Commutative Algebra

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Lecture - 33
Uniqueness of Primary Decomposition

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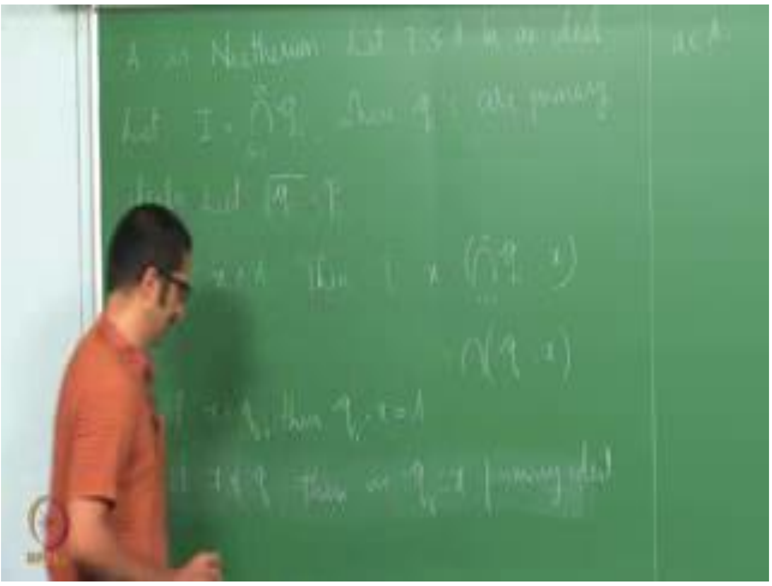

Let us try and see if we can. So, now, rings is Noetherian and suppose I have an ideal I be an ideal proper ideal. Then we know that it has a primary decomposition. So, let I be equal to intersection q i i from 1 to n, where q i's are primary ideals. Let the radical of q i be equal to p i prime ideals. So, what we have already seen is that q 1 q 2 up to q n need not be uniquely determined, because yesterday we saw x square comma x y has decomposition x intersection x square comma y and another decomposition is x intersection x square comma x y comma y square. So, we the components need not be uniquely determined.

So, suppose I have an element x in A then what can you say about I colon x. This is intersection q i colon x i from 1 to n and that is equal to the property of colons and intersection. This is equal to intersection q i colon x. So, I have this ideal q i colon x. Again if x is in q i then this is q i colon x is equal to A. Suppose x is not in q i. The question is here I have an ideal I and I have an intersection here, whether these are all irreducible ideals; if they are all irreducible then this will be a primary decomposition of this ideal.

Therefore, the question is whether these are all irreducible. Now we know that in Noetherian these are all; I mean irreducibles are primary. Let us see if we can say that this is primary or not. See if they are primary then this will this will again give me a primary decomposition, whether this is minimal decomposition or not that will come up later, but first whether this is primary or not. So, the question is, whether then is q i intersection x primary ideal.

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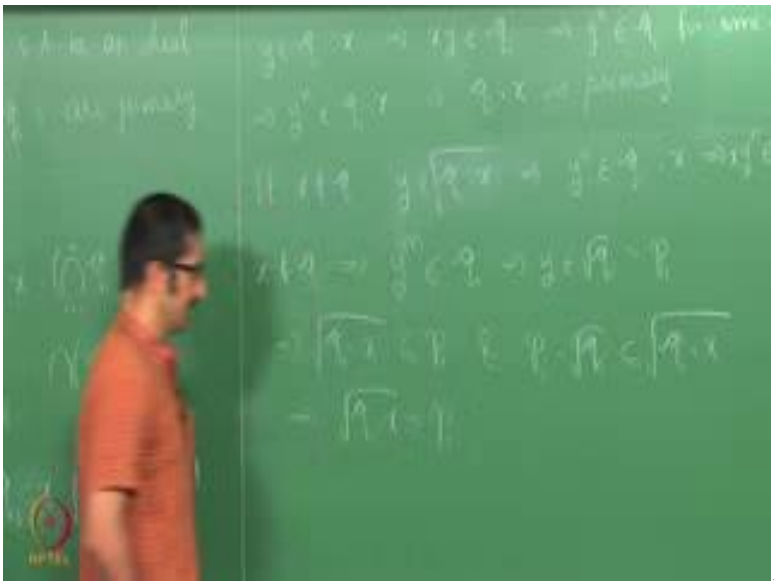

Let us see if y belongs to q i intersection x; q i colon x that implies that x y is in q i. And we are assuming that x is not in q i and that implies that y power n is in q i for some i. So, that would imply that y power n is in q i in I mean colon x, because it is q i is contained in q i colon x.

