Commutative Algebra
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Lecture – 32
Primary Decomposition (Continued)

Let us continue.

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Last time we proved that if the radical of an ideal is a maximal ideal then I is primary. So, last time we saw an example where radical is prime does not really imply that the ideal is primary, but if the radical is maximal then I; the ideal is primary. And one more thing that one can say is that I is primary that need not necessarily imply that I is a power of a prime ideal. Straightforward example is; I is I think did we look at this example in k x y, k a field and polynomial ring k x y this is a primary ideal, but this is not power of any prime ideal. There is only one prime ideal containing I, what is that prime ideal containing I?

Student: (Refer Time: 02:30).

The maximal I deal generated by x y, that is only prime ideal containing. Or in other words radical of I is m, we are in this situation radical of I is m. But m is not equal to I, m square is contained in I, what would be m square? This is x square, x y, y square; m is, so take m to be x y, then m square is this. So, m square is contained here and I is contained in m. So, m square strictly contained in I; strictly contained in m. And m is only prime ideal containing I. So, I is not equal to p power n for any prime ideal.

Yeah.

Student: (Refer Time: 03:39).

This is properly contained in, because y is not here, x y is here because y is there in I so x y is there. So, this is contained here but this I is strict containment this is also strict contain, so I cannot be equal to any prime ideal; I mean power of prime ideal.

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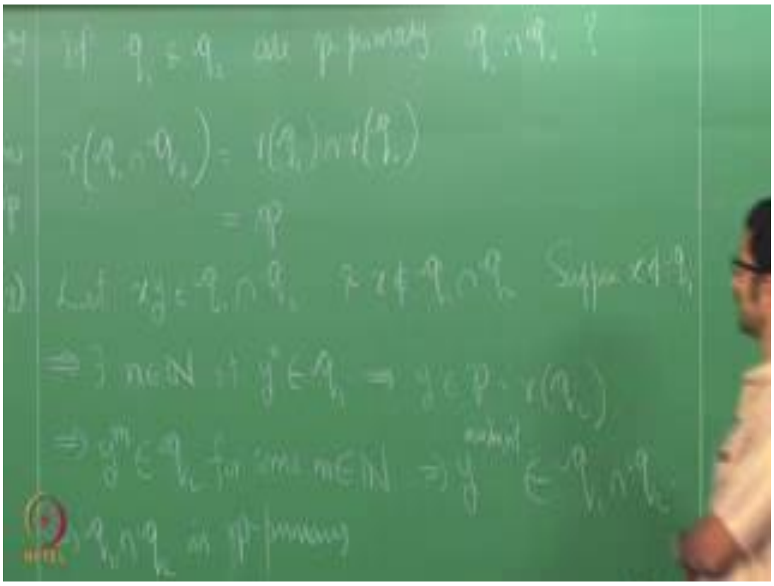

So, what we were looking at is kind of a generalization of the prime decomposition of integers into the case of ideals in Noetherian ring. So, first let us observe something, suppose if q 1 and q 2 are p primary, what can you say about q 1 intersection q 2? First of all let us look at the radical of q 1 intersection q 2; what is radical of q 1 intersection q 2? Properties of by properties of radical this is equal to radical of q 1 intersection radical of q 2, but both of them are p primary. So therefore;

Student: (Refer Time: 05:32).

This will be p. So, this implies that this is equal to p, sorry. But that does not mean that the intersection is primary. So, let us see if we can prove that it is primary or not, I mean let us check, let us verify with it. And x is not in q 1 intersection q 2. We want to see if some power of y belongs to the intersection. Now x is not here implies, x is not in either q 1 or q 2. Suppose x is not in q 1, x is not in the intersection means x is not in one of them. So, suppose x is not in q 1. Now, x y is in q 1 x is not in q 1, and q 1 is p primary.

So, there exist N in N such that y power n belongs to q 1. But what we want is some power of y in q 1 intersection q 2, how do you get it? Y power n belongs to q 1, what does that mean? Y belongs to.

Student: (Refer Time: 07:30).

Radical of q 1 which is p, but now you can; y belongs to p and p is also radical of;

Student: Q 2.

Q 2; it is also radical of q 2. That means some power of y belongs to;

Student: Q 2.

Q 2, if you take maximum of m and n that will be in q 1 intersection q 2. So, that implies q 1 intersection q 2 is p prime. So, if I take 2 primary ideals and intersect them we get a p prime; I mean again if I take p primary ideals and intersect them we still get a p primary ideal.

Now, which are the ideals that can be written in the form of intersection, such ideals are called reducible ideals or decomposable ideals.

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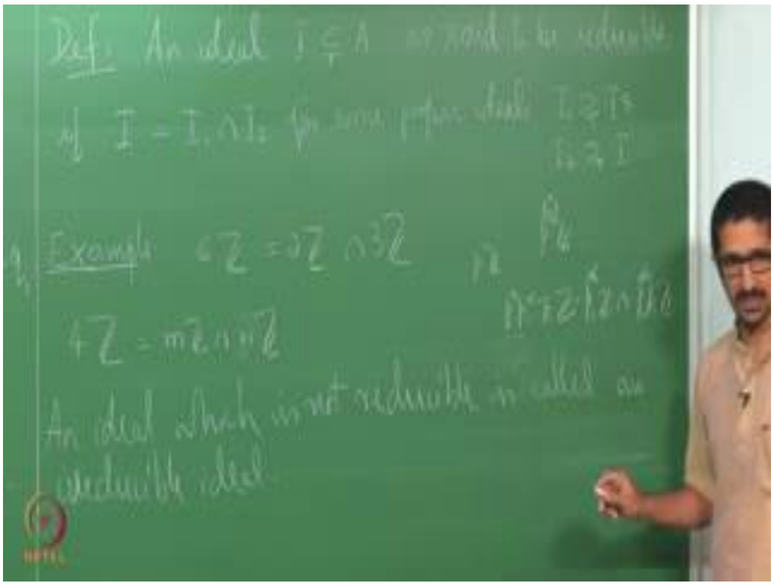

So, an proper ideal is said to be reducible if I is equal to I 1 intersection I 2 for some proper ideals I 1 properly containing I and I 2 properly containing I. That means, it is intersection of 2 bigger ideals, but bigger proper ideals. For example, in Z if you take 6 Z can you write 6 Z as?

Student: 2 Z (Refer Time: 10:18).

2 Z intersections 3 Z; this 6 Z is reducible ideal. What about 4 Z?

Student: (Refer Time: 10:38).

See in Z this is easy to see, what is this intersection?

Student: (Refer Time: 10:53).

LCM of:

Student: (Refer Time: 10:55).

Plus is GCD of m and n intersection is LCM of m and n. So, LCM of m and n is 4 implies either m has to be 4 or n has to be 4. Therefore, this is not reducible; such ideals are called irreducible ideals. So, an ideal which is not reducible is called an Irreducible ideal.

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If the ideal is irreducible, if I is equal to I 1 intersection I 2 then either I 1 is equal to I 2 or I 1 is, I mean I is equal to I 2 or I 1 I is equal to I 1. That is I is irreducible for some proper ideals I 1 comma I 2, I is equal to I 1 intersection I 2 implies I is equal to I 1 or I is equal to I 2. So, 4 Z is an irreducible ideal in Z. Can you generalize this? What are the irreducible ideals of Z?

Student: (Refer Time: 12:51).

P power n;

Student: (Refer Time: 12:57).

How would p itself, p Z is this.

Student: (Refer Time: 13:10).

This is irreducible, because this is maximal.

Student: Here n is not equal to 1.

So, if p power n Z if n is equal to 1 even then it is irreducible.

Student: (Refer Time: 13:31).

Then it is?

Student: Reducible.

Reducible; or in other words what are the irreducible ideals of. See if I can write p 1 power alpha 1 p 2 power alpha 2 if there exists something like this, then you can and you know what are k, this is k, where k does not have p 1 and p 2 factors, then you can always write this as p 1 power alpha. So, this Z I can write it as p 1 power alpha 1 Z intersection with p 2 power alpha 2.

Student: K.

K, Z; I can always; so if there are 2 distinct primes in the prime factorization of that integer that ideal will be, the ideal generated with that integer will be a reducible ideal. Or in other words an irreducible ideal is?

Student: (Refer Time: 14:35).

In Z is?

Student: P power.

