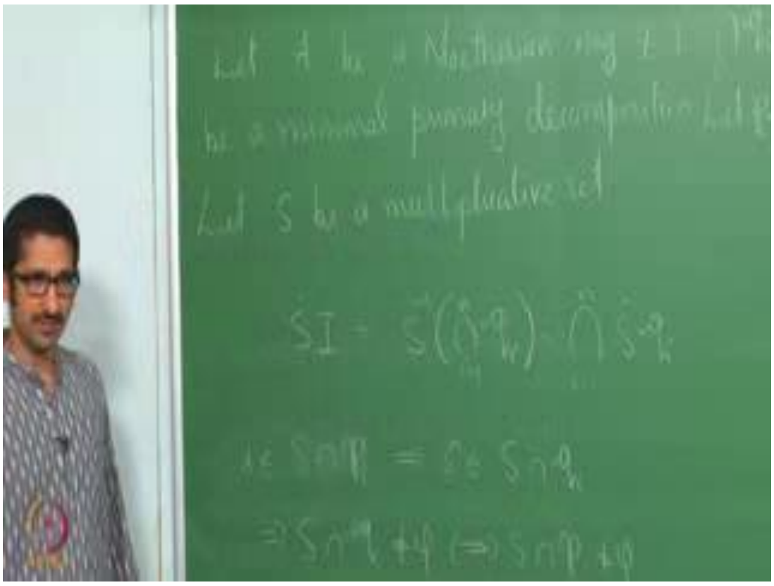

So, we were talking about primary decomposition, so as A be a Noetherian ring. So, the results that I proved yesterday, the uniqueness and some of the results, we only need primary decomposition. Once the primary decomposition is assumed you can go ahead and do that; prove so for example, if you look at the; I think a 5th chapter of Atiyah McDonald, there once the ideal is decomposable and has a primary decomposition then. So, they assume this and then go ahead and prove a lot of results which we discussed here in the Noetherian setting.

So, many of those results you do not really require Noetherian property, but in Noetherian property the primary decomposition is assured. Once the ring is Noetherian every ideal has a primary decomposition, therefore all the results that was discussed assuming a primary decomposes primary decomposition of an ideal is valid.

So, let A be a Noetherian ring and I equal to intersection q i, i from 1 to n be a minimal prime primary decomposition. Let P i be the radical of q i, suppose S is a multiplicative set. So, again when we talk about S inverse I, S inverse I we are I mean S inverse I is equal to S inverse intersection q i this is finitely many. So, therefore, this is equal to intersection i from 1 to n S inverse q i.

Now, if S inverse q, I mean S if S does not intersect with q i, this will be a proper ideal, now the question is whether this will be a primary ideal? Suppose it is primary is it true that the radical if this is S inverse P i. So, let us look at one thing see S intersection q i, suppose some S is in S intersection q i, this implies that some S power n S is in q i; that means, some S power n.

Student: (Refer Time: 04:34).

Sorry, I mean I want to say intersection P i, suppose S is in S intersection P i then this implies that some S power n is in.

Student: Q i.

Q i, but S power n is certainly there in S as well. So, therefore, S power n is in S intersection q i. Now conversely if there is an element in S intersection q i that is naturally there in S intersection P i because that is q i is contained in P i. So, this is; therefore, S intersection q i is non empty is equivalent to saying that S intersection P i is non empty, the advantage here is that we have a prime ideal.

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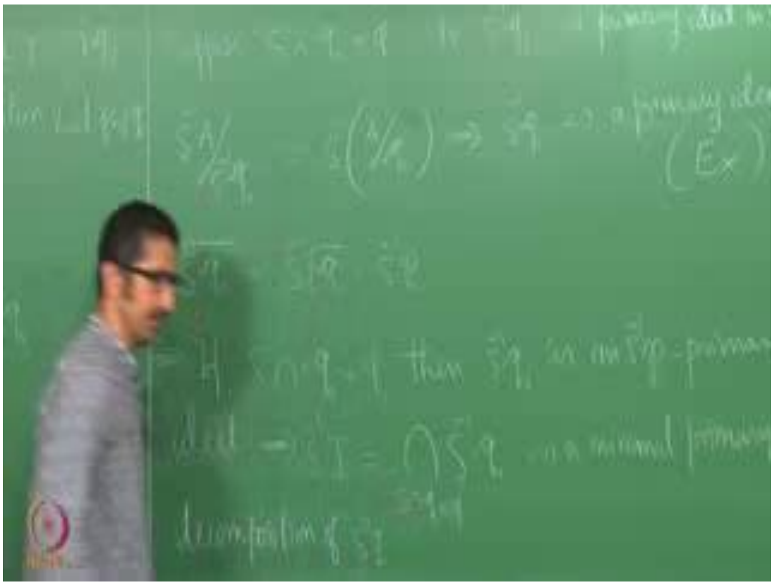

Now, let us come back to this now first of all is this is is S inverse q i. So, assume that suppose S intersection q i is empty so; that means, S inverse q i is a proper ideal now why can we say that is S inverse q i a primary ideal in S inverse A. So, how do you check that?

Student: (Refer Time: 06:34).

We want to say every, so, if you look at S inverse A mod S inverse q i every nonzero divisor should be nilpotent in this ring, but what is this ring? This is isomorphic to S inverse of a mod q i. So, if I look at an element here let us say x bar I mean x by S bar you can write it as some x bar by S where x bar belongs to this if x bar x bar by S is a 0 divisor here, this x bar will be a 0 divisor here and that is nilpotent because q i is primary in A.

Therefore, it is nilpotent the element x bar by x by S whole bar is nilpotent here or else you can start with you know start with an element x by S times y by t belongs to this the normal way one can proceed one can show that it is primary. So, I will leave the exercise to you to check that S inverse q i is a primary ideal either way one can conclude this. Now, what is the radical of S inverse q i?

Student: (Refer Time: 08:32) S inverse.

S inverse of that again we have seen S inverse of radical of q i which is S inverse of P i. So, therefore, if S do not intersect with q i S inverse q i is a S inverse P i primary ideal. So, what we have proved is that if S intersection q i is empty then S inverse q i is an S inverse P i primary ideal.

Therefore, this is indeed a primary decomposition and if we have to write, this will imply that if that is a minimal decomposition I can write this S inverse equal to S inverse q i S intersection q i is non sorry, empty I look at intersection over the q i's which are which are in the complement of S i take this is a minimal primary decomposition minimal we have not discussed by I leave it to you to check minimal primary decomposition of a S inverse I to check minimality what we need to show is that S inverse q i do not contain intersection of the rest of them that follows from the minimality here if I am starting with minimal decomposition here that this will follow I mean it is direct verification.

Now, as a consequence of this observation we have a we have another kind of uniqueness theorem, first uniqueness theorem that we proved yesterday said that if I have a primary decomposition of i, i equal to intersection of q i then the radical of the primary components is uniquely determined.

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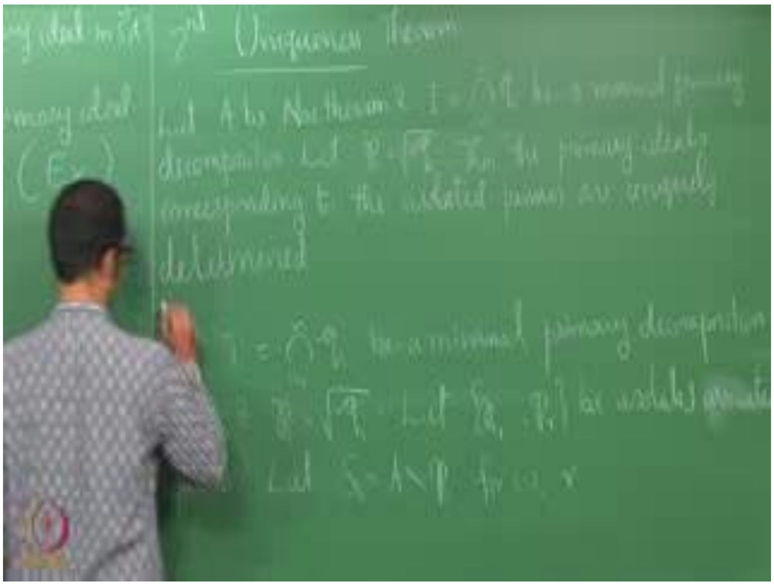

Now, suppose I have a; I will write down the statement let us let I be a primary decomposition minimal primary decomposition of I and P i equal to radical of q i.

So, we have I mean these are all the associated primes of A mod I now among them there are minimal primes that is isolated primes and embedded primes. So, look at the isolated primes let P 1 up to P r be isolated prime ideals of isolated associated primes of; that means, these are the minimal ones.

Now, suppose I take S be equal to A, the complement of S i I will just write this S i, S i is A without P i, i from 1 to r. I look at the multiplicative set S i equal to the complement of these minimal associated primes then what would be S inverse I this I mean S i inverse I this would be?

Student: S i inverse q i.

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So, this is equal to intersection of S i I mean intersection of S i let us write this as q j, but now this is one of the minimal primes here and I am taking the complement of the or this minimal prime. So, this S i will intersect with all other primes because this is let us say S 1 is A without P 1, S 1 will intersect with all other prime ideals; that means, S inverse I is equal to S i inverse q i that will be the only one that will survive since S i intersection P j is non empty for all j not equal to I which is equivalent to saying that S i intersection q j is non empty for all j not equal to i.

Therefore, this will in this intersection only one of them will survive which is the primary components which are; now forget about this part if you look at this part this is you know we are taking some multiplicative set this is independent of the primary decomposition I am taking see we have proved that the primary components are unique the prime I mean prime ideals associated to I are unique. So, I take those prime ideals among them I look at the minimal prime ideals look at I take one of them and look at its complement and apply the localization.

What we get here is S inverse q i this says that- however, your primary decomposition is primary decomposition the primary ideals involved need not be unique, but then the minimal primary com I mean the primary ideals corresponding to the isolated prime ideals they are uniquely determined right S inverse q i is now uniquely determined. So, therefore, the second uniqueness theorem says- let A be Noetherian and I be I equal to intersection q i and be a minimal prime decomposition primary decomposition let P i be equal to q i radical of q i and let then the primary components the primary ideals corresponding to the isolated primes are uniquely determined.

So, S i is q i, therefore the primary components given by phi inverse of S i i where this is components corresponding to. So, let me corresponding to the isolated primes are given by phi I inverse of S i inverse of I where S i is complement of P i and phi I is the natural map from A to S i inverse of A. So, this set has nothing to do we are not talking anything about the primary decomposition of I when we are describing the primary components corresponding to the minimal I mean the associated minimal associated primes or the isolated components.

Student: (Refer Time: 20:31) components so.

Components of S inverse I; sorry, components of primary components of I.

Student: (Refer Time: 20:42).

So what we are here see earlier when we try to prove the uniqueness of the first uniqueness theorem how did we prove it we proved that the primes that appear in the in the as radical of primary components is they are precisely the prime ideals of the form radical of I colon x where x varies over A. So, that has nothing to do with we have expressed that set independent of what primary decomposition you are taking.

So, there that that is how it is you know we proved the uniqueness similarly here we are saying that if I take the primary components corresponding to the ideal S inverse I are precisely of this I mean the minimal primes are precisely of this form. So, this does not really talk about because you know once the primes are uniquely determined we can use that uniqueness to get S i, S i we are not talking about q i's how ever your primary decomposition is S i's is are uniquely determined.

Now, you just look at phi I inverse of S i i where S i is the complement of minimal prime minimal associated prime. So therefore, this implies that the. So, that proves the assertion this implies that the minimal components are uniquely determined. So, this is another uniqueness theorem. So, this one we saw yesterday in the example of x square comma x y x square comma in x square comma x y there was x intersection x square comma y that was one decomposition and the other was x intersection x square comma x y comma y square or y power in for any n.

So, the primary component corresponding to the embedded primes are varying, but the primary component corresponding to the isolated component is x itself.