Therefore, this says that q i colon x is primary. What does it is radical? If x is not in q i, so let us look at the radical of q i colon x. What would this be? This would be; so let us look at an ideal an element y in q i colon x radical of this that implies y power n belongs to q i colon x, and that would imply that x y power n belongs to q i. And x is not in q i, and that would imply that some another power of y will be in q i. But this implies that y is in the radical of q i which is p i.

So therefore, radical of q i colon x is contained in p i. What about the converse inclusion? P i equal to a radical of q i this is contained in radical of q i colon x, right. Ideal is contained in the colon. So therefore, p i is contained in the radical of q i colon x and here we have proved that q i colon x is contained in p i, so that implies that. So, if x is not in q i these are all p i primary ideals again. So, what is it we have proved here now, I take an arbitrary element x and look at the colon I colon x then I colon x is this intersection of q i colon x and for each x not in q i this is p i primary. Or in other words this is primary decomposition of this ideal.

Now suppose here primary decomposition is minimal, did I define what is the minimal primary decomposition? No. So, minimal is what could possibly be minimal in the natural sense that there are no redundant components here. In the sense that, no q i is contained in the intersection of the rest of the q j's, then it is called a minimal decomposition or it is even called irredundant decompositions.

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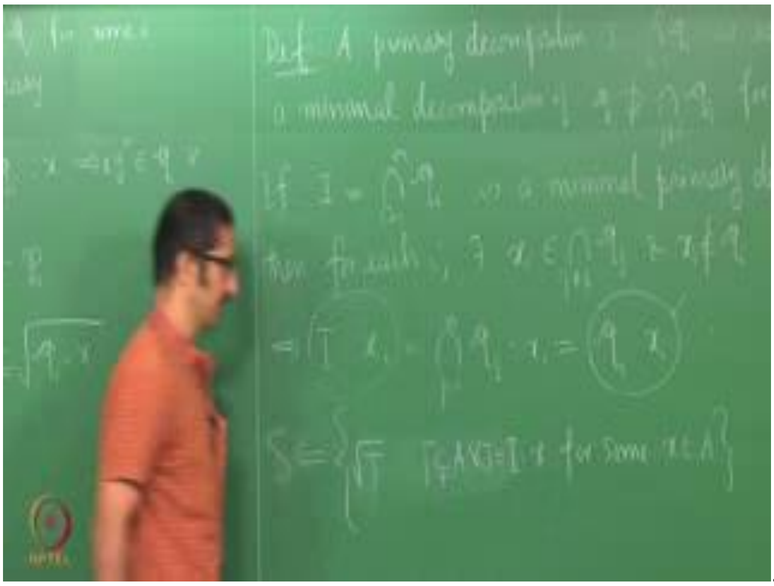

So, let me define what is a primary decomposition; I equal to intersection q i i from 1 to n is said to be a minimal decomposition if q i does not contain intersection of q j j not equal i for each i from 1 to n. None of them is redundant I mean if q i if for say for example, if q 1 contains the intersection of q 2 up to q n then that q 1 does not play any role because q 1 intersection the rest will be the other part q 2 up to q n. So therefore, q 1 does not play any role we do not really need it. Suppose I write this as a minimal decomposition, then what does this say- see if it is a minimal decomposition for each I this does not contain this rest of the intersection. What does that say? There exists an element here which is not here.

So if I equal to intersection q i i from 1 to n is minimal primary decomposition then for each I there exists x i in the intersection of q i i not equal to q j, j not I and x i is not in q i, because for each I this does not contain the intersection of this. Then what happens to this I colon x i. Let me just write use a different symbol q j colon x i j from 1 to n, but now if j is not equal to i what would this be?

Student: Whole.

Whole of A, because x i is contained in q j for all j not equal to i, so therefore q j colon x i would be.

Student: (Refer Time: 13:32).

Only q i; and only q i q j colon x i would be a for all j not equal to i and the only proper components here would be q i colon x i. This would be equal to q i colon x i. And we know that this is a primary ideal, and this is a p i primary ideal. So, what we are seeing here is that for each i if I take a minimal decomposition; for each I there exist a p i primary component. If I take the minimal primary component ideal there exists a p i primary component.

See this is independent of your primary decomposition. How are you take primary decomposition? If I take these proper ideals in this collection; proper ideals in this collection these are the ideals which will be maximal. If I take all the proper ideals of this form I colon x where x varies over , I look at all the proper ideals. These are the maximal elements of that set, because if x is in two of them; if x is in I then this would be A. If x is not in i; that means it is missing from some of these components. If it is missing from only one and included in all others this will be the ideal; I colon x i will be precisely this. If it is involved in more than one component this will be intersection, there will be more.

So these are the maximal elements. So, if I consider the set S equal to set of all J proper ideals of A such that J is equal to i colon x for some x in A. I consider this set then these are all maximal elements of the set S. Therefore, the primary components we do not have control on q i as such, but what we have the control we have is radical of j is prime ideal. These are maximal elements in this set, and what would be radical of this? Radical of this is p i.

So, the prime ideals are unique. Now if I look at, slightly if I modify this set instead of S if I write the following set. A radical J where J is a proper ideal and J equal to i colon x for some x. I look at this collection.

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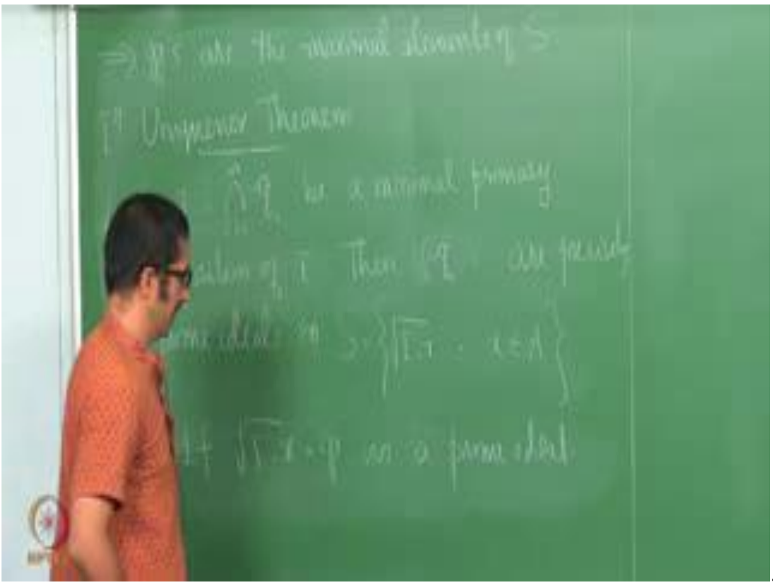

Then p i's are the maximal elements of maximal elements of S. So, the primary components need not be uniquely determined, but the prime ideals associated with the primary decomposition are uniquely determined; because see, if I look at this description has nothing to do with the primary decomposition.

Student: (Refer Time: 18:53) equivalently we can said that if I colon (Refer Time: 18:57).

Depending on see, that will again depend on x right where x is. It has a primary decomposition that comes from being the ring being Noetherian.

Student: Each component should be.

If it is proper; if x belongs to one of the primary components that will become holding, if it is a proper ideal then yes. So, this description does not say anything about the primary come decomposition; this will always be. So, let me write down, this is called first uniqueness theorem. Let I equal to intersection q i i from 1 to n be a minimal primary decomposition of I. Then radical of q i's are precisely the prime ideals in the set S radical of I colon x, where x varies over A.

Here what we proved, let me just bring it in one line the proves. So, what we have proved, I mean what we are saying is that radical of q i's, the prime ideals are uniquely determined because they come from the prime elements here. So, from there what we have shown is that, if x belongs to all except to one primary ideal then this is a prime ideal. So, any p i will be here. Now we are saying that they are precisely the prime ideals here. Suppose radical of I colon x this is a prime ideal. Suppose this, what does that say? That says that see we want to say that p is equal to radical of q i for some i.

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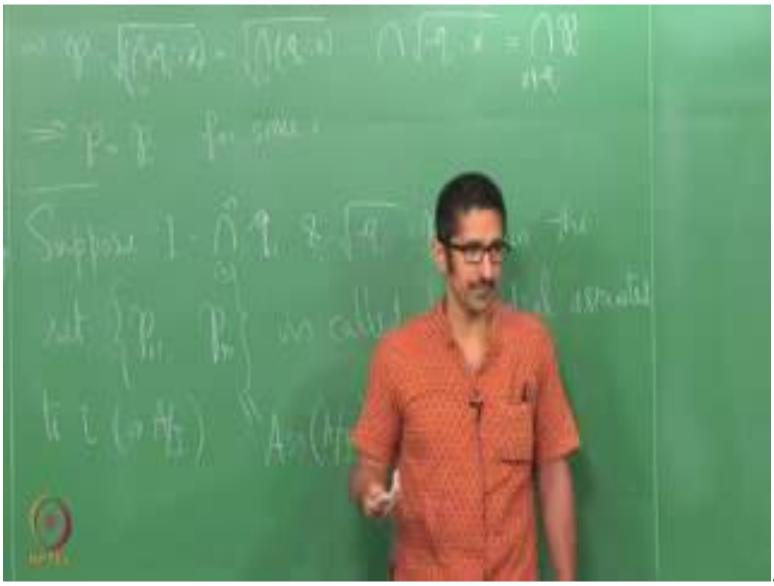

But now, what is this? Radical of; so I is q i right q i colon x. So, let p is equal to radical of intersection q i colon x, but this is equal to radical of intersection of q i colon x. Now radical of I intersection j the radical of I intersection radical of j, these are all finite by the way. So, this is equal to intersection of radical of q i colon x, but we have shown that if x is not in q i this has to be a p i primary ideal, therefore radical this is equal to intersection p i; so x not in q i. Intersection of those prime ideals for which x is not there in q i. P is equal to a finite intersection of prime ideals. Properties of prime ideals say that, p has to be one of them.

So, any prime ideal is there in this set that we showed there, and if p is any prime deal of this form then p has to be one of those. So the prime ideals which are radicals of the primary ideals involved in the primary decomposition are unique; they are uniquely determined. That is the uniqueness we have. Is this clear? See.

Student: (Refer Time: 25:19).

This is, yes.

Student: (Refer Time: 25:25).

Sorry.

Student: (Refer Time: 25:28) radical of q i’s (Refer Time: 25:32).

No, I am saying see we want to say that these are p i’s are precisely the prime ideals in this set. There we proved that any p i belongs to this set, because if you take an element x which is there in all q j’s except q i then I colon that x will give me q i colon x and it is radical will be p i. Therefore, any p i is here.

Now, we want to show that if a prime occurs in this form it has to be one of those. It cannot be any other prime but it has to be one of those p i’s, that is what we are showing here. So, I start with suppose I colon x is p, now I am writing i is this. So, I write p equal to i colon x i is this I substitute I equal to intersection q i. Now I use the properties of colon and radical that is all. And then use the properties of prime ideals; p equal to finite intersection of prime ideals, therefore it has to be equal to one of them.

So this is one uniqueness property that we have. The prime ideals; suppose, I is equal to intersection q i i from 1 to n and radical of q i is equal to p i, then the prime ideals the ideals or the set is called prime ideals associated to I or A mod I. In fact, there is another way to look at is as we said earlier for every ideal, ideal has primary decomposition is you can go modulo I and talk about primary decomposition of the 0 ideal.

So, if you look at think of this as 0 ideal, I mean the primary decomposition this will be primary decomposition of 0 in this one. Basically, one uses when one generalizes this to the case of modules what one uses is the primary decomposition of 0. So, this set is denoted by the associated primes of A mod I. So, this writing A mod I is because we are; there is another way to look at this is that, see these are the prime deals that occur as radical of I colon x. Or in other words in A mod I these are the prime ideals that occur as radical of 0 colon x. Or in other words these are prime ideals which are assassins of, now which kills x, some power skills x.

Therefore, that is in fact another reason for the notation Ass, assassins or associated primes.

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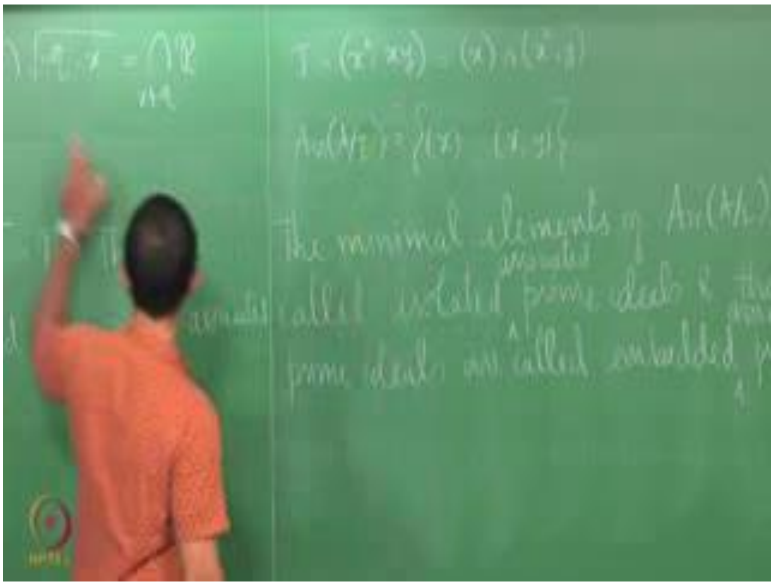

Now, in this set for example if I have I equal to x square comma x y, this is x intersection x square comma y. So, the associated primes of this m mod I this is x and x y. The primary components you do not have such an inclusion, but this is a primary component, this is a primary component, and this none of them contain the other. But then here we have x is p 1 is contained in p 2. In such situation p 1 the minimal elements of this set is called isolated components and the rest is called embedded components.

The minimal associated primes are called isolated prime ideals, and other prime ideals are called embedded prime ideals; embedded associated prime, isolated associated prime ideals. So, these are not simply prime ideal, but associated. In this case this is isolated primes and this is an embedded prime. The terminology isolated and embedded comes from geometry. So, when we take an ideal that corresponds to a variety in the corresponding affine space, and that will the primary decomposition gives rise to a reducible decomposition of the variety in the sense you can write it as union of sub varieties. These isolated components are the irreducible component varieties whose union is the original variety, and the embedded components are the sub varieties of these irreducible.

So that you know I will come back to this after doing what is called null tensor, because to understand the connection between variety defined by q i and p i we need null tensor; I will come back to that. So, this gives me some kind of uniqueness.

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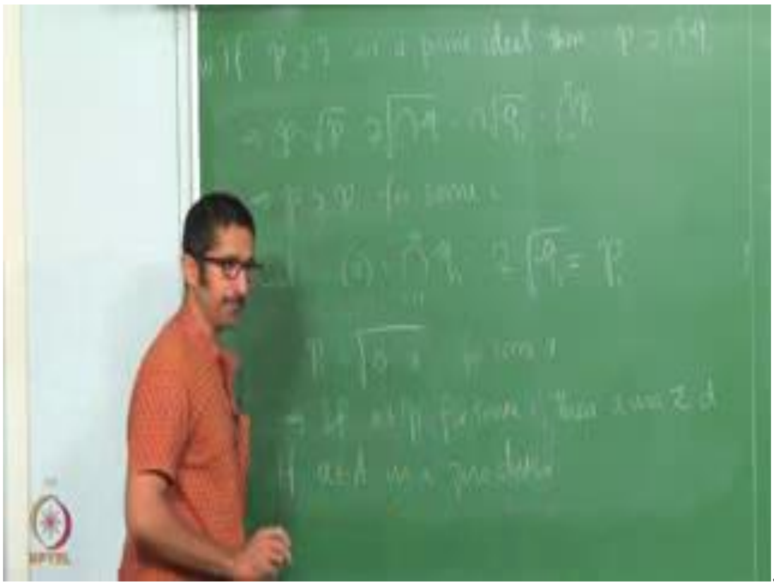

Now, suppose I take any prime ideal if p is a prime ideal, some you know observations related to the primary decomposition. If p is a prime ideal, then what does that say that says that p contains intersection of q i i from 1 to n, and that implies radical of p again equal to radical of p which contains intersection of; so radical of the intersection which is intersection of the radicals q i which is intersection of p i. That means, p contains intersection of all these are finite that implies p contains p i for some i.

What does it say if a prime ideal contains I then one of the associated primes is contained in that primary. If I is contained in p then there exists some associated prime which is contained in p. That is one easy observation. Now if I look at; suppose I take the primary decomposition of 0 ideal in a ring a. I mean we are in Noetherian ring case. So, every ideal has a primary decomposition unique; sorry, every ideal has a minimal primary decomposition which with some uniqueness property as we proved.

So, suppose this is a minimal primary decomposition, then what does it say there exist. So, suppose I and q i equal to, sorry radical of q i equal to p i. So, what are p i's? P i's are precisely the radical of 0 colon x for some x right, that is what we have proved now. So if I take any 0 divisor; if a in A is a 0 divisor; that means first observation is that which means if a is in p i for some i then a is a 0 divisor. Because a is in p i means a power n annihilates x; that means, A annihilates a power n minus 1 x. Therefore, A is a 0 divisor.

So, every element of p i is a 0 divisor. Now suppose a in A is a 0 divisor.

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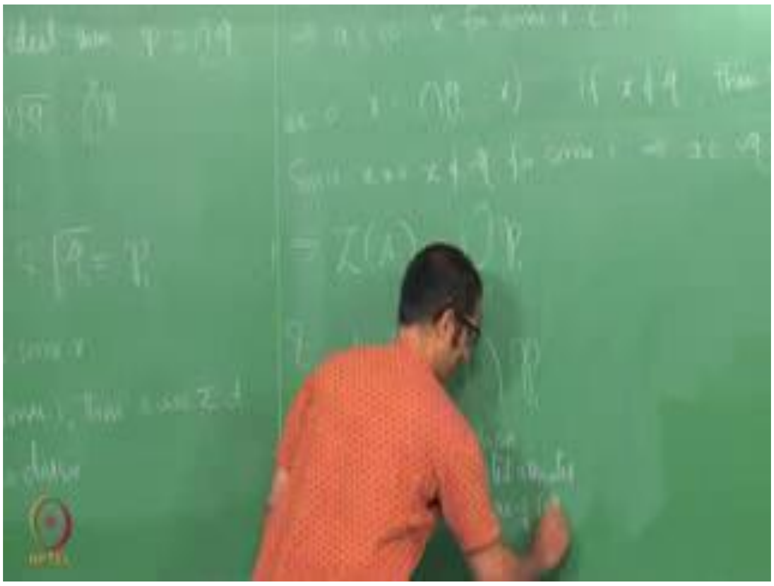

This means that a belongs to this implies that a belongs to 0 colon x for some x in A. But then, see what we indeed showed there was that the prime ideals in the 0 colon x or I colon x they are the maximal elements of this set. Therefore, there exists a prime ideal that is that contains this one in the set S that we described earlier. So, this implies that a belongs to p i for some associated prime. So what we have shown is that, the set of all if I take Z of A to be the set of all 0 divisors of A. If this is Z of A denotes the set of all 0 divisors of A then this is precisely equal to union p i i from 1 to n.

Student: (Refer Time: 40:54).

Maybe I will repeat this. See if I take a in 0 colon x what is 0? 0 is 0 colon x is intersection of q i colon x. So a is here, now this is equal to; I mean I will just write like this q i colon x. So, what we have shown earlier is that if x is not in q i then q i colon x is a.

Student: (Refer Time: 41:57).

This is a p i primary ideal right. So that means, a belongs to q i colon x.

Student: Sir, in x does not belongs to q i and x belongs to other q j's, then it is?

No, see if x is not in q i then this is p i primary. Now this says that a belongs to 0 colon x implies that a is in some of these. See x is nonzero; that means it is not there in at least in one q i. That means, this a belongs to one of those q i colon x where x is not there. Therefore, for a belongs to that prime ideal, we can say that a belongs to at least that prime ideal. See a is here, since x is nonzero x does not belong to q i for some I, because intersection of q i is 0; x is nonzero means if x is not in. So, this implies that a belongs to q i colon x which is contained in p i.

So, what we have shown is that Z of A is union p i i from 1 to n. The 0 divisors are precisely the union of all associated primes; 0 divisors of a are precisely the union of all associated primes.

Student: What is x belongs to intersection of (Refer Time: 44:30) q i.

Student: What is intersection all q i?

Student: To 0.

Ok. Now there is a nice observation here. What is nil radical of A? By definition this is set of all nilpotents or by characterizations this is set of all.

Student: Prime.

Prime ideals intersection of all prime ideals. But now we proved something here, if p is a prime ideal containing I then there exists an associated prime ideal which is contained in p. So, if I take any prime ideal here it will contain an associated prime of 0; it will contain an associated prime of 0. That means, in this intersection there would be only associated primes. And among them you can further reduce if I have this situation I do not need to look at this one, I only need to look at the isolated primes.

So, I can write i where p i is an isolated associated prime. Suppose I have a primary decomposition of i and S is a multiplicative set, then what would this be? What would be there? Again I am in Noetherian ring and I is an ideal.

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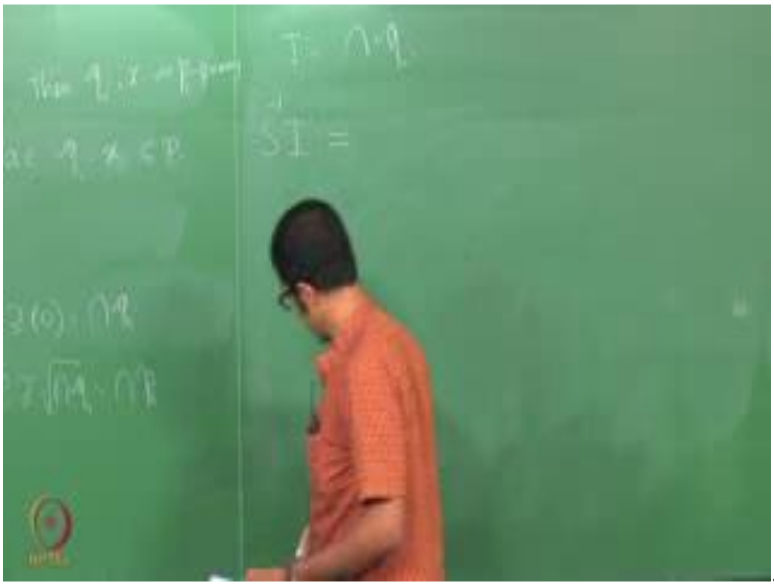
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therefore, I has a primary decomposition. Now look at a multiplicative set S, s inverse what should be S inverse i. So, what is that one would be expected?

Student: (Refer Time: 46:58).

It is we would probably expect that S inverse I is equal to S inverse q i, but then you have to prove few things. What would be S inverse q i? S inverse q i is it same as; I mean when would this be equal to?

Student: The whole ring.

The whole ring, when would this be a proper ideal, and if it is a proper ideal will it be a primary ideal, if it is a primary ideal is the radical same as, what does suppose S inverse q i is primary what would you expect it is radical be.

Student: (Refer Time: 47:45) S inverse q i.

S inverse q i.

Student: (Refer Time: 47:49).

You would expect that radical of it is S inverse p i. So, we have to prove or verify check whether all these things are true. We will do that tomorrow.