Commutative Algebra

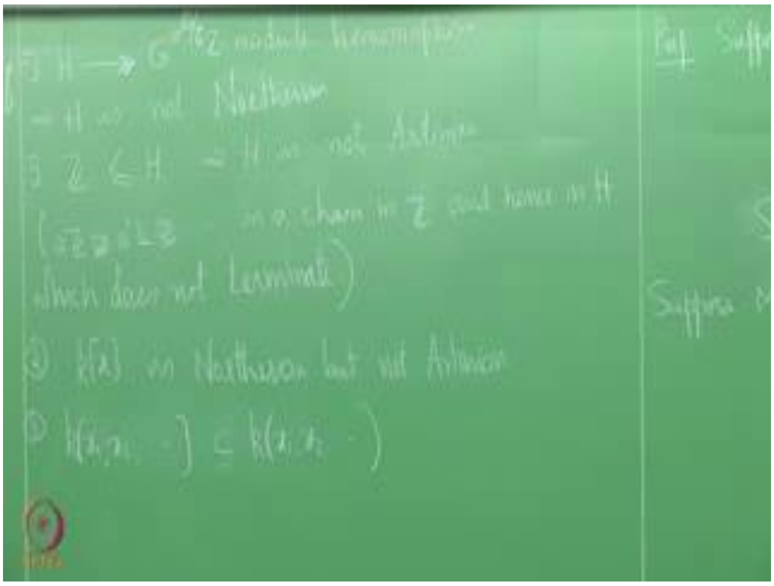
Prof. A. V. Jayanthan
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 28
Noetherian and Artinian Modules

(Refer Slide Time: 00:27)

So, let us recall that an A module M is said to be Noetherian if it satisfies the ascending chain condition ascending chain condition and M is said to be Artinian if it satisfies the descending chain condition so we saw some examples of Artinian modules Noetherian modules, modules which are not Noetherian, but Artinian and modules which are not Artinian, but Noetherian. Another example did we discuss this example - look at this fix a prime p and take H to be set of all a by p power n, a belongs to Z and n bigger than to 0. H is neither Noetherian nor Artinian right. Because it this has if you take, so, you can think of this see this is a Z module. The group that we discussed last time that g is all elements of order a power of prime m power of p.

(Refer Slide Time: 02:50)



So, you have a surjective map from H to G and you take any ascending chain that will lead to an ascending chain here which does not terminate so here an ascending chain cannot terminate, but any descending chain here descending chain also cannot terminate. If I take what are the subgroups of H there are no finite subgroups. Here because if you take some a by p power n is there then you know.

Yeah, n a by I mean m a by p power n is also there for all m and z so you can easily see that now one can produce a descending chain H is not Noetherian, there exists on to Z module homomorphism. Z is not therefore; Z is not Noetherian. And you can see this other way H is not Artinian because Z is a sub module of H right. Z is a sub module of H implies H is not Artinian. It has all Z here right.

Student: (Refer Time: 05:20).

See if I have a; you take an ascending chain here; suppose we know that every ascending chain terminates.

Now look at that ascending chain, in G, in G it cannot terminate. There exists a chain in G and we should say we should. In fact, say that way like there exists a chain in G which does not terminate right. G 1 which is group of all G 1 is the identity element G 2 is all yeah all elements of order p square then G 3 is all elements of order p cube and so on. Now look at it is pullback in H and then that will be a strictly ascending chain which will not terminate at all so therefore, this is not Noetherian. Similarly, Z is contained in H so I can think of this 2 Z contained in, 2 square Z contained in you know this is a chain in H in G and hence in H which does not terminate. This is a strict chain of sub modules in Z right.

Student: We can say that.

In Z. Another example is k x is Noetherian, but not Artinian.

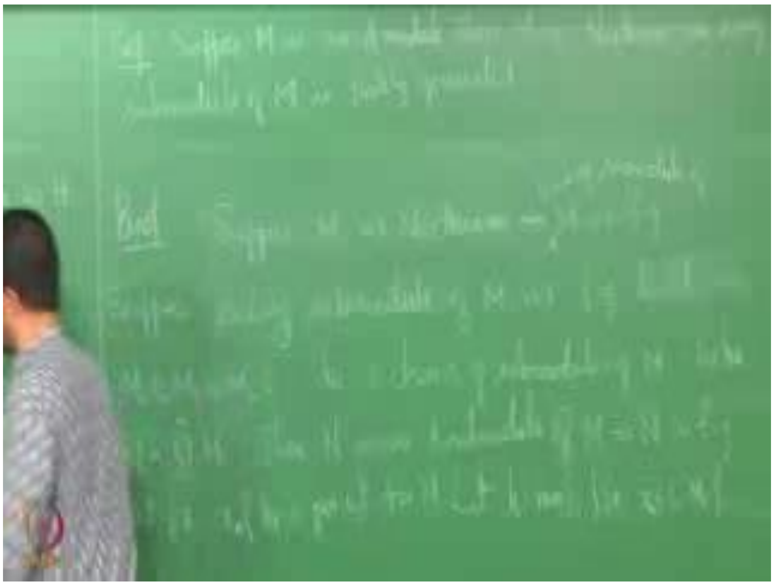
Student: Sir, do we (Refer Time: 08:13).

G is the, so, G is sub group of Q mod Z with all elements of order which is a power of p, k x is Noetherian, but not Artinian. And if you look at this see Noetherian property is not, it does not pass on to the substructures always.

For example, if you take k x square x y x y square this ring this, this is a sub ring of k x y well so let me not do this right now because we do not know whether k x y is Noetherian or not so let us look at this k x 1 x 2 infinitely many variables. This is not Noetherian. It is this is neither Noetherian nor Artinian. Because you can have this chain x 1 contained in x 1 x 2 contained in x 1 x 2 x 3 and so on. That will be a strictly increasing chain which never terminates so therefore, this is not Noetherian, but you can have this if you look at the field, the fraction field of this ring. Any field is Noetherian because there is only one ideal in a field one proper ideal which is 0.

So, therefore, if you look at all sub modules of I mean the field it is there, there is only one proper sub module there is only one. Or if you collect all sub modules there are only 2 which is the whole ring and 0 itself so therefore, this a field satisfies ascending chain condition descending chain condition that is it but so therefore, this is Noetherian, but this is not sub rings need not be Noetherian, even if you are the original ring is Noetherian, but now in the case of modules this is not precisely the case.

(Refer Slide Time: 11:04)



Suppose I have a Noetherian. M is a Noetherian a module. Sorry M is an A module. So look at the collection of all, see if a module is not finitely generated. Suppose a module is not finitely generated then which means that there exists a collection of infinites infinitely many elements which forms a minimal generating set and that gives me an infinite chain which see if I have a if this is you know alpha in i let us let us just write n, n and n. Suppose this is a sub collection of generators minimal generators then I can always and I if I take M n to be the module generated by x 1 up to x n then M 1 is contained in M 2 contained in M 3 contained in M 4 and that will be a strictly increasing chain and that never terminates which means if your module is not finitely generated it cannot be Noetherian.

Now if a module is Noetherian can we say that it has to be a finitely generated a module how would you say. So let us suppose M is yeah M is Noetherian that implies that M is finitely generated. That is what we showed just now and if you if it has if it is not finitely generated then they there exist a infinite chain.

Now, suppose, we are trying to see if it is if and only if, suppose M is a finitely generated a module we are not saying we have not said anything about A it is a finitely generated A module, can we say that M is Noetherian. See for example, in this case suppose my if this is my A and this is my M then is M finitely generated a module.

Student: (Refer Time: 14:50) Z.

