Commutative Algebra
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Lecture - 03
Ideals in Commutative Rings

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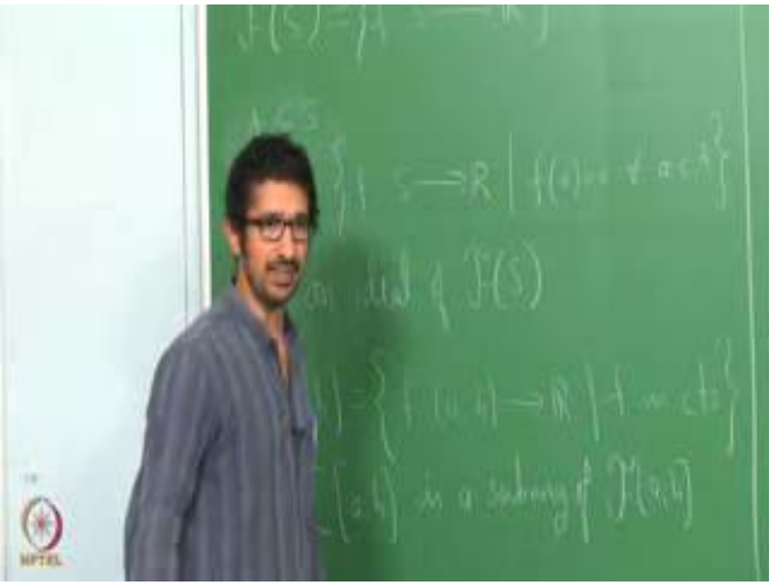

So, this is for any set S look at set of all functions from S to R. As we saw earlier this has a natural ring structure with point wise addition and point wise multiplication. Now we already saw as an example of an ideal, look at set of all. So, take a subset A of S and look at I of A to be set of all f from S to R such that f of a is 0 for every a in A.

This is an ideal of f of S: my question was can you think of a prime ideal here?

Student: (Refer Time: 01:31)

Maximal ideal; and if you are not spend some time thinking about I will leave it, we will come back to this example later. Think about that it is a nice example. If you put some additional structure on this you can look at a sub ring of this called. So, again if you have you need to have some structure on the set. I look at C a b my; S is the closed interval a b, this is set of all continuous functions, then C a b is a sub ring of F a b.

Now, it is interesting to think about ideals of C a b, it has little I mean it has little more structure because this has a topological space it has a subspace topology coming from that usual topology of R. So, with that you can think more about functions. There are you know; there would not be functions and not all functions like this appearing here will be here. For example, if S is, and if you look at in this one you can define all functions f of 0 f of x equal to 0 if x is rational and if f of x is 1 if x is irrational. It will be here, but not here.

Similarly this has more structure, and hence it is more interesting, and the ideals in particular the maximal ideals of this r of very particular form. We will come back to that later.

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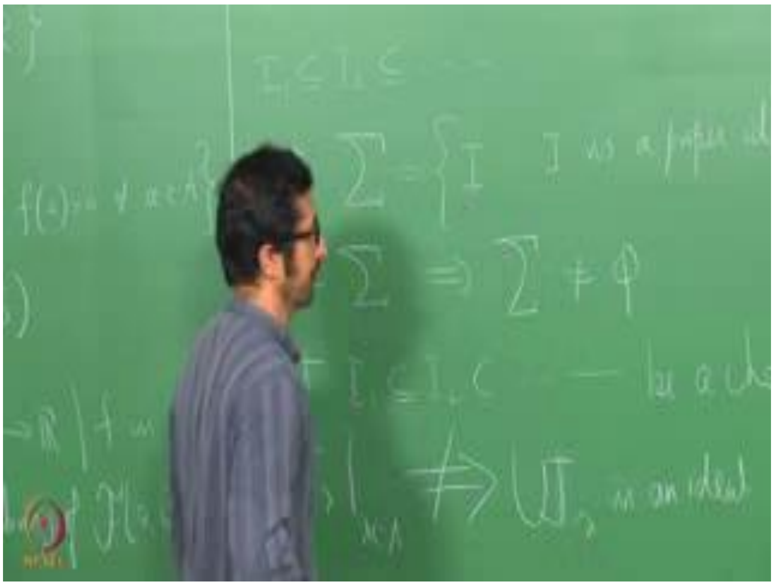

So, the question is suppose you have a ring commutative ring with identity, does it have a maximal idea at all? You know maybe there is a possibility that I have ideals I 1; the only ideals of a are like this you know. A chain which never stops, there is no ideal which is at the top of all other ideals or you know at least there is nothing between. And there is no ideal proper ideal which contains a lot of it or which is not the proper ideal of any other sub ideal of any other ideal. How do I make sure that there exists something like that?

Let us take the collection of this be set of all ideals I is a proper ideal of a. So, let us assume A to be a non-zero ring. Now, can you see that this is non-empty, why is it non empty? The 0 ideal belongs to sigma, so this says that this is non empty. This is collection of all ideals, so this has a natural partial order; the inclusion. So the idea is, see I am trying to see if I can get a maximal ideal. If I can get a maximal element in this one, naturally we expect that it is a maximal ideal.

So, the question is can we have a maximal element in this one, how do we think of a new; get a set which is partially ordered set and you are looking for a maximal element what is the result that comes to your mind Zorn's lemma. So, to apply Zorn's lemma you need to say that any chain of elements will have a maximal element in sigma. So, let us take a chain in sigma. Let I 1 contained in I 2 be a chain in sigma. And you want to look for a maximal element in sigma; what should be can you think of a natural candidate for this? I am looking for an ideal which is proper.

Student: (Refer Time: 07:26).

So if you look at, in general if you take I 1 and I 2 to ideals or you know collection of ideals their union need not be an ideal. If you have a collection I lambda, lambda belongs to some indexing set. This need not imply that union I lambda is an ideal. Union of two ideals need not be an ideal. Can you tell me a quick example? Whenever you think of examples I would expect that you think of Z. In Z if you take 2 Z and 3 Z take their union right it is not an ideal. Why is it not an ideal?

Student: (Refer Time: 08:28).

Well, for that you need to prove more things, but it is immediately you can prove that you can say that it does not contain certain elements; 2 is there, 3 is there, but 5 is not there. So, that immediately says that it is not an ideal.

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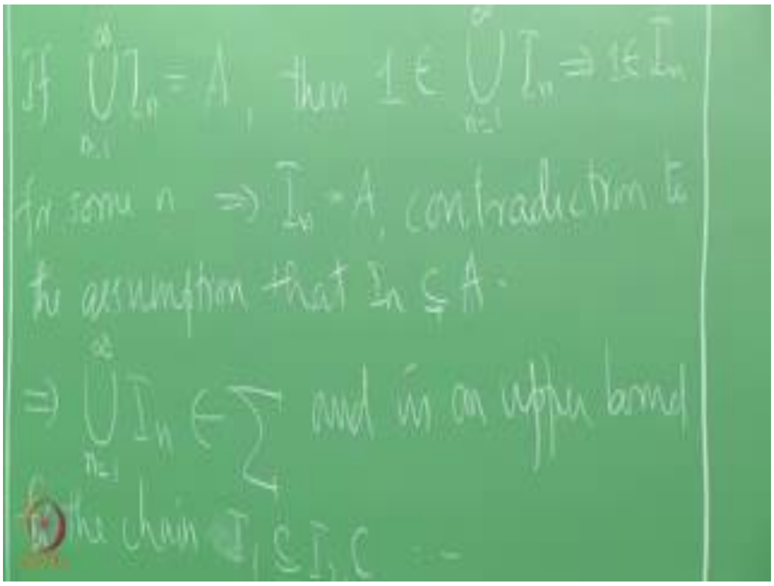

Therefore, union need not be an ideal, but if you have an increasing chain of ideals; union, this maybe I will use a different notation here because here I am using I for this. I n, n from one to infinity is an ideal, because this is since I 1 is an increasing chain. And of course, you need to verify that it satisfy all the properties; usual properties of. Now the question is, is this a proper idea? See we need to given any chain; I need to find an upper bound for this chain in sigma. For this ideal to be in sigma we have to show that it is a proper ideal.

Why is this proper ideal? This is not a proper ideal it is equal to the whole ring. When do you say a union is the whole ring? In this ideal, this is an ideal an ideal is whole ring if and only if a unit belongs to it. So, ideal I n has a unit, it has one element the element identity is in the union therefore it is in one of them, but if it is in one of them if an ideal contains identity then it is the whole ring; that is a contradiction. If union I n is equal to A then 1 belongs to the union, that implies one belongs to I n for some n, that implies I n is equal to A which is a contradiction. Contradiction to the assumption that I n is proper.

Therefore, this union belongs to sigma and this is an upper bound; upper bound for the chain I 1 contained in I 2. So, what we have seen or what we have shown now is that increasing chain in or given any chain in sigma it has an upper bound in sigma.

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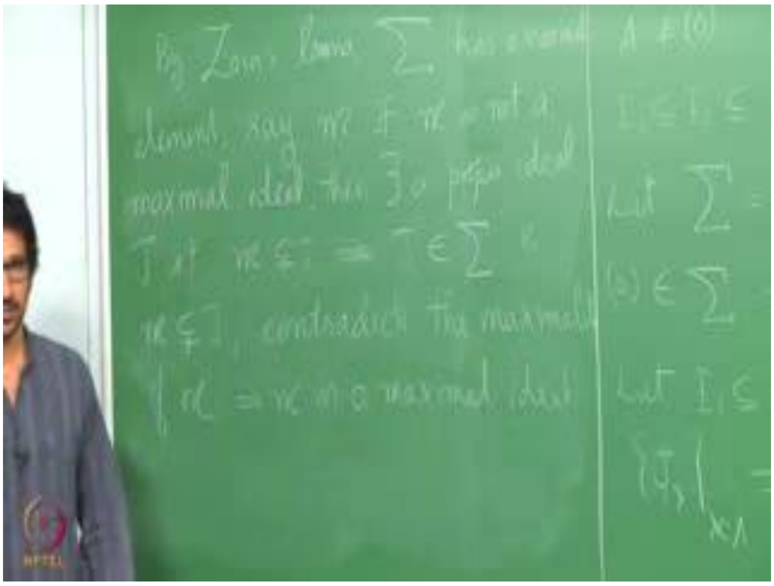

Therefore, by Zorn's lemma sigma has a maximal element. This has a maximal element. Let us say denote that by m and we expect this to be a maximal ideal. Let us see if this is a maximal ideal. What does meant by this is not a maximal ideal?

Student: (Refer Time: 13:17).

There exists an ideal let us say J which is a proper ideal of a and properly containing m, but that directly contradicts the maximality of m in sigma. So, J belongs to m sigma and that will contradict the maximality of m. Therefore, if m is not a maximal ideal then there exists a proper ideal J such that m is strictly contained in J. This implies that J belongs to sigma because it is a proper ideal and m is strictly contained in J, and these two together contradicts the maximality of m. Contradicts to the conclusion here that m is a maximal element of sigma.

Therefore, m is indeed the maximal ideal. So, what did we prove now? If A is a commutative ring with the identity; A is a non-zero commutative ring with the identity then A has a maximal ideal.

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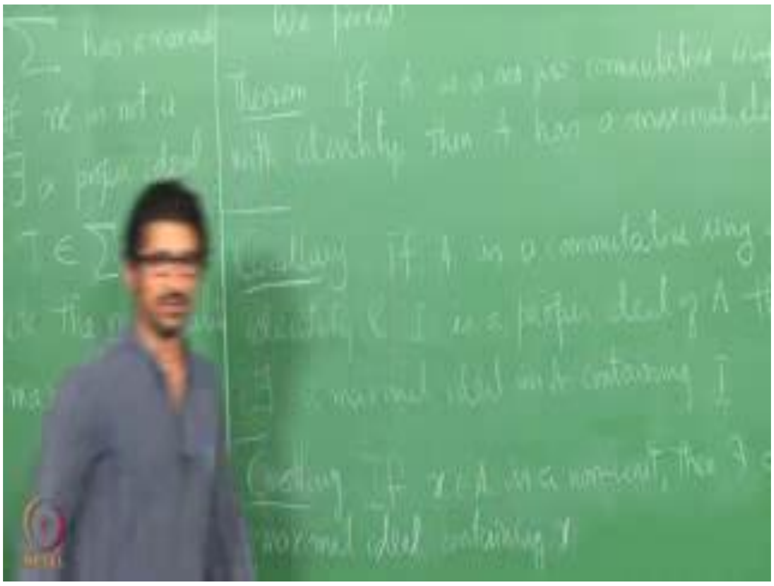

So therefore, we proved if A is a non-zero commutative ring with the identity then A has maximal ideal. So, this method of finding maximal element, we will see a couple of times in this course; not couple of times I mean more than that. Getting a collection of ideals or modules and then looking for a maximal element in that using Zorn’s lemma. This is something the standard tool in commutative algebra.

This has a nice corollary, suppose I is a proper ideal of A; if I is a proper ideal of A, what we have shown now is that given any commutative ring with the identity it contains a maximal ideal. Now I can demand more. I take an ideal I, can we have a maximal ideal containing I? There are two ways of doing it: one that you can repeat the proof; you can consider collection of all proper ideals containing I and then go through the same process. If my collection I modified to be that the whole thing goes through exactly as it is, there is no difference. That is double work; I mean you are duplicating your work. Much simpler is look at the ring A mod I. If you look at the ring A mod I this is again a commutative ring with the identity non-zero because I is a proper ideal, therefore this has a maximal ideal.

Now, you have a natural map from A to A mod I, using this you can show that the inverse image of the maximal ideal here will be a maximal ideal here containing I. So therefore, a corollary of this is if A is a commutative ring with identity and I is a proper ideal of A, then there exists a maximal ideal in A containing I.

This corollary has one more nice corollary, actually an observation. If you take any non-unit element; if you start with some element of A which is not a unit element, then there exists a maximal ideal containing this element. Can you see that? Every non-unit is contained in some maximal ideal. Can you see that?

Student: (Refer Time: 20:22).

Look at the ideal generated by x and apply this corollary. So, every non-unit is there in some maximal ideal or there exists a maximal ideal containing a non m. So, if x in a is a non-unit then there exists a maximal ideal containing x.

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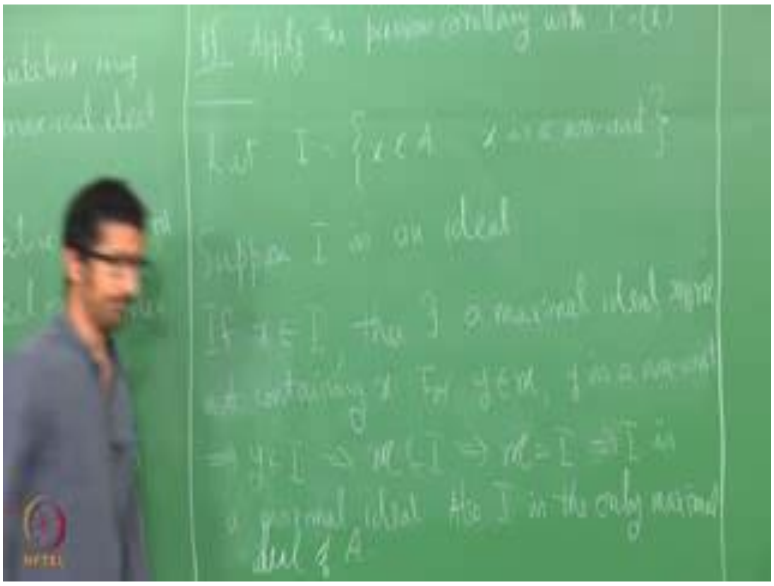

So this is, apply corollary the previous corollary with. So, what we have shown now is that, given any non-unit it is in some maximal ideal. So now, suppose I collect all non-units suffering, will that form an ideal?

Student: No, Sir.

Need not necessarily foraminate. For example, can you think of a ring where this is not true? That all non-units do not form an ideal; collect all non-units of that ring that is not an ideal. Can you think of a ring with this property?

Student: Let Z be (Refer Time: 22:29).

Z itself right. In Z except 1 and minus 1 you collect all of them, that is not an ideal. Therefore, the collections of all non-units do not form an ideal, but suppose it is true. Suppose the collection of all non-units forms an ideal, what kind of ideal will it be? Let I be equal to set of all x in a x is a non-unit. So, our basic assumption on a is that a is a commutative ring with the identity. I am collecting all non-units of a, will it be a proper subset of a?

Student: Yes.

Yes, because 1 is not there. Now suppose this is I is an ideal, can you make some more observation about the ideal I? Will it be a maximal ideal? You take any x in I then by the previous observation there exists a maximal ideal containing this element. But now if you look at the elements of that maximal ideal all elements in that ideal should be non-units. That means, all of them should be here, this is a proper ideal and a maximal ideal is contained in this one which means they have to be equal. Or in other words I is a maximal ideal. If collections of all non-units form an ideal it has to be a maximal ideal.

