Communicative Algebra

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Lecture – 29
Properties of Noetherian and Artinian Modules, Composition Series

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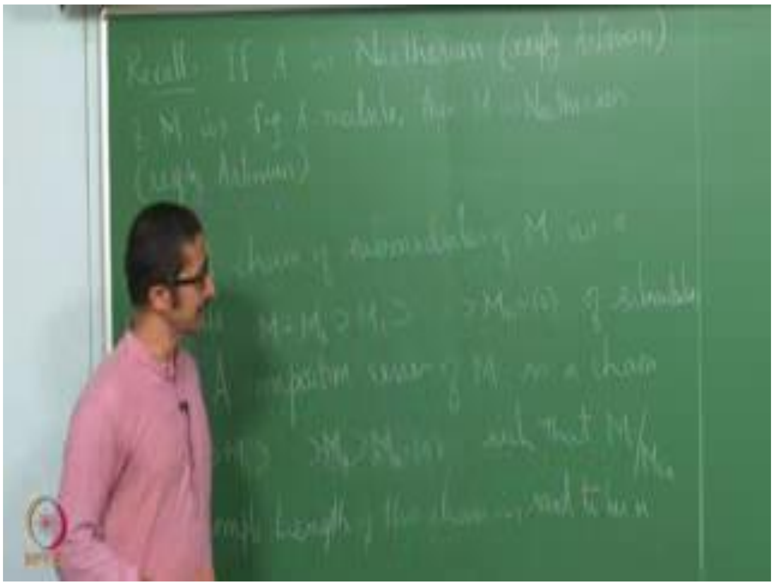

So let us just recall that if A is Noetherian respectively artinian and M is finitely generated a module then M is Noetherian respectively artinian is follows directly from the, on to homomorphism A into M. A chain of sub modules of M is a sequence M naught M could M naught contained in M 1. So, here each one is a strict inclusion of sub modules of M. And a composition series of M is a chain such that M i mod M i plus 1 is simple. What do you mean by simple? It has no nontrivial subgroups other than 0 and itself. So, there are no subgroups sub modules of this module other than 0 and itself.

So, if let me maybe I will write this as M minus 1 M n minus M n equal to 0. I mean I am talking about composition series that terminates at 0; that means, M 1 is a maximal sub module of M naught M 2 is the maximal sub module of M 1 and so on. And the length of this chain is said to be n.

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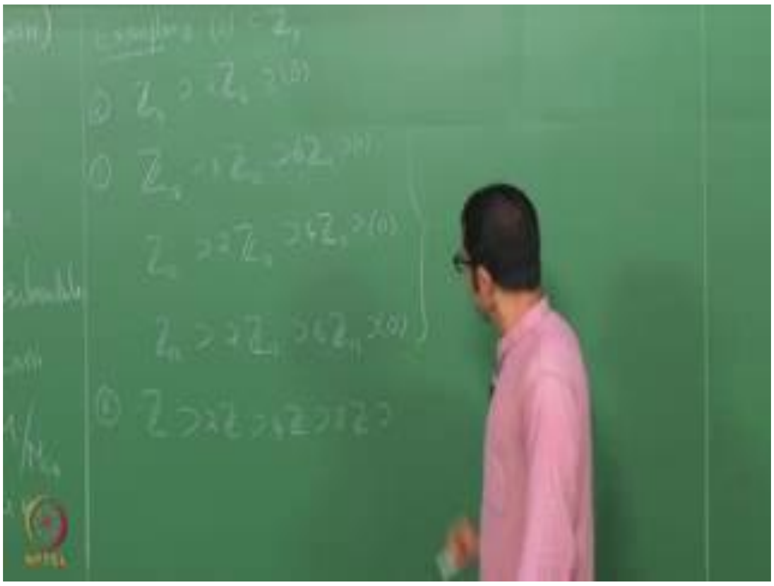

Let us look at 1 or 2 examples. So, if the module itself is a simple module, then your composition series now this is a composition series right Z p is a simple Z module. So, this is a composition series. Suppose I take Z 4 let say. Can you give me a composition series for Z 4?

Student: Z 4, Z 2.

Z 4.

Student: Z 8.

Z 8.

Student: Z 2.

Z 2.

Student: Z 4 0 2.

See when we talk about Z 2 contained in Z 4 etcetera.

Student: The ideal is generated by 2 and 0.

The ideal generated by 2 2 Z 4 and Z. You should not really say that Z 2 contained in Z 4. Now that has a lot of scope for confusion. So, this is a composition series of Z 4.

Another composition series of let say Z 12; Z 12 then 6 Z 12, 3 Z 12 sorry other way round right 3 Z 12, 6 Z 12 0, what is yeah can you give me another composition series for Z 12?

Student: 4 Z 12.

2 Z 12.

Student: 4 z 12.

4 Z 12.

Student: 8 z 12.

What is 8 Z 12? 8 Z 12 is will again be 4 Z 12, there are no proper sub modules of 4 Z 12 right. 4 Z 12 is 0.

Student: 4 8.

4 8. It does not have a proper sub module. Similarly, I have one more right. I can say 2 Z 12, 6 Z 12, 0 there are composition series.

So, the composition series of Z 12 all of them have length 3 right. What about take the Z module Z itself. Does it have a composition series? I can start with you know if I want to construct a composition I should find a maximal sub module. So, let say 2 Z a maximal sub module of this 4 Z, a maximal sub module of this 8 Z and so on. It never terminates. So, Z does not have a composition series. If a mod I will come back. So, the first thing that one you know tends to think about, suppose you take a module which has a composition series, and suppose it has different composition series is it true that all of them have same length in this case 8 Z 12 has same.

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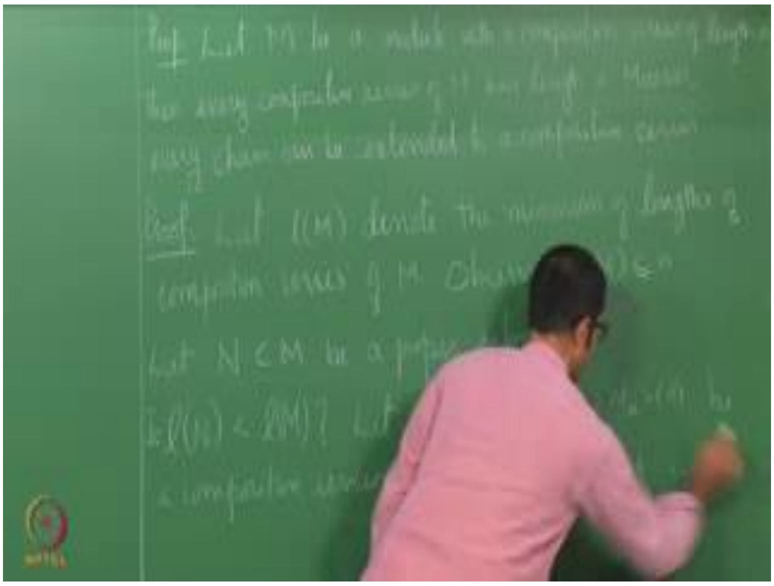

So, this is indeed true. Let M be a module with a with composition series of length n. Then every composition series of M has length n. Moreover, every composition series, every chain can be extended to a composition series. So, M can have many composition series.

So, let us denote, let l of M denote the minimum of lengths of composition series of M. I want to say that every composition series is of this length. To start with I know that you know that n is bigger than equal to M right. So, observe that length of M is less than equal to n. Let n be a sub module of, can we say that l of N is less than l of M. First of all, will this have a composition series at all, will n have a composition series. How can we construct a composition series for n?

Student: Take a composition and intersection.

Yeah, take a composition series for M and then intersect with N each N that will give me a composition series for N. So, therefore, a N will have a composition series. Now what will be the length of N will it be strictly less than M I mean will this always hold true.

Let us look at this. So, let M equal to M not contained and M 1 be a composition series of M. Now I take N i to be n intersection M i. Now what can we say about see then.

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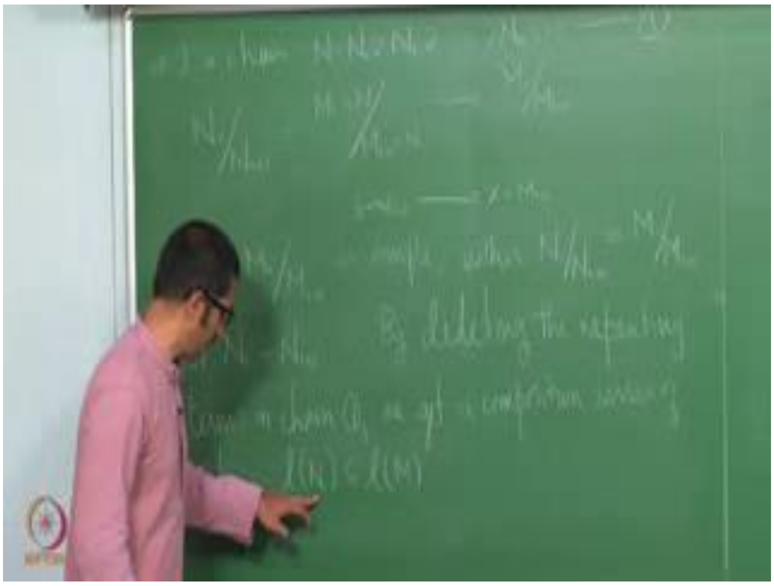

We have then there is there exists a chain, N N naught is N because that is M naught is M N is a sub module therefore, M intersection N is N naught, I mean N itself this is contained in N 1 N in equal to 0. So, here see in this one, we will have to put less than I mean subset or equal to because we do not know how there can be 2 of them which when intersecting with N can give rise to only one sub module. So, we will just put like this.

Now, what can you say about N i intersects N i mod N i plus 1. What is this? This is M i intersection N mod M I plus 1. Intersection N, now this this injects into M i mod M i plus 1, right this one if I take any x bar here. So, x plus N i plus 1 if I map to x plus M i plus 1 this will be an injective a module homomorphism. So, therefore, this injects into M i mod M i plus 1, but then this is a simple a model. So, therefore, this is either equal or 0. Since M i mod M i plus 1 is simple either N i mod N i plus 1 is equal to M i mod M i plus 1 or N i is equal to N i plus 1. So, if N i is equal to N i plus 1, I just remove it. So, by deleting the corresponding I mean repeated terms, what I get is a composition series, by deleting the repeating terms.

So, let me call this chain 1, we get a composition series for composition series of N. Therefore, length of N is less than or equal length of M. Right now suppose I take what is you know, what is mean by see our aim is to show that it is strictly less N if N is a proper sub module then this is strictly less what is meant by this these 2 are equal.

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If l of N is equal to l of M, what does that say that says that here none of them is repeating; that means, nowhere this happens.

Or in other words N 1 what is N sorry what is N n minus 1 look at what is N n minus 1 this is n intersection M n minus 1. So, this is not equal to. So, N M minus 1 mod N M is equal to M N minus 1 mod M N, but N M is what is N m that is 0 therefore, N M minus 1 is M N minus 1 right similarly N M minus 2 is now N M minus 1 is M N minus 1 therefore, pulling back ultimately we get and that implies N M minus 2 is M N minus 2. So, on ultimately we get N is equal to M.

So, therefore, therefore, if N is a proper sub module of M, then length of N is strictly less than length of N. See this has a very interesting consequence. Let M equal to M naught contained in M 1 be a composition series of M. Then length of M is bigger than or equal to length of M 1 plus 1 right, from the previous discussion. This is bigger than or equal to length of M 2 plus 2 length of M n plus n, but M n is 0 therefore, this is n.

So, l of M is bigger than or equal to n if I take any composition series l of M is bigger than equal to n, but l of M by definition what is it? Minimum of lengths of composition series which means if you take any composition series that has length l of N.

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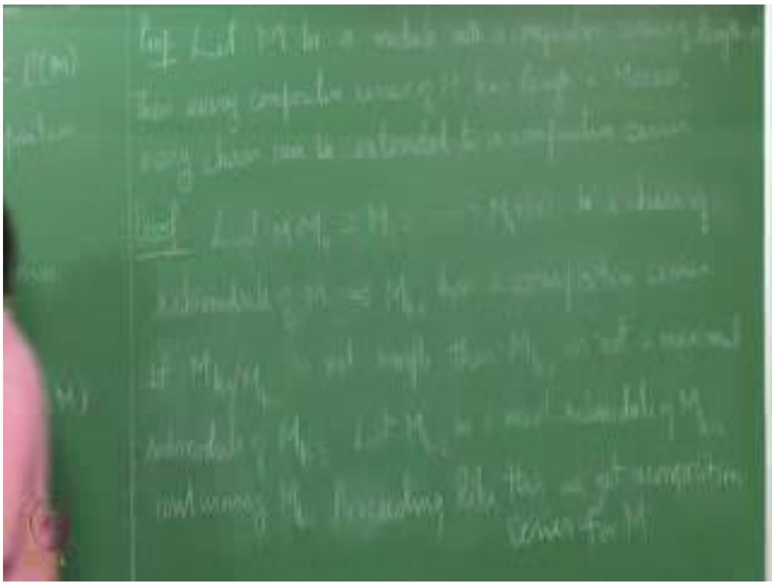

Now, to prove that last part. Suppose I take let M k sorry M naught is M, M 1 be a chain of sub modules of M, then we know that length of M k minus 1, this is less than length of M. So, when you say that, it means M k minus 1 has a composition series and it has finite length you know this is a well-defined. So, I can find a composition series for M k minus 1.

Now, if M k mod M k minus 1 is not simple; that means, M k is not a maximal sub module of M maximal sub module of M k minus 1 is not a maximal sub module of M k minus 2 and length of M k minus 2 is again finite because it is less than length of M naught. Therefore, it has a composition series take the I mean M k is not a maximal sub module therefore, you can you know extend it keep doing this and find the composition series for I mean what we are doing is basically we are inserting modules whenever it is not a maximal sub module. This the fact that length of any if M has a composition series then any proper sub module has a composition series and the length is less than the length of the composition series of M makes us I mean enable us to complete this process. So, less than length of M then M k minus 1 has a composition series if M k minus 2 M k minus 1 is not simple, then M k minus 1 is not a maximal sub module of.

So, if I take, now the maximal sub module of, M k minus 2. So, choose a maximal sub module of M k minus 2 contain M k minus 1 and I have kind of apply in direction you keep doing this. So, let M k minus 2 prime be a maximal sub module of M k minus 2 containing M k minus 1 and keep doing this. So, therefore, this length has proceeding like this.

So, what we have seen is that, if a module has a composition series then every composition series is of same length.

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So, therefore, if M has a composition series, then there is a unique integer associated with M, which is the length of the composition series that is unique. The length of the composition series of M is called the length of M and if M does not have a composition series, then we say length of M to be. So, we have already seen examples where it is finite length. So, what is length of Z 12?

Student: 3?

3, length of Z p 1, length of Z 4 2. Suppose I take M to be take a ring A and look at the maximal ideal m power n. I look at m is a maximal ideal in A and look at the module m equal to A mod m power n.

Now, can you give me a composition series for this?

Student: m mod n.

I should, I want a sub module of m.

Student: A divided by m.

m, which is same as m right, m mod m power n contained in, I mean containing m square mod m power n. What does a mod m power n modulo m mod m power n is this a composition series?

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See how do I say this is a composition series, this is I call this m 1 m mod m 1 is this simple, what is m mod m 1?

Student: a mod m.

M mod m 1 is this is isomorphic to a mod m right. That is field therefore, this is simple. What about m mod m square, m mod m power n? Module of m square mod m power n. What is this isomorphic to? m mod m square, this need not necessarily be isomorphic to a mod m. Or this is m mod m square is an A mod m module. It is an A module at the same time it is an A mod m module, but m mod m is a field. So, this is a vector space over a mod m right this could be a finite dimensional vector space. This need not be of dimension one like this.

So, there could be many now this need not be simple. If this is an n dimensional vector space, then suppose this is generated by x 1 bar up to x m bar, then I can within this I can have this x 1 bar up to x m bar x 1 x 2 bar up to x m bar x m bar right to 0. This itself is a composition series of m mod m square right. So, therefore, this is this need not necessarily be a composition series, I can I can you know in insert in between and get a composition series, but then each time what we are going to do is insert finitely many well assuming that m is finitely generated. If m is finitely generated each time I will have a I will have finite number of modules in between, and this will be a module with composition series with finite length.

So, this implies that if m is a finitely generated a module finitely generated ideal more generally if I assume A is Noetherian. Then length of a mod m power n is finite. A mod m power n has a composition series that composition series need not be what I wrote there that now A mod m A mod m power n containing m mod m power n containing m square mod m power n and so on. That need not be a composition series, but we can insert in between there could be many, but we can insert and ultimately get a finite composition series.

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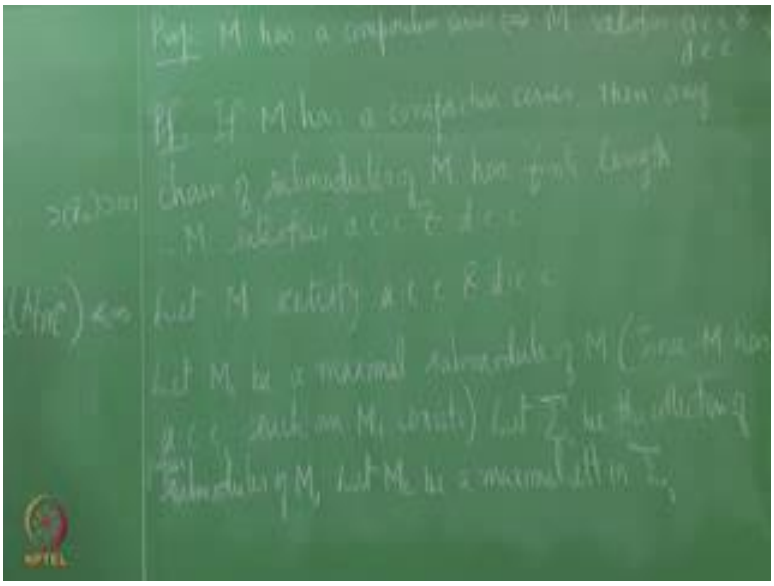

So, this is another important property of composition series modules with composition series. M has a composition series if and only if M satisfies acc, ascending chain condition and descending chain condition both. If a composition series M has a composition series, then it should satisfy ascending chain condition and descending chain condition. So, I mean one way is immediate. If M has a composition series any chain can be extended to a composition series, that is what the previous result says, which means any chain has to be finite whether it is ascending or descending whatever you wherever you start with it has to be.

Student: Terminate.

It has to terminate right. So, therefore, this if m has a composition series, then any chain of sub modules of M has finite length therefore, M satisfies acc and dcc. Any chain will have only finitely many elements. So, ascending chain condition and descending chain condition both are satisfied.

Now, suppose I conversely let M be M satisfy ascending chain condition and descending chain condition. I want to say that; it has a composition series. So, we construct a composition series. Now let M 1 be a maximal sub module of M. I can say this because M satisfies ascending chain condition. What is meant by ascending chain condition it is any increasing chain terminates, equivalently any collection of sub modules of M has a maximal element, right and equivalently every sub module is finitely generated. These are 3 equivalent conditions. Now I am start I know that M satisfies ascending chain condition therefore, I can choose a maximal sub module of M.

So, therefore, I now I look at M 1. I look at the collection of all sub modules of M 1 which are anyway sub modules of M, sub modules of M 1 again because of M satisfying ascending chain condition, this will have a maximal element. M 2 be a, so, we can since M has ascending chain condition, M 1 such an M 1 exists. Then now sigma 1 be the collection of sub modules of M 1, this is collection of proper sub module of M 1 let m 2 be a, in fact, I should have chosen M 1 like this let sigma be the collection of all proper sub modules of M let M 1 be a maximal element, now let m 2 be a maximal element in sigma 1 then I have so, continuing like this I get a chain of sub modules.

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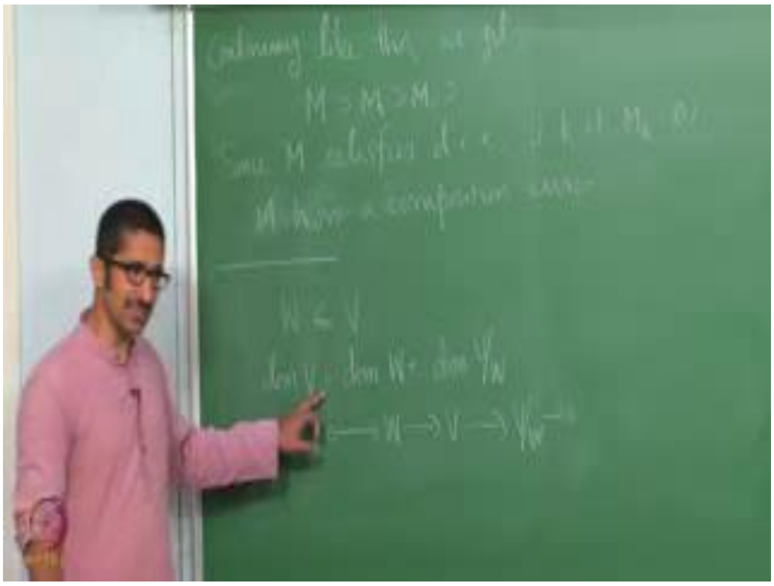

We get M contained in M 1 contained in M 2 and so on. Since M satisfies descending chain condition there exists k such that M k is 0. Because if it is nonzero I can you know if it has a nonzero proper sub module maximal sub module I still can go further down. So, therefore, the chain terminates. M has a composition series. So, having composition series is equivalent to saying that, it has both ascending chain condition and descending chain condition.

So, this is like you know length is pretty much similar to the dimension of vector space. It is ring theoretically very close. If I have a sub module you know if I have a subspace W of V if it is a proper subspace, then dimension of V is strictly bigger than dimension of W. Moreover, we have we know this dimension of V is equal to this is rank nullity theorem you can say dimension of W plus dimension of V mod W. Right I mean how does is come. So, if I look at I have this exact sequence. What this, what rank nullity theorem says is that dimension is additive on the short exact sequence. Dimension of W minus dimension of V plus dimension of this is 0. So, is this true for the case of modules?

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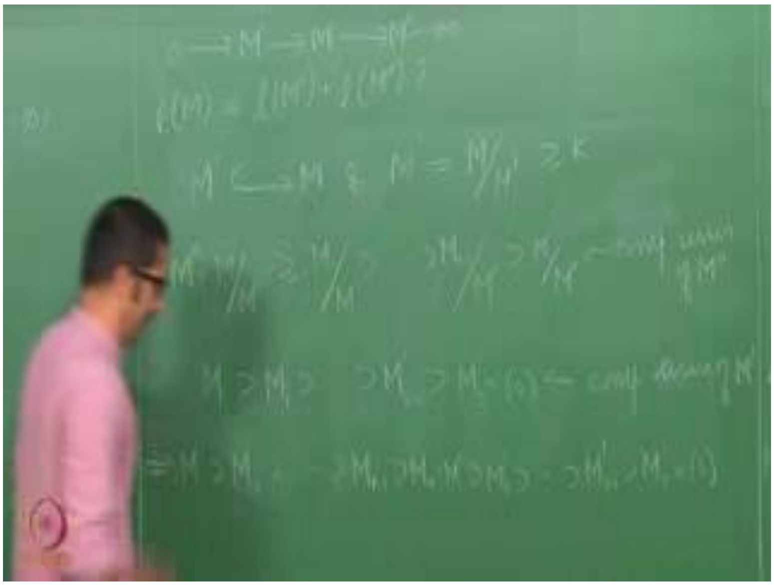

So, suppose I have you know M prime to M to M W prime to 0. Modules of all of them have modules of I mean all of them are modules of finite length.

Now, suppose I have a, say I want to say that is it true that length of M mod length of M is equal length of M prime plus length of M W prime. I want we have to produce one composition series of which satisfy this property. Composition series of M with length equal to this plus this. Can you think of one? So, here you can think of m prime as a sub module of M and M double prime as M mod M prime right. So, you can think of this M prime as a sub module of M and M double prime equal to M mod M prime M mod M prime has a composition series.

So, let us look at M double prime. See this will be this has a composition series. Any sub module of this is of the form some you know sub module of M mod containing M prime modulo M prime. If I have a sub module K of this then K I can think of this as some N mod M prime, where N is a sub module of M containing M prime right.

So, I have I can write like this M prime, which is m mod m prime contained in M 1 mod M 1 mod M prime M r minus 1 mod M prime and 0. What is 0? This is M prime, mod M prime right. This is a composition series of M W prime. Now suppose I have a composition series of m prime m prime contained in M 1 prime M r M s. So, this is M s minus 1 prime contained in M s prime which is 0.

Now, can you tell me a composition series for M? This composition series of M double prime, this is composition series of M prime.

Student: (Refer Time: 47:43).

M, right. M, M 1 contained in M r minus 1, M r which is same as M prime contained in M 1 prime m s minus 1, prime M s prime equal to 0.

Now, look at each quotient M mod M 1. M mod M 1 is same as this modulo this, but this is a composition series. Therefore, this modulo this is simple. This modulo, this is M mod M 1. Therefore, this is simple. Similarly, at each stage the corresponding quotient is same as the corresponding quotient. Therefore, each time to simple therefore, this is a, length is additive on short exact sequences.