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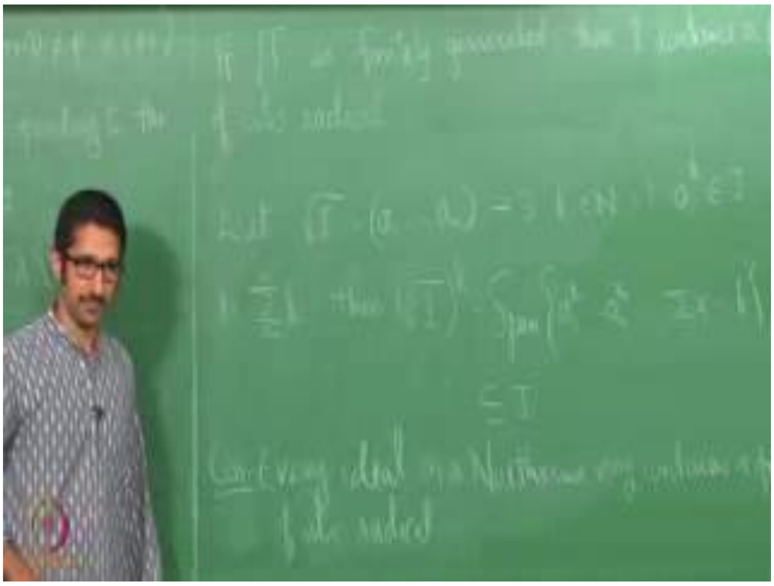

Now let us look at some more interesting properties of A ideals in Noetherian rings. Suppose I is; this is if I is finitely generated then I contains a power of its radical, if I is finitely generated then the ideal contains a power of its radical; that means, there exists see I is contained in radical of I. So, I have I is contained in radical of i, but what this says is that there exists some m such that this is true this is when I is finitely generated how do you prove this if I is finitely generated, I guess I will require a radical of I is finitely.

Student: (Refer Time: 24:54).

If radical of I is finitely generated then I contains the power of its radical. So, let this be equal to a 1 up to a n, each element in the radical there exists a power which is contained in I. So, this implies there exists k I in n such that a I power k I belongs to I now if I take my k to be summation k I i from 1 to n then radical of I whole power k will be contained in I right how what is this see this will be generated by radical of I is generated by a 1 up to a n. So, whole power k will be generated by this is span of a 1 power r 1 up to a n power r n where summation r I is k now k is summation k i. So, therefore, at least one for at least one I r I will be bigger than to k i. So therefore, for each therefore, this each element will be in I therefore, the whole of this in I.

So, corollary of this is that every ideal in a Noetherian ring contains a power of its radical because in a Noetherian ring every ideal is finitely generated.

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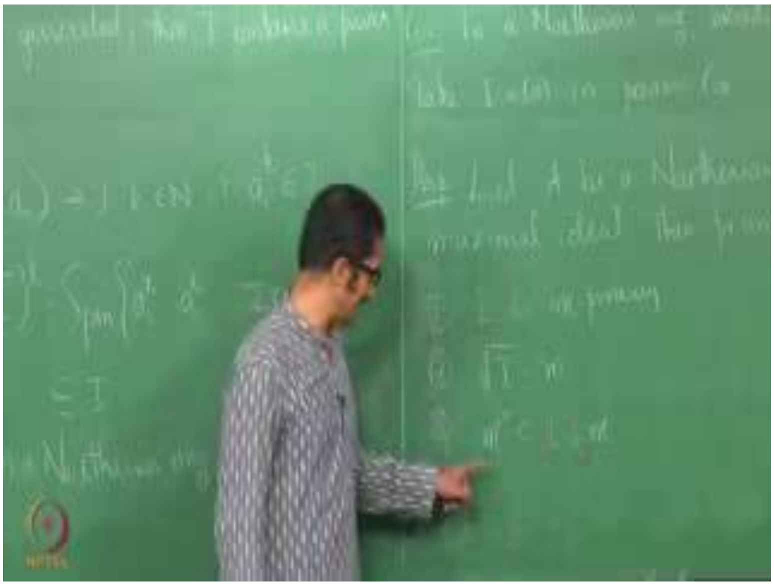

So, you start with any ideal its radical is finitely generated take its radical and take the generators and do this and another corollary to this is in a Noetherian ring nilradical is nilpotent take I to be equal to 0 what is nilradical collection of all.

Student: Nilpotent.

Nilpotent elements; that means, set of all n such that x power n is 0 which means radical of the nilradical is radically of the 0 ideal. So, therefore, if I take in corollary in the previous corollary take I equal to 0 then in a Noetherian ring it contains some power is contained in some power of its radical is contained in the ideal therefore, some radical of I which is the nilradical; nilradical whole power something is 0 another interesting the same time not very difficult let A be a Noetherian ring and m be a maximal ideal then for an ideal I the following are equivalent I is m primary radical of I is m and this is m power n is contained in I contained in m now if I is m primary it means its primary and it is it is radical is m. So, 1 implies 2 is straightforward radical of I is m then of course, I is contained in m and we are in Noetherian ring. Therefore, every ideal contain a power of its radical. So, therefore, m power n is contained in it.

Now, suppose m power n is contained in I contained in m, we want to prove I is m primary this equation says that apply radical on this equation, what do you get? Radical of m power n is contained in radical of I is contained in m. Now what is radical of m power n?

Student: (Refer Time: 31:16) m.

M itself, because for m power n m is the only ideal maximal ideal containing m power n, therefore, m power n radical of m power n which is m, m is contained in radical of I contained in m which means radical of I is m if radical of I is a maximal ideal then it is primary this is what we proved something. So, that proves three implies one I leave it you to complete that.

Now, let us go ahead and learn some properties of Artinian rings.

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So, let us recall a ring is Artinian if every descending chain of ideal terminates that if I 1 I 2 is a chain in a chain of ideals in a then there exists k such that I n is I n plus 1. So, on one when we talk about the Artinian rings one important property that we proved earlier is that this is recall suppose a is a ring such that the 0 ideal is a finite product of a some maximal ideals of a is a ring whose 0 ideal is a finite product of maximal ideals of a then a is Noetherian if and only if it is Artinian this is something that we proved some time back. So, I if this is m 1 up to if I can write 0 as m 1 up to m n then I can I have a chain product m 1 up to m n contained in m, m 1 up to m n minus 1 contained in m 1, m 1 up to m n minus 2 each quotient is a vector space over a mod corresponding m I.

So, now by induction we can say that each module I mean each ideal product is as a module over a it is Noetherian if and only if Artinian and by induction we prove ultimately reach m 1 and then a that is the that is how we proved this result now Artinian rings are and the all the examples that we looked at they are all you know we will close to being Noetherian or even stronger than being Noetherian. So, we will see that that is it is true in general but before that.

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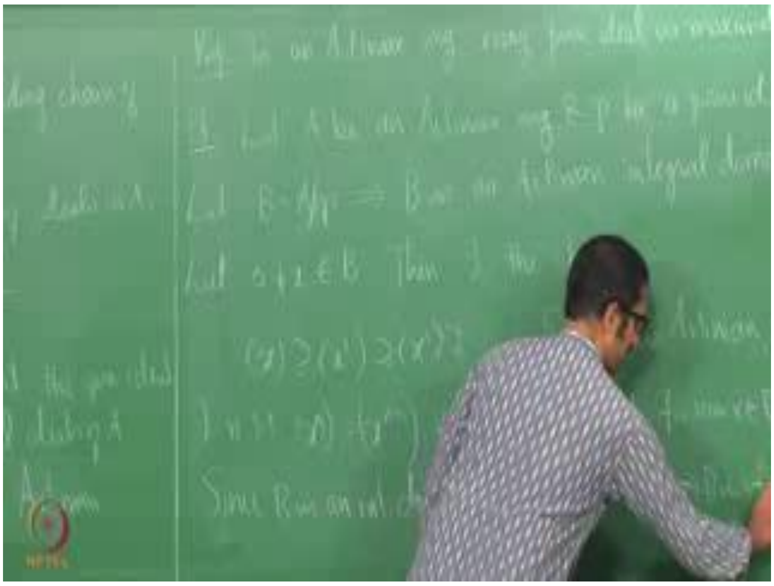

Let us look at some nice properties of Artinian rings in an Artinian ring every prime ideal is maximal suppose I take a prime ideal let a be an Artinian ring and P be a prime ideal of; we want to say that P is maximal or in other words we want to say that a mod P is a field. So, let B be equal to a mod P we want to say that B is.

Student: Field.

Field we know is what we know is that B is.

Student: Integral domain.

Integral domain now we also know that B is Artinian because A is Artinian and P is an ideal in a therefore, B is Artinian. So, B is an Artinian integral domain we want to say that B is a field. So, let us start with a nonzero element B then we have you know then there exists this chain the chain x contained in x square contained in x cube and. So, on; that means, since B is Artinian there exist some n such that x power n is equal to x power n plus I for all I bigger than to 1. So, I can write x power n is contained in the ideal plus 1 or in other words. So, I can write this is equal to r times x power n plus 1 for some r in B now B is an integral domain.

Therefore, I can cancel x power n; that means, since B is an integral domain one is equal to r x that implies x is a unit and that implies B is the field and that is exactly what we wanted to prove. So, what we have proved here is that an Artinian integral domain is a field and this is this can be thought of as a corollary to that result an Artinian the proof is if you start from here what we have proved is an Artinian integral domain is a field and this is. In fact, a corollary of that result so, in an Artinian ring there are no non maximal prime ideals every prime ideal is maximal.

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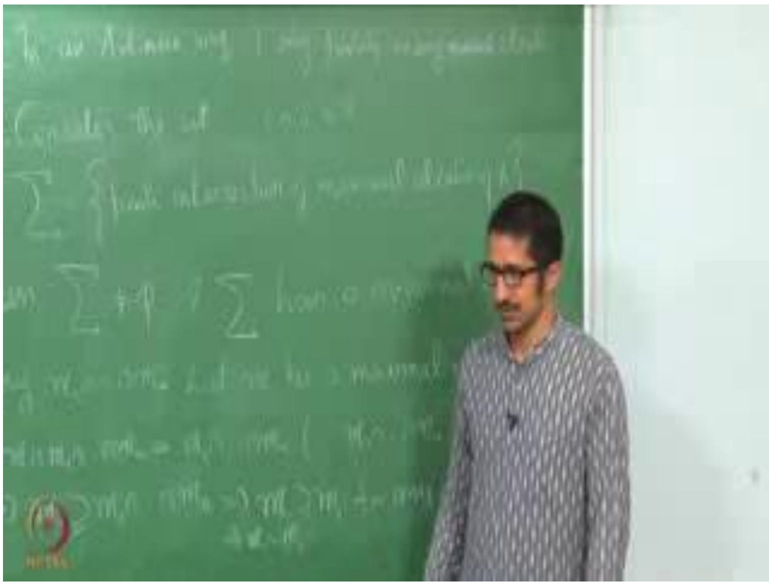

Now, equally interesting property of Artinian ring is that there cannot be too many of them that is there exists in an Artinian ring there exists only finitely many prime ideals or maximal ideals now I will write as maximal because there cannot be too many of them. For this look at this set consider the set sigma equal to m 1 up to m n set of all I will write in words that is easier to finite intersection of maximal ideals I take some finite maximal ideals intersect and put them in this, this is certainly a non empty set I would assume that it is not a field I am starting with an arbitrary Artinian ring.

So therefore, every collection of ideals has a minimal element in Artinian ring in Noetherian ring any collection of ideals has a maximal element here in Artinian ring any collection has a minimal element then sigma is non empty and sigma has a minimal element. So, I will denote this by m 1 intersection m n. Now, I take any other maximal ideal let m be a maximal ideal then this m intersection m 1 intersection m n this is an ideal that is contained in m 1 intersection m n this is which means this is smaller than the minimal it cannot be because this is indeed the minimal.

So therefore, these two are equal this implies this is equal to m 1 intersection up to m n this is an element in sigma and this is contained here, but then this is contained in if there is a comparison between two ideals then this is contained in. So, therefore, these 2 are equal that implies that m contains m 1 intersection m n I intersection j is equal to I if and only if j is contained in j contains I. So therefore, this m is m contains m 1 intersection m n, but when a prime ideal contains a finite intersection.

Student: (Refer Time: 45:14).

One of them is contained here this means that m contains m I for some i, but that means, m is equal to m i. So, if m is if m is any maximal ideal of a it has to be one of them; that means, the only maximal ideals here are.

Student: M 1 up to m n.

M 1 up to m n. So, that proves that it has only finitely many maximal ideals. Now, more properties of Artinian rings I will take up on in the next class.