I will make one more observation. If x belongs to I then there exists a maximal ideal in A containing x, say m. For each y in m y is a non-unit, because it is in a maximal ideal and that implies that y belongs to I, this implies that m is contained in I, this implies that m is equal to I. And that implies I is maximal ideal.

Now in this case we proved little more, can you see we proved that I is maximal ideal yes, but we proved little more than that. See you here we happen to take a maximal ideal of a and we proved that it is indeed equal to I; any maximal ideal is equal to I. That is what we indeed proved. You started with some element of I or in fact I could start with any non-unit of a, then I have a maximal ideal containing that non-unit. And we proved that that maximal ideal is indeed equal to I.

Therefore, what we have proved is that, there is only one maximal ideal which is I. So, such ring where there is only one maximal ideal is called a Local Ring.

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So therefore, what we proved is the folder. Let me make one more observation here; that also I is the only maximal ideal of.

Student: 1 minus 8 6 (Refer Time: 28:03).

I will come back to this. What we have proved now is that, if the set of all non-units say I form a proper ideal of; form an ideal we do not really need to see proper because it is naturally proper; from an ideal of A then I is the unique maximal ideal in A. This is what we proved here, that is the only maximal ideal of A. A ring is said to be a local ring if it has only one maximal ideal. Can you think of some examples of local rings? Is Z a local ring?

Student: No.

No, can you make a local ring from z? Can you construct a local ring from z? In general if you take basic example any field right. If you take any field it has only two ideals either 0 or the entire ring; 0 is a maximal ideal there. So, that is the only maximal ideal. But now can you give me an example of a ring which is not a field, but a local ring. Again think of the correspondence between.

Student: Quotient.

Quotient and the ring; from Z can you construct a ring which is local? Not fields you know; again do not tell me Z 2, Z 3, Z 5. Give me an example of a ring which has only one maximal ideal.

Student: (Refer Time: 31:14).

As a basic example let us look at Z 9. What are the maximal ideals of Z 9? So this is isomorphic to Z mod 9 Z, maximal ideals of this ring would be of the form p Z by 9 Z where p is a divisor of 9 there is only one which is.

Student: 3.

3, ideal generated by 3 in this ring. That is the only maximal ideal.

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So, more generally Z mod p power n Z where p is a prime, this is an example of a local ring. Now we have one more nice characterization of local rings. If you take; this is let A be a commutative ring with identity, so I will just say let A be a ring this means A is a commutative ring with the identity. And m be a maximal ideal. If 1 plus x in m is a unit for all x in m then m is the unique maximal ideal of A.

So, m is a maximal ideal. So, how do you say that 1 plus x is a unit? How do you say that 1 plus x is a unit in; I am sorry we are start assuming that 1 plus x is a unit in m how do you say that m is the unique maximal ideal. Suppose I have let x be, see how do you say that this is the unique maximal ideal? If I can show that all non-units are here, then we are true.

If I take a non-unit, if it is not in m then we get something. Let x be a non-unit, this if x is not in m then see m is a maximal ideal, therefore if I take the ideal generated by m and x it has to be the entire ring. Then m plus x has to be whole of A. Or in other words there exists some u in m and some r in a such that 1 is equal to u plus r x. U is in m, this implies 1 minus u is equal to r x. U is in m, therefore 1 minus u by assumption is a unit, which means r x is a unit, which means x is a unit. That is a contradiction.

Therefore, x has to be in m, which means all non-units are in m which means m is the collection of all non-units and hence it is a maximal, the only maximal ideal.

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This implies by hypothesis x belongs; sorry 1 minus u is a unit. And hence 1 minus u is a unit, this implies r x is a unit, this implies x is a unit; that is a contradiction to the assumption that x is not a unit. This implies x belongs to m and that implies m is a set of all non-units; that implies m is the unique maximal ideal. Is this clear? So, for any maximal ideal 1 plus x is a unit for every x in the maximal ideal will imply that it is a local ring and m is the unique maximal ideal.

We had been talking about collection of units. Now there is a sub collection right, we have seen that, not units we were talking about non-units. Now a sub collection of non-units interesting sub collection are set of all 0 divisors and set of all nilpotents.

So, first let us collect all the nilpotents. So, set of all x is a nilpotent. We saw that collection of all non-units do not necessarily form an ideal. What about this collection? What is the definition of nilpotent? There exists a power n such that x power n is 0. Now will this form an ideal? This is certainly a proper subset of a, because none of the units in particular one cannot be in n. Therefore, this is indeed a proper subset is this an ideal. For example, if I take if x comma y belong to n, can we say that x plus y belong to n? Or what does this mean? Then there exists n m such that x power n is 0 and y power m is 0. The question is, can we say x plus y is nilpotent?

Student: (Refer Time: 40:23) y power n plus m.

If I take this implies x plus y whole power n plus m this will in the binomial expansion any of the monomial term will contain either power of x power n or a power of y power m.

Student: Y power m.

Therefore, this has to be 0. This implies x plus y belongs to n. So, what are the other conditions? Associativity, if x belongs to n, can we say minus x belongs to n? Straightforward right, the same power will work.

Student: Sir.

Yes.

Student: If you take m plus n minus 1 (Refer Time: 41:21).

Sorry.

Student: (Refer Time: 41:24).

Will it go through? There will be a power, minus 1 might also go through. I mean see you do not really have to worry about what power, asymptotically it is 0. Some high power is 0 that is all you have to worry about.

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So if I take, if x belongs to n then minus x belongs to n; that is pretty straightforward. What about if x belongs to n and r belongs to A, then is r x in n; again the same power the; if x power n is 0 then r x whole power n will be 0 again. Therefore, x power n is 0 will imply r x power n is 0, this implies r x is also nilpotent. So, what we have shown now is that, the set of all nilpotent elements form a proper ideal of; this is what we proved.

Now there is an interesting thing to observe here; there is an interesting thing to observe here is that. If I take a nilpotent element x in n, then x power n is 0 by definition. But then 0 is an element that belongs to all ideals. So, this for any element in n some power belongs to all ideals. In particular, if I take a prime ideal what does this say? 0 belongs to prime ideal any ideal it belongs to; so in particular if I take a prime ideal x power n belongs to the any prime ideal. What does that imply?

Student: (Refer Time: 44:39).

X belongs to p or x power n minus 1 belongs to p.

Student: Minus (Refer Time: 44:49).

Now, you can keep coming down.

Student: (Refer Time: 44:51).

X belongs to p. So, this says that all the nilpotent elements are in all the prime ideals.

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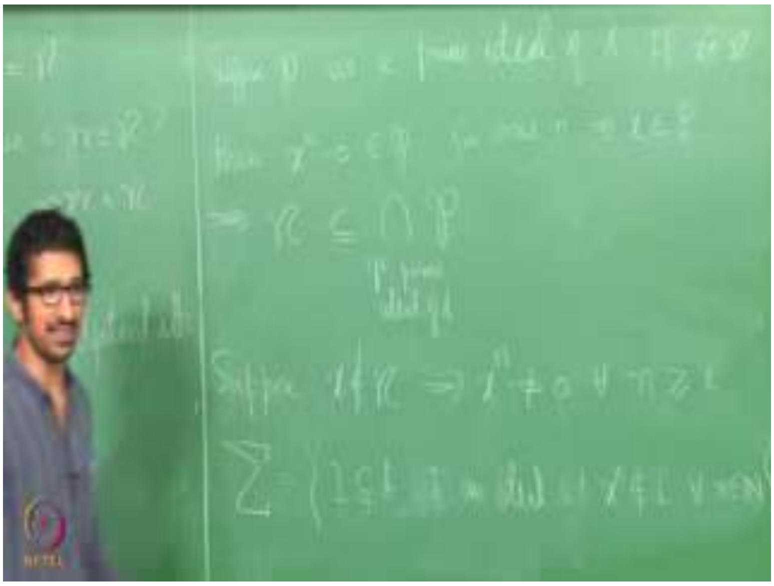

If p is a prime ideal of A; suppose p is the prime ideal of A. If x belongs to n then x power n is 0 which belongs to p for some n and p being prime; prime this implies that x belongs to p. Or in other words what we have shown here now is that this n is contained in intersection of r. Every element of n is contained in all ideals; all prime ideals. Or in other words this is contained in the intersection.

So there is a natural question, whenever you see an inclusion naturally one tends to ask whether this is a proper inequality or actually inequality. So, the question is, are they equal? So, let us try. Suppose, I want to show that every element here is here. Or in other words if I show that if x is not here then it cannot be here. Now how do I show that it cannot be here? It is not in the intersection, if I want to show that it is not in then the intersection.

Student: (Refer Time: 47:13).

I have to produce a prime ideal which avoids that element. So, let us start with an element which is not in n. This implies x power n is not equal to 0 for a every n bigger than to 1. I want to get a prime ideal which avoids x. Again this is as I mentioned earlier we apply the Zorn’s lemma tool in this case. Let us look at this collection of all proper ideals of I an ideal, such that x power n is not in I for every n. I just look at all the ideals which do not contain any power of x, is this non empty?

Student: 0 (Refer Time: 48:40).

The 0 ideal right x power n is non-zero for every n, which means 0 ideal is in this one.

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The ideal 0 belongs to sigma, this implies sigma is non-empty. Now how do you show and how do you proceed? Take a chain of ideals, be a chain in sigma. What is the natural upper bound that you can think of?

Student: Union.

Union: let I be equal to union I n, n from 1 to infinity, then I is a proper ideal. So, I will leave that as a exercise to verify for yourself. Now the question is whether I belongs to sigma. What is meant by I is not in sigma? There exists some power of x in I, but if an element belongs to I it has to be in one of the I n's; that again contradicts. Therefore, an I belongs to sigma, so will just write it as verify. So, what this is an upper bound for this chain. So, this I is an upper bound for the chain. So, by Zorn's lemma sigma has a maximal element.

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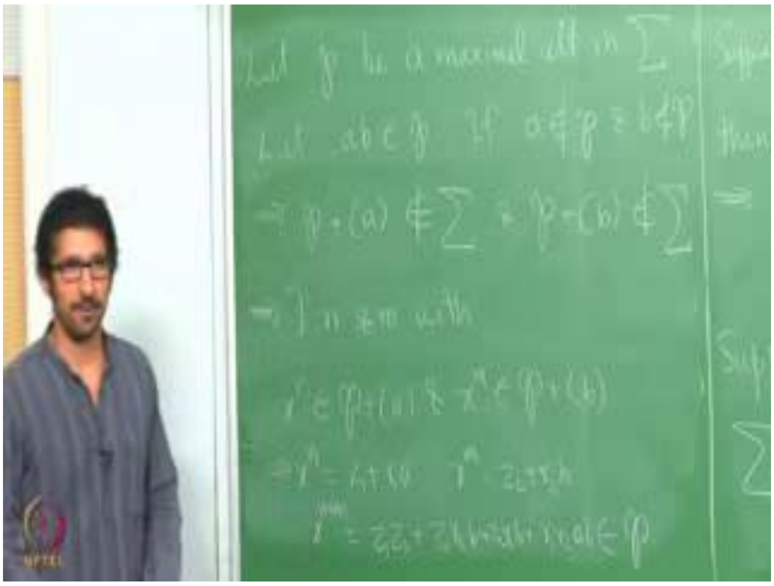

Let me call this maximal element p: p be a maximal element. I claim that this is a prime ideal. So, let a b belongs to p. If a is not in p and b is not in p, if you want to show that either a belongs to p or b belongs to p, suppose, both are not in p that would imply that p plus a is not in sigma and p plus b is not in sigma, because both are strictly bigger ideals. This will imply that a power of x belongs to this ideal and a power of x belongs to this ideal. This implies there exists n and m with x power n belongs to p plus a and x power m belongs to p plus b. So, I can write x power n as some Z 1 plus r 1 a and x power m as Z 2 plus r 2 b.

Now, if I take their product what do I get? X power n plus m will be Z 1 Z 2 plus Z 1 r 2 b plus Z 2 r 1 b plus r 1 r 2 a b. Now what can you say about this product? What can you say about a b? A b is in p Z 2 is in p therefore this is in p, this is in p, this is in p, the whole thing is in p; that is a contradict. Therefore, p is a prime ideal and p does not contain any power of x, in particular x is not in p. So, we have obtained a prime ideal that avoids x. If x is not nilpotent there exists a prime ideal that avoids x.

So, what we have proved now is that, n is indeed the intersection of all prime ideals. Stop here.