Of the form p power alpha for some prime p and for some positive integer alpha; so if n Z is irreducible then n is equal to p power alpha for some prime p and alpha in N. So, does that suggest something? These are irreducible implies primary; these are all the primary ideals of Z. So, let us start with irreducible ideal. Let I be an irreducible ideal. So, the question is see in the case of Z the irreducible ideals are all same as primary ideals, but Z is a PID, it is it is much much stronger than being Noetherian ring.

So, a priori there is no reason to believe that in Noetherian rings irreducibility will imply property or not, but still let us see what happens. So, let us start with x y be in I. I want to see if whether I is primary or not. One way to do this slightly easier to handle the situation is. So, we are looking at I ideal in A, I is irreducible if and only if; so for 0 is irreducible in A mod I. If I is reducible then I can be written as I 1 intersection I 2 where I 1 and I 2 are proper ideals properly containing I. So, I can take their images into A mod I and write 0 bar as.

Student: (Refer Time: 17:54).

There I 1 bar intersection I 2 bar, similarly the converse. So therefore, 0 being irreducible or reducible in A mod I is equivalent to saying that I is being reducible or irreducible in A.

Student: Sir it may be possible that I 1 bar (Refer Time: 18:18) I 2 bar is strictly contained in I.

So, that is what we are saying; if I is equal to I 1 intersection I 2 that would imply that you take the same equation to the; I bar is I 1 intersection I 2 bar.

Student: We are taking I contained in I 1.

I is reducible implies there exist ideals I 1 and I 2 containing I properly containing I and proper ideals such that I is equal to this one. Now, look at this equation modulo I, look at this in A mod I. In A mod I this is bar is 0 bar, 0 bar will be written as I 1 bar intersection I 2 bar. That means, 0 bar 0 is reducible in A mod I. Conversely if 0 is reducible A mod i that means I can find 2 ideals I 1 bar and I 2 bar, such that 0 bar is equal to I 1 bar intersection I 2 bar. Proper ideals properly containing 0 bar. But then any ideal of A mod I is of the form.

Student: (Refer Time: 19:37).

Some j mod I where j is an ideal containing I; so therefore, just pick those inverse images of I 1 bar and I 2 bar bring it back to a you get I is equal to this one. Therefore, 0 being reducible or irreducible in A mod I is equivalent to the ideal I being reducible or correspondingly irreducible in the I ring A. So therefore, we can instead of this we will assume that our ideal I is 0 in A. We want to check whether it is primary in A. So, let x y be equal to 0 and x is non zero; we want to check whether y power n is 0 for some n.

Student: (Refer Time: 20:51).

No we are, so that is what I am saying- certainly we can look at A mod I instead and then bring it back to I instead we just assume that I is 0, because once we prove this the other one can be deduced from here. You start with some irreducible ideal (Refer Time: 21:12) and then use this result. Suppose x y is 0 and x is nonzero; now you have a nice increasing chain of ideal, sorry annihilator of y see our aim is to prove that y power n is 0. We have a chain of ideals. Once you see an ascending chain what comes to your mind, whether it terminates or not. So to make sure, suppose it terminates that is the condition on A if A is Noetherian then this will terminate.

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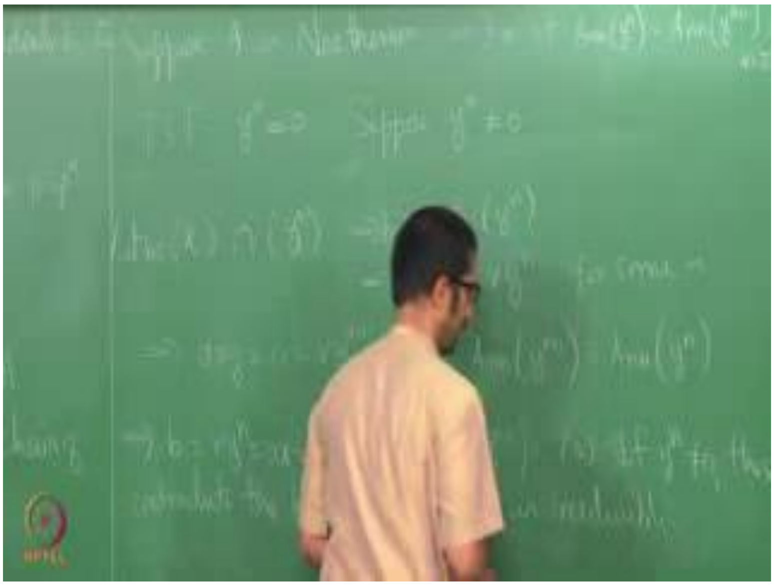

Suppose A is Noetherian, then there exist n such that annihilator of y power n is equal to annihilator of y power n plus I for all I bigger than to 1. The sequence terminates implies it is after finite state everything is equal. I want to say that y power n is 0. Suppose y power n is non zero.

Now, we know that you know x y power n is 0, because x y itself is 0. So, see we have said that 0 is an irreducible ideal that is our hypothesis; we have not used this fact at all. Now y power n is non zero. Suppose we can get a contradiction that should be on his hypothesis, because you know this is the something that we have not used yet. So, one way check is whether 0 can be written as intersection of proper ideals. What we know is that x y is 0. But this ideal generated by x and y this need not necessarily be 0, because the product x y is contained here but that need not be this. For example, in the case of Z intersection is, intersection of m and n is LCM of m; it is not the product of this generated by not need not necessarily be generated by the product.

So, we cannot expect this, how about y power n? Suppose I have a belongs to x intersection y power n, what does that mean? I have a x is in the ideal generate by an element. You start with some element b in the intersection b is in the ideal x. So, there therefore, b is of the form a x, but that is there in y power n. So, I can write this implies that a x is equal to r y power n for some n. Now multiplied by y on both sides, what do you get?

Student: (Refer Time: 25:39).

R a x y; a x y is 0 because x y is 0 that will imply that a x y equal to 0 equal to r y power n plus 1. And that would imply that r belongs to annihilator of y power n plus 1, but annihilator of y power n plus 1 is same as annihilator of y power n. And that implies r y power n is 0 or in other words a x is 0 or in other words sorry I am I wrote b; b is equal to r y power n that is 0; that means this intersection is 0. This implies b equal to r y power n which is equal to x equal to 0. This implies x intersection y power n plus 1 this is 0.

So, this is 0 not only that both of them are nonzero. If y power n is nonzero this contradicts the assumption that the hypothesis that 0 is irreducible. So, the contradiction implies that some of our assumptions are wrong. So, the only assumption that we have made is y power n is non zero.

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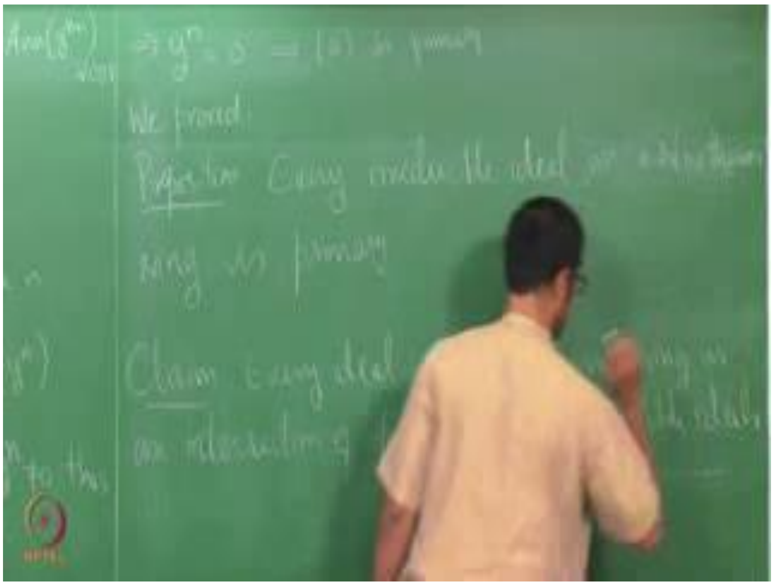

So, this implies that y power n is 0 and that implies that 0 is primary. So, if we include this in our hypothesis that our ring is Noetherian then what we have proved is, every irreducible ideal is primary. So, ideal in; is this clear. In a Noetherian ring every irreducible ideal is primary.

So now what we know is that; suppose you start with a reducible ideal which means you can write it as intersection of I 1 and I 2 some proper ideals properly contained in A and properly containing I. Now look at I 1 and I 2, suppose I 1 is irreducible and I 2 is a reducible we are through I mean the decomposition stops there, you cannot do further. Suppose I 1 is reducible we can further decompose. Suppose I 2 is reducible you can further decompose. How far you can go? I mean can this process continue infinite times? The question is whether given any ideal in a Noetherian ring can be express it as intersection of finitely many irreducible ideals.

Student: Sir, can we say that it will be good within number of (Refer Time: 29:59).

Now we are no more in Z, we are in. In Z one can easily write down right. Suppose your prime factorization is p 1 power alpha 1 up to p k power alpha k then the prime factorization is; sorry, the primary decomposition is precisely p 1 power alpha 1 Z intersection so on. So, the question here is whether we can you know given an ideal in a Noetherian ring can we express it as intersection of finitely many irreducible ideals. Let us say suppose we cannot do that; that means there exists some.

Student: (Refer Time: 30:45).

So, what is that? So, let us write like this; let I be equal to I 1 intersection I 2. So, the claim is; what is the claim? Every ideal in a Noetherian ring is an intersection of finitely many irreducible ideals. So suppose not, what happens? So, I mean, how do you write down a proof, in the sense that you know. I mean suppose this is not true; that means, there exists an ideal with an infinite intersection.

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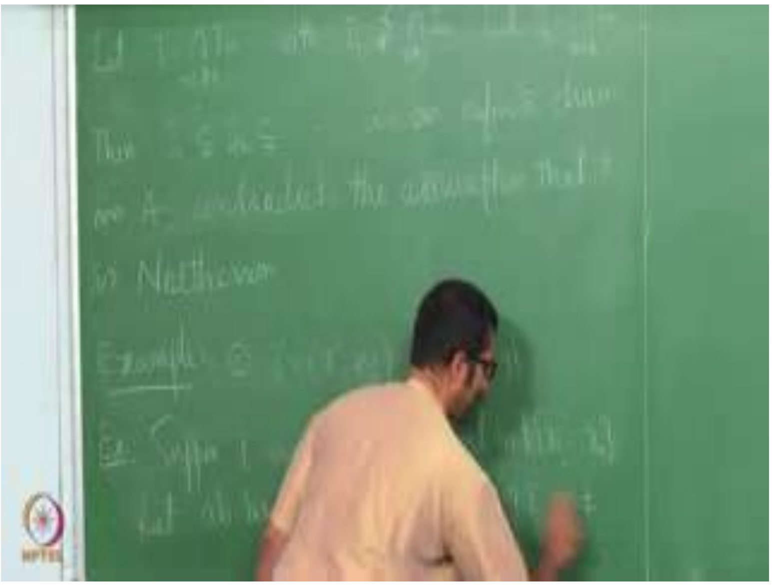

So let us write, suppose I is equal to I n n in N; let us say let us assume it is countable infinite then what is the. Your J 1 is I then contained in, so take J I to be intersection.

Student: (Refer Time: 32:28).

J from I to infinity; J 2; so how do you say that this is;

Student: (Refer Time: 32:43) then we are done.

No here we have infinitely many. So, here why is this properly contained in; J 2 is properly contained in?

Student: (Refer Time: 33:05).

We can we can assume that. So, this is we have to assume that if I is of this form with such that I j does not contain intersection of the rest, because if I j contains intersection of the rest then you can remove I j and keep doing this. So, let I be equal to intersection I n n in N with I j not containing intersection of I n n not equal to I.

Let J k equal to intersection n from k to infinity I n, then J 1 is properly contained in J 2 properly contained is an infinite chain in A. So, that is contradicts the assumption that A is Noetherian. So, that says that every ideal in a Noetherian ring is intersection of finitely many irreducible ideals, but in Noetherian ring we have proved that every reducible ideal is primary. So, in the Noetherian ring every ideal has a primary decomposition; the primary decomposition is precisely what we said.

Student: Sir, (Refer Time: 35:31).

This assumption; see if I j contains this intersection the rest of them then I j; I mean if I have some ideal I is contained in J then I intersection J is same as J, sorry I. So, here the role of J is not there I mean; J need not be there at all.

So, in this case when you say that there exists a prime ideal which cannot be written as intersection of finitely many irreducible ideals, that is what we are claiming. Or in other words there exists an ideal with such an infinite intersection does not have a finite intersection of reducible ideals. And from here for convenience we do not want redundant ideals there. I mean, which containing the rest then say for example, if I 1 contains the intersection of the rest of them then I 1 it is not needed in the intersection at all.

So we just throw away; I mean just for the sake of argument to make it simpler we can assume that. And we will see what are the remaining when we throw out the remaining cannot be finite, because our assumption is that I is an ideal which does not have a finite primary decomposition. So, this has certainly an infinite; not only that it has finite does not have finite decomposition any primary decomposition has to be infinite.

Student: (Refer Time: 37:29).

So, this is n not equal to sorry, not I this is J.

See this condition will ensure that there are no redundant ideals not only that, this condition will say that at each step we have a proper containment. Let us look at some examples of primary decompositions. What is the primary decomposition of x square y x square x y sorry. X square y is a primary ideal itself; so x square x y this. So, for the case of monomial I mean finding primary decomposition is not always a trivial task, but in the case of monomial ideal it is very easy. Let me write down one exercise. Suppose I is a monomial ideal, what is meant by a monomial ideal? In polynomial ring k x 1 up to x n some polynomial; monomial ideal is ideal generated by monomials, polynomials with only one constant. Polynomial is as many terms. So, polynomial is some of monomials.

Suppose I is a monomial ideal in k x 1 up to x n; let a b be a monomial generator of I such that GCD of a comma b is 1, they are co prime, no common factor.

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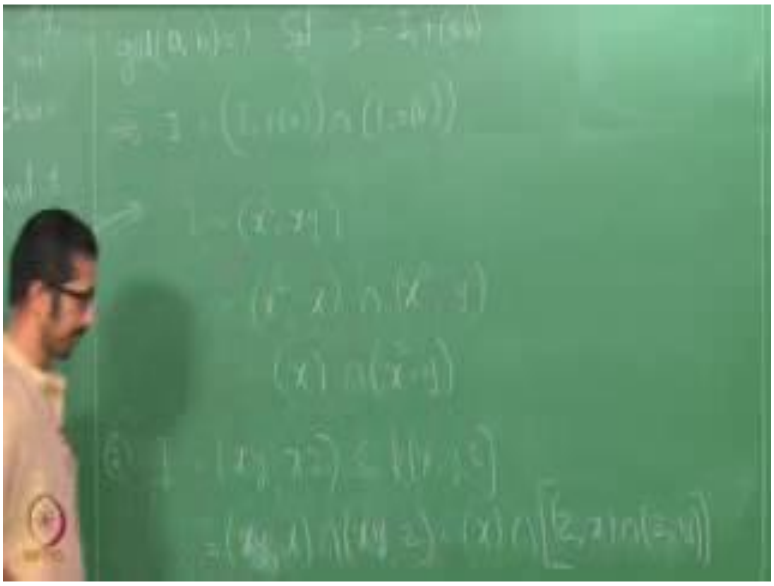

Then know set j I equal to I 1 plus ab. So, a b is a generator. So, I can write it as say for example, this one I can write it as ideal generated by x square plus x y ideal generated by x y. Then I is equal to I 1 plus a intersection I 1 plus b. This is; leave it to you as an exercise. Do you understand the assertion? I take out; so I will give you I will take look at an example. Let us do this example itself. I equal to x square comma x y. So, here our I 1 is the ideal generated by x square a is x and b is y. So, this is equal to x square comma x intersection x square comma y, but what is this?

Student: X.

This is x; so this is x intersection x square comma y.

Now this is primary, this is primary. Let us look at another example, I equal to x y comma x z in k x, y, z. So, apply the previous result recursively, what do you get? This one this result, what you get is this is equal to sorry, x y comma x intersection x y comma z.

Student: (Refer Time: 42:00).

But now this is same as.

Student: (Refer Time: 42:03).

Now, x y see this ideal; what is this ideal?

Student: (Refer Time: 42:11).

Can you see what this ideal is?

Student: (Refer Time: 42:17).

Ideal generated by x itself, because x y is there in the ideal generated by x; the second generator. So, it is like looking at the ideal generated by 2 and 4 in z; 4 is contained in the ideal generated by 2. So therefore, this is ideal generated by x itself. But the second one this is we can again use the previous result; z comma x intersection with z comma y. This one I can think of see; z as I 1 here and this as ab. So, z comma x intersected with z comma y.

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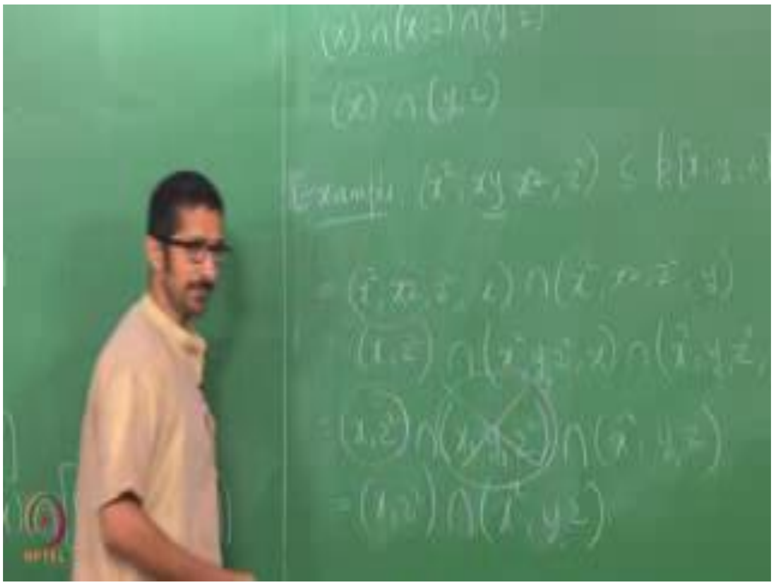

So what we have here is now, x intersection x comma z intersection y comma z. But now, do you see some redundant ideals there? What is this intersection?

Student: (Refer Time: 43:35).

X only; so this intersection we do not need, this one this is redundant we can just remove. This is equal to x intersection y z. Of course, if you look this little carefully you can write down that one very easily; x square xy x z, z square in k x, y, z.

So, we can take one of these as a b and the rest of them I 1. So, this is equal to x square x z z square x intersected with x square x z z square y. Now what is this ideal?

Student: (Refer Time: 45:01).

These both are contained in the ideal generated by x. So, these 2 elements are not required at all in the generating set of I. So, this is equal to x comma z square intersection. Now we have to split this again.

Student: (Refer Time: 45:21).

Not z, x z.

Student: (Refer Time: 45:25).

Because, there is a generator x.

Student: (Refer Time: 45:31).

Now x square y z square, now we split this x intersection x square y z square and z; x z square and what is this ideal, this will be?

Student: X, y, z square.

X, y, z square, and this ideal would be?

Student: (Refer Time: 46:08).

X square y z. So, using this property one can decompose monomial ideals very easily. I will stop with one (Refer Time: 46:31) to show you that sorry.

Student: (Refer Time: 46:34).

Or we can further reduce it.

Student: (Refer Time: 46:38).

This is contained.

Student: (Refer Time: 46:41).

This is redundant. So, this is x comma z square intersection x square y z. See the decomposition need not necessarily be unique.

(Refer Slide Time: 47:14)

For example, the one that we started with x square comma x y, yes;

Student: (Refer Time: 47:22).

There is no y.

Student: (Refer Time: 47:28).

This is x, we just removed these 2.

Student: (Refer Time: 47:37).

This one?

Student: Yes sir.

No. See this ideal contains this one; I is contained in j therefore I intersection j is I itself. Therefore, this is not required in the intersection that is a redundant component. Whether it is there or not it is not going to make any difference.

So see, look at this ideal this one we looked at this primary decomposition x intersection x square comma y. One can also write this as x intersection x square comma xy comma y square. This is also equal to this intersection. In fact, this is equal to y power n for any n bigger than equal to 1; any n in N or n bigger than equal to 1.

So, this says that primary decomposition is not unique. So, what is meant by not unique or what kind of uniqueness can we expect in primary decomposition.

Student: (Refer Time: 49:44).

The decomposition; see in the primary decomposition in z what we have is or even in a normal UFD, what we have is some you know associates. Or in other words the primary components are unique in some sense. Whether we have something similar like that in this case of Noetherian rings, we will see that later.