No I am saying this ring is A which is also equal to M. It is finitely generated because 1 is a generator right, but is M Noetherian? No. So we want something more. If I take A is you know an arbitrary ring and M to be a finitely generated A module that need not necessarily say that M is Noetherian. So this is not enough. What we have is, suppose every sub module of m is finitely generated, every sub module is finitely generated, can we say that M is Noetherian? So I want to say whether M satisfies ascending chain condition or not. So let us start with a chain. Let M not contained in M 1 be a chain of sub modules of M.

Now, how do I say whether this is finitely generated or not I mean whether this terminates or not, see from this chain I have to find a sub module. What should be, see if I get a sub module I can say that it is finitely generated.

Student: Sum (Refer Time: 17:22).

Ah.

Student: sum of the all M.

So, if I take N to be sum of M i, i from 1 to infinity 0 to infinity what if I take union.

Student: It is a chain.

It is a chain this is same as union as well. Then this is indeed a sub module of N, take N to be this then N is a is an A sub module of M. That implies N is finitely generated and that implies so N is finitely generated means it has a finite generating set. So let x 1 up to x m be a generating set for N. Now what do I do?

Student: (Refer Time: 19:06).

I look for the least k which contains all of them. They are finite. This set is finite therefore, it has to be somewhere at some finite stage, this will be contained in all of them; that means, N will be contained in that particular module, sub module; that means, N will be equal to that which means from there it terminates. So let k be the minimum i such that x 1 to x m is contained in m I therefore, x k will be sorry m k will be equal to if those elements are there in m k that would imply that N is contained in M k.

(Refer Slide Time: 20:06)

But by definition all M k's are contained in n that implies N is equal to M k and that would imply that N is equal to M k equal to M k plus 1 and so on. So therefore, the chain terminates. That implies M is Noetherian.

So we are saying here that, if every sub module of M is finitely generated then M is Noetherian. Here we showed that M is Noetherian implies M is finitely generated. Can we modify this and say that every sub module of M is finitely generated? The proof just simply goes through right. Instead of suppose there exists a sub module which is not finitely generated we still can generate the same in the same way you can generate the ascending chain condition which never terminates. So therefore, we can say that every sub module of M is finitely generated. So what we proved now is that then M is Noetherian, if and only if every sub module of M is finitely generated. I leave it to you to write down this part of the proof. Complete it. Now, suppose I have a module M and a module N. If I have a strictly increasing chain which never terminates here, that will give me a strictly increasing never which chain which never terminates here.

I have an onto map or in other words, if this is Noetherian, then this is Noetherian. Similarly, if I have a descending chain which never terminates here, then I have a descending chain which never terminates here. Look at the pullback see if I have any sub module here I can have a phi inverse N i. If this is if I have a chain like this here, that will give me a chain like that here and this is never terminating will imply this is also never terminating. Or in other words if this is Artinian, this has to be Artinian. So let M and N be A modules. And phi from M to N be an onto A module homomorphism. Then if M is Noetherian, respectively Artinian, then so is N, that is N is Noetherian and respectively Artinian, follows directly from the in surjectivity property. If this is Noetherian then this is Noetherian this is Artinian, this is Artinian. Now what happens if I have, suppose I have an exact sequence. By the way is the converse true. If this is I mean Noetherian for example, if N is Noetherian can we say M is Noetherian.

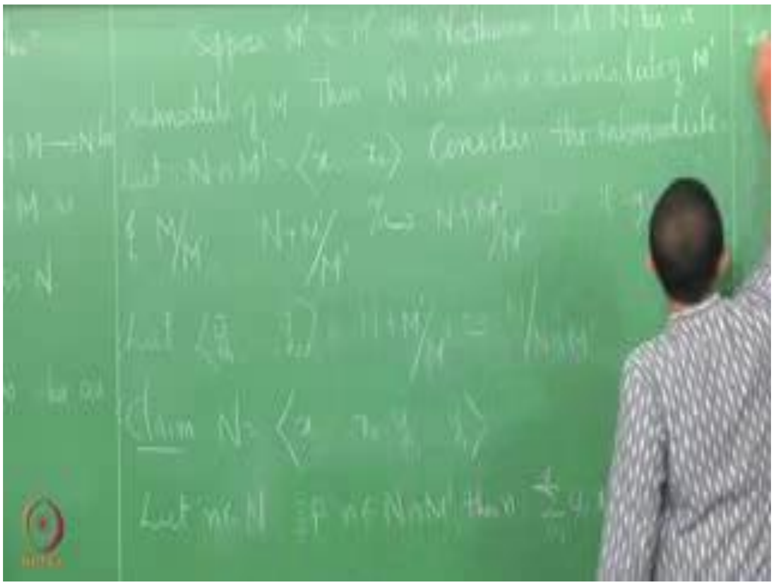
Student: We can take g and suppose we map first c 2 as to some set and all the c 1 to same set. We can do it injectively.

What they need is little more, but this is see if I have a surjective map, I mean for example, I will tell you this, k x 1 x 2 so on to k x x 1 there is a surjective map right, x I homomorphism it says put x I equal to 0 for all I bigger than equal to 2. That is the surjective homomorphism from here to here. This is Noetherian, but this is not. So the converse of this is not true, that if N is Noetherian, then M is Noetherian we cannot say. So what more do we require? Let us look at, consider the exact sequence. So let 0 to M prime to M to M double prime to 0 be an exact sequence of a modules. Now if M is Noetherian this is Noetherian, right. Now M is Noetherian every, what can you say about M prime.

Student: (Refer Time: 28:24).

How do you say that? We need to use the previous proposition. If m is Noetherian every sub module of m is finitely generated. In particular, if I look at the collection of all sub modules of M prime that is a subset of the collection of all sub modules of M. Or in other words every sub module of M prime is finitely generated. Which means M prime is Noetherian by the previous proposition. So if M is Noetherian, this both of them are Noetherian. What about if M prime M double prime is Noetherian? We have already seen the example that that need not necessarily imply M as Noetherian. What if we impose both of them are Noetherian? Suppose we say that this and this are Noetherian can we say this is Noetherian?

(Refer Slide Time: 29:30)



Suppose M double prime and M prime are Noetherian. Can we say that M is Noetherian? The idea is what you said, how do you prove that it is Noetherian, we can one way is to use this that every sub modulus finitely generated.

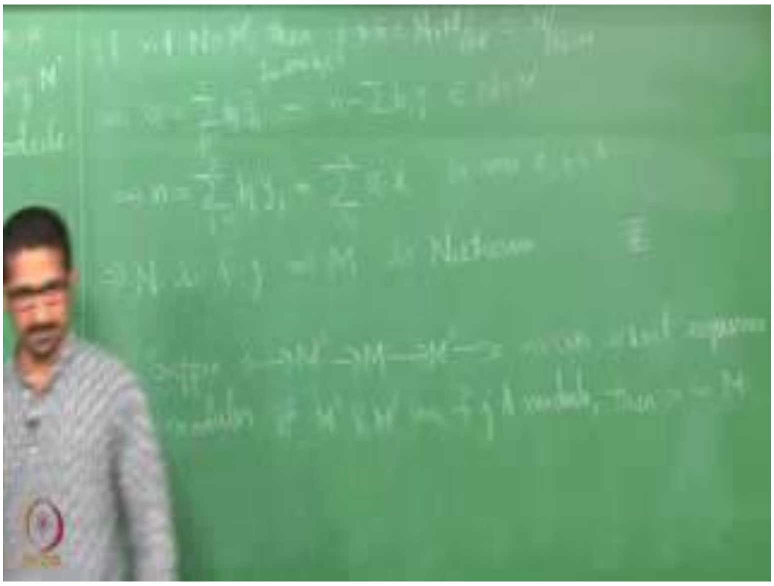
So, