Commutative Algebra
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Lecture - 30
Further Properties of Noetherian and Artinian Modules and Rings

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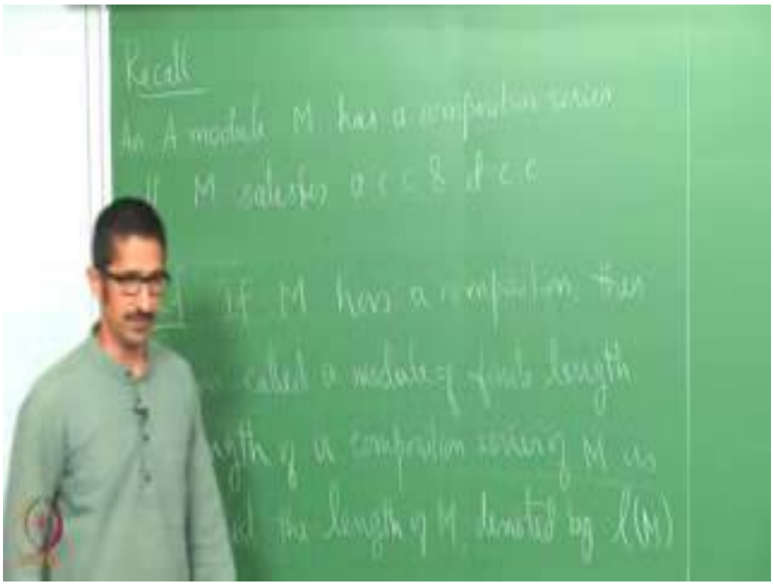

So, let us begin, proved this last time that an a module as a composition series if and only if it satisfies both ascending chain condition and descending chain condition and earlier we proved that if M has a composition series then every composition series will have same length. So, when if M has a composition series then M is called a module of finite length, not only that; this finite length, there is a unique integer associated with M which is the length of any composition series of M. And length of a composition series of M is called the length of M, denoted by and there are l of M. This is uniquely determined because of the theorem that we proved earlier that any composition series can be will be of same length.

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So, this length is unique. For example, we have already seen; what is; so if I take let us look at this k x mod x square. Let us look at this M to be this is a over k x itself, you can think of A to be k x itself or k itself. Then you have k x mod x square, ideal generated by x mod x square containing 0. What is this; can you tell me, what is ideal generated by I mean this x mod x square?

Student: mod powers.

So this is an exercise. If I have a ring A and an ideal and an element a in A which is a non zero divisor. What is meant by non zero divisor? That a x is non zero for any non zero x in A. Then this is isomorphic to what? Isomorphic to, can you looking at this are you not tempted to say; do something?

Student: A by a.

Do a right; try to see if you can prove this? Suppose you want to prove this how do you go about doing it?

Student: we have to (Refer Time: 05:17).

Ha.

Student: (Refer Time: 05:22).

From where to where?

Student: A to here.

From here to?

Student: A.

Send a map; A going to multiply I mean A I mean; sorry, x going to A x bar and look at what is the kernel. So, because of this, what would this be? Suppose this is true, x is a non zero divisor in k x. So, if I assume this is true, then what is this? This is same as x k x ideal generated by x square is x square k x, that is same as this. So, this is isomorphic to?

Student: k x mod.

K x mod.

Student: (Refer Time: 06:15).

Ideal generated by x, but what is k x mod ideal generated by x?

Student: k.

So, this is a maximal sub module of this. This is a maximal ideal here; x ideal generated by x is a maximal ideal here. So, this is a simple; this modulo, this is simple. Similarly this k x; this is say isomorphic to k x mod x which is k which is a simple sub module. You do not have any k x sub module of k. So, therefore, this this mod this is simple. So, what is the length of M here? What is the length of M?

Student: (Refer Time: 07:11).

2. So, now, what is length of k x mod x power n? What is length of n? Prove this, oh! not factor n.

So, now, let us look at; yeah.

Student: some contained in composition series.

This is yeah. So, how do you say this is M 2 contained in M 1 contained in M naught equal to 0? When do you say this is the composition series? If M 2 mod M 1 is simple, and M 1 mod M naught is m M 1 mod M naught is M itself. So, M 1 should be simple and M 2 mod M 1 is simple. Now what is M 2 mod M 1 here? M 2 mod M 1 is k x mod x square mod x k x mod x which is k x mod x which is k. It is a simple sub module, any vector space of dimension one is a simple sub module. Particularly we are looking at modules over k x. So, it is it does not have a proper sub module k x a sub module contained in k. So, therefore, this because every element is a unit. If you take any element and them sub module generated by that it will have to contain k; complete k. So, k is a simple any field is simple module over itself. So, this is M 2 mod M 1 is simple now look at M, M 1 itself is?

Student: k x mod x.

K x mod x which is k again, so, this itself is k, isomorphic to k. Therefore, this is also simple. Therefore, M 1 is simple. Therefore, this is a composition series.

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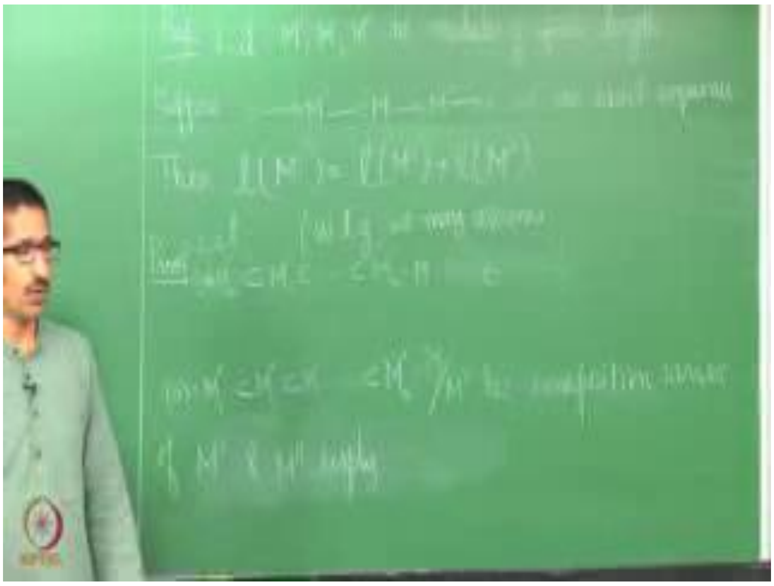

Let us look at some properties of length, let M prime M M double prime be modules of finite length. Suppose 0 to M point to M to M double prime to 0 is an exact sequence.

Let us reserve the; you know conclusion. Let us look at what we can say about lengths. What can you say about? So, suppose I have a, so, I can think of you know M prime as

M prime is isomorphic to a sub module of M. So, I can as well assume that M prime is a sub module of M. Similarly M double prime is isomorphic to M mod M prime, I mean M mod image of M prime. So, I can assume as well as seen that M double prime is indeed M mod M prime with M prime a sub module of M.

Now, suppose I have a, M prime is a sub module of M. Now I have, So, this is M naught prime contained in M 1 prime up to M n prime. This is M prime. This is a; suppose this is a composition series of M prime. Think of them. So, M prime is a sub module of M. So, all these are sub modules of M as well. So, therefore, these all will be this; these all will be I mean sub modules of M and this will be a composition; a part of composition series of M itself. Because each of them is sub module of M, and each quotient is simple. Simple module does not really matter whether we are thinking of a sub module of M prime or M. If it is simple, it is simple.

So, this will be a sub module of; I mean composition series of parts of composition series of M. Now let us look at you know composition series of M double prime. M double prime is M mod M prime. So, this I will call this as M M double prime, I have you know composition series like this, M 1 double prime contained in M naught double prime which is 0.

Now, what is M naught double prime here? Which is 0, what is meant by 0 here? M prime mod M prime, now if I look at M 1 double prime, it is a sub module of M mod M prime. How does any sub module of M mod M prime look like?

Student: (Refer Time: 13:48) If we take a composition series.

No, forget about composition series, how will this look like? This will be a sub module of M mod M prime, how does any sub module of M mod M prime look like?

Student: subset of m containing (Refer Time: 14:03).

So, it will look like something of the form M 1 mod M prime. So, this is M prime mod M prime. Now the next one. So, M 2 double prime this will look like M 2 mod M prime, and so on up to this is M mod M prime.

Now, look at this quotient, M 1 I mean M 1 double prime mod M naught double prime, what is that? Is same as M 1 itself, M 2 mod M 1; M 2 mod M 1 is same as M 2 mod M prime mod M 1 mod M prime therefore, that is simple M 2 mod M 1 is simple. Similarly, M 3 mod M 2 is simple up to M mod M M minus 1 is simple. So, what does that give you now? So, you start with 0 here, go up to M prime from there you jump to M 1, M 2 and so on up to M. That will be a composition series for M. So, what have we shown?

Student: (Refer Time: 16:04).

Length of yeah; so, length of composition series is called as length of the module itself. So, length of M is equal to length of M prime plus length of M double prime. So, let us write down that first.

So, let this be a composition series of; let this and this be a composition series of M prime and M double prime respectively. So, there we can first we can say without loss of generality, we assume that M prime is a sub module of M and M double prime is, So, here without loss of generality we may assume.

Student: (Refer Time: 17:25).

We proved this last time.

Student: last.

Oh.

Student: this was the last.

I thought we proved only we have ended with.

Student: (Refer Time: 17:37).

Oh, we proved that.

Student: Sir that statement was not written ((Refer Time: 17:41).

Oh, fine. So, we then let us; so this is a; then this will be you know M naught prime, which is 0 contained in M 1 prime containing up to M n prime contained in M 1, M M is a composition series of for m and that gives the result.

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For k vector spaces, this is for finite dimensional k vector spaces, the following are equivalent.

V is finite dimensional. Yeah V has finite length. V satisfies ascending chain condition and V satisfies descending chain conditions. All these are equivalent. So, what we are saying is that length is nothing but generalization to the case of to the case of modules and log S to the dimension of a vector space.

So, this is if this V has finite length. So, we are talking about V as a module over k. 2, if and only if 3 and 2, if and only if 4, we already know. If sorry, yeah, 1 implies 1 if and only if 3, 1 if and only if 4, directly follows some linear algebra.

Because any subspace of finite dimensional. So, you start any ascending chain or descending chain, it has to terminate. Each step we will I mean in ascending chain; each step dimension will increase at least by 1. So, it cannot go more than the number of it cannot go more than the number of steps as much as in the dimension of V you start with 0. Similarly for any descending chain, you can have at most dimension V number of steps.

So, 1 if and only if 3 and 1 if and only if 4 is straightforward; so we only have to prove 1 implies 2 is again straightforward, why is that so? If it has a finite dimensional vector space, I can take the basis V 1, V 2, up to V n then you can have a composition series V

1 up to V n, contained in containing V 1 up to V n minus 1, containing V 1 up to V n minus 2 and so on up to V 1 and then to 0. Each quotient will be isomorphic to k it will be a 1 dimensional vector space. So, therefore, it will be simple. So, therefore, one implies 2 is a basis then.

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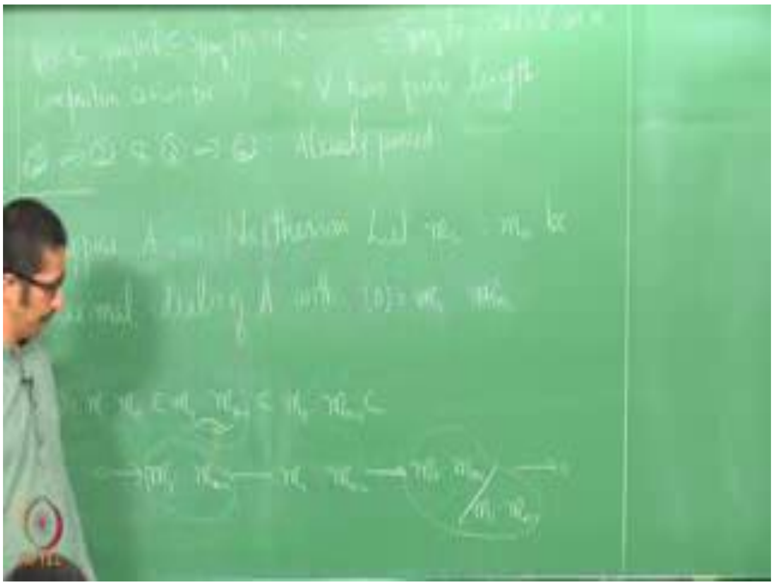

So, 0 k span of is a composition series, for we has finite and we have already proved that, 2 implies 3 and 2 implies 4 has already been proved. Because a module has finite length

if and only if it satisfies both ACC and DCC. So, 2 implies 3 and 2 implies 4 is already proved. So on vector spaces finite dimension is equivalent to saying that it is finite length, which is equivalent to saying that you know ascending chain condition descending chain condition are all satisfied.

So, see what we are scene is that if a ring, suppose your ring is Noetherian, when can you know when can we say that length I mean this has a composition series, can we say suppose we have some nice property is true, can we say length of A is finite. We know that this is not always true for example.

Student: (Refer Time: 24:54)

Z k x or take Z, you it does not have a composition series because you take I mean like you start with any ideal, I mean there are no simple sub modules of Z you take any sub module it contains a proper sub module. For any n Z you have infinitely many you know chain starting with any n Z you have infinite chain descending. So, descending chain condition is never satisfied in Z. So, length of Z is infinite or you know it does not it is not a module of finitely, but can we say something nice. If you know if suppose with some nice property.

Suppose you have M 1 up to M n be maximal ideals of A, with 0 equal to the product. The product of the maximal ideal is 0. For example, suppose you take Z 6. What are the maximal ideals of Z 6; 2 and 3. The product is 0? So, if you have some nice structure on A the product is maximal ideal. Some product of maximal ideal is 0. Then look at this sequence contained in M 1 up to M n minus 1. What can you say about this? What is this a simple module. Forget about this part what would be this mod this.

Student: k x (Refer Time: 27:34)

No this modulo this. See if I call this ideal or you know ideal I, this is nothing but I mod M n I. This is a module over a mod M n or in other words it is a vector space if it is of dimension one then.

Student: k.

We will have the previous proposition will say that it has this is simple or whatever it be, if I look at this mod this, it is a vector space over A mod N. So, it will have finite length.

Now, this need not be simple, but this this has finite length, I mean this has this is a finite this satisfies ascending chain condition. Or you know it has finite length. Now look at M 1 up to M n minus 2.

Student: a by (Refer Time: 28:54)

This 0 to M 1 up to M n minus 1, you have this inclusion map M 1 up to M n minus 2 to M 1 up to M n minus 2 mod, M 1 up to M n minus 1 to 0. This see this by induction this is a module of finite length, not by induction, this we have seen that this is a module of finite length. Because this modulo this is a finite dimensional vector space. So, here why is it the finite dimensional vector space? We are making use of the fact that this is Noetherian. What is this if I this is I i mod M n I is a vector, it is generated by generators of I i is an ideal in a Noetherian ring. Every sub module is finitely generated or every ideal is finitely generated. So, therefore, this is finitely generated therefore, this has ascending chain condition. Because your vector spaces this modulo this. So, this itself is a A mod M n vector space it has finite length which is equivalent to saying that it is finite dimension is equivalent to saying that finite length it satisfies ACC it satisfies DCC.

So, this is this has finite length. Similarly, this is a finite dimensional vector space over a mod M n minus 1. So, therefore, this satisfies DCC, ascending chain condition descending chain condition by the previous proposition. So, in the when we proved that length is additive, what we proved is that if I have a composition series of this and if I have a composition series of this, I can just merge them and get a composition series of this which means this length is finite. I can keep going up to so, now, keep going to M 1 and then contained in A.

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So, I have this 0 equal to M 1 up to M n M 1 up to M n minus 1 M 1 to A. What we are showing here is that this is 0. This has finite length, make this has finite length comes from the fact that this is a finite dimensional A mod M n vector space. The by induction the next one or you know by recursively taking it to the next one this has finite length, M 1 has finite length and therefore, a has finite length.

So, if a is Noetherian and M 1 up to M n are maximal ideals with their product being 0 then length of a is finite.

Student: product of whole maximal ideal or some maximal ideal.

I mean some maximal ideal they let this be maximal ideals be some maximal ideals. Need not be all maximal ideals. See if we are not use making use of that pattern, just give there should be maximal ideals whose product should be 0. Then you can have a you know this is not a composition series by the way, this need not necessarily be a composition series. Because this, yeah?

Student: (Refer Time: 33:34).

Sorry.

Student: (Refer Time: 33:36).

Yeah, I mean I just wanted to make clear that you know we are not emphasizing that product of all maximal ideals. And a priori we do not need to say that. So, what we have shown here is that, suppose M 1, M 2 up to M n be maximal ideals of are maximal ideals of A such that m 1 the product m 1 up to m n is 0 if A is Noetherian then A is artinian. It satisfies the descending chain condition. What about converse suppose A is artinian.

Student: (Refer Time: 35:20).

Again we can look at the you know see here what we are what we are saying is that each I mean each quotient look at this one this mod this is an A mod M n vector space therefore, what we are saying is that since this is finitely generated it satisfies DCC. Conversely see once we are looking at M 1 look at any m i m i mod m 1 up to m i plus 1 this is a vector space over a mod m i m i plus 1. So, this one satisfies ascending chain condition, if and only if descending change condition if and only if this is a finite dimension if and one if it is a finite length. So, on this module all these are equivalent. So, therefore, we can swap the roles of ACC and DCC in this argument, that this mod this is a finite dimension is a vector space therefore, it satisfies ACC if and only if it is DCC.

Now, look at the next one. This mod this is a vector space over a mod M n minus 1 therefore, this satisfies ACC if and only if it satisfies DCC keep going. So, ultimately what we get is A satisfies ascending chain condition if and only if it satisfies descending chain condition. Or in other words a is Noetherian if and only if a is artinian. So, what we have proved is more than this, let me modify the statement. Then A is Noetherian if and only if A is artinian. So, this you should you know write down write down the complete proof it is I mean it, it is the same we have not just that I have not systematically written down, but you should be able to do that.

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So, a ring A is Noetherian if and only if every a satisfies ascending chain condition, every ideal is finitely generated, and every chain of us yeah every collection of ideals of a, has a maximal ideal; maximal element.

Now, if A to B be an onto ring homomorphism. Or in other words, let us to start with; let I be an ideal of A, if A is Noetherian then what can you say about A mod I?

Student: (Refer Time: 40:24).

A mod I will also be Noetherian. Because every ideal of a mod I looks like some j modulo I where j is an ideal in a containing I, but j is finitely (Refer Time: 40:40) So, if I take those generating set and look at their images in a mod I that will generate j mod I right. So, therefore, every ideal of a mod I is finitely generated therefore, this is Noetherian, property 1 may be or observation remark whatever.

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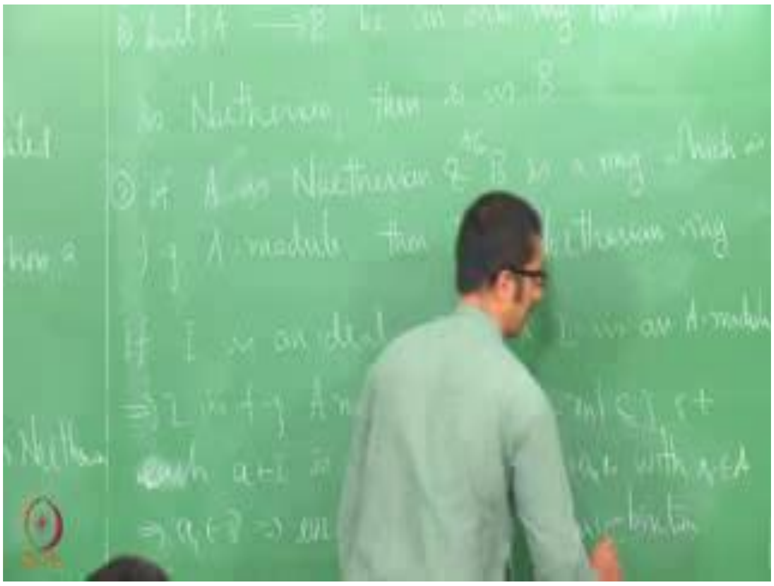

As a corollary of remark 1, one can say then, so, let me you know into ring homomorphism. If A is Noetherian then so, is B. If I have A ring homomorphism onto ring homomorphism then B is isomorphic to, let me just say f B is isomorphic to A modulo kernel f, first isomorphism theorem B is isomorphic to a mod kernel f. If A is Noetherian then a mod kernel f is Noetherian which means B is Noetherian.

Suppose, so if A is Noetherian and B is finitely generated, B is a ring, which is finitely generated A module. Then B is a finitely generated A module.

Student: (Refer Time: 43:24).

How do you say this?

Student: and in somewhere it will have actually finding generated implies.

So, here to say that it is a; this is a Noetherian ring. See as a ring, it is Noetherian we want to say see as a mod that we have already seen see if A is Noetherian and B is finitely generated a module, then we have already seen that it is a Noetherian module Noetherian A module, but we want to say that as a ring it is Noetherian right. So, how do you do that? We have many ways of saying this. We want to say that every let us say for example, every ideal is finitely generated. If I take an ideal in A in B, it is an A module, but B is already a Noetherian A module therefore, this ideal is finitely generated as an?

Student: a module.

A module, in particular, it will be a finitely generated B module, as an ideal. See if I take. So, I am looking at A contained in B or a yeah, and we need to have multiplication compatibility structure. So, if I is an ideal of B then I is an A module. This implies that I is finitely generated A module. What does that mean every element? So, there exists a finite generating set let us say x 1, x 2, up to x n such that every element of I is written as a linear combination of the form a x 1 plus a 1 x 1 plus etcetera a n, x n where each a i is coming from A.

Student: (Refer Time: 46:02).

But A is already there in B. So, therefore, it is a every linear; every element is a linear combination of x 1 up to x n which means I as an ideal is finitely generated.

So, which means in I such that every each a in I is of the form a 1 x 1 plus etcetera a n x n with a i in a which implies a i's are in b and that implies every element of I is a B linear combination of this same set x 1 up to x n.

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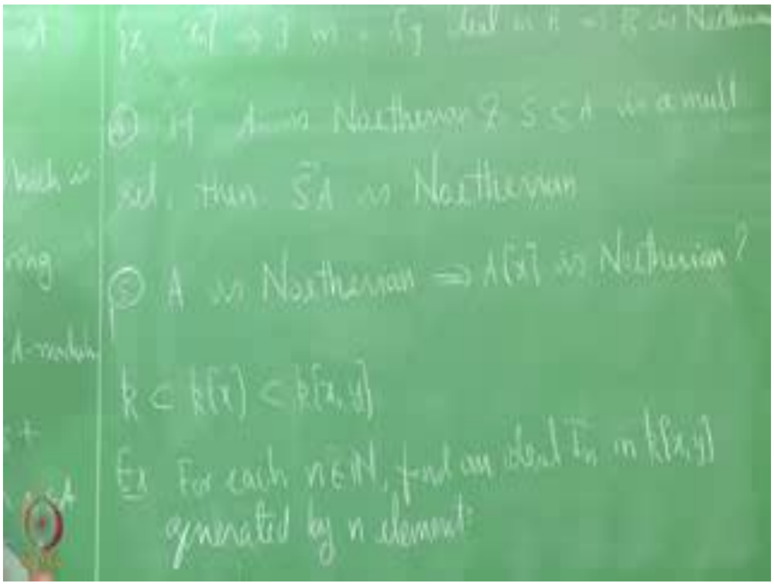

Which means I is a finitely generated ideal in B; implies B is Noetherian. If A is Noetherian and S in A is a multiplicative set, then what can you say about S inverse A? How does any ideal of S inverse A look like?

Student: s inverse.

S inverse I where?

Student: I intersection S is empty.

I intersection S is empty. Now I is an ideal in A. A is Noetherian therefore, it is finitely generated. So, therefore, S inverse I is also finitely generated.

So, this means this is Noetherian. So, now, comes the question A is Noetherian, k is Noetherian, any field is Noetherian, it has only 2 ideals. So, every ideal is finitely generated, what about k x, is it Noetherian?

Student: yes.

Noetherian ascending chain condition, or finite every ideal should be finitely generated, is that true, k x?

Student: (Refer Time: 49:36).

K x is PID, it is a principal ideal domain; every ideal is generated by.

Student: (Refer Time: 49:42).

One element, so therefore, this is Noetherian.

Student: (Refer Time: 49:49).

What about; yeah, every PID is Noetherian. What if I put k x y? Here we know and we do not know anything about this, can you think of given any n; given any n, my exercises for each n in N, find an ideal I n in k x y, generated by n elements; that means, however, large your n is 10 million, whatever it be, the ideal is generated by; there exists an ideal generated by that many ideal. So, are we thinking that this is not Noetherian? Are we saying that it is not Noetherian?

Student: (Refer Time: 51:10).

We will see it in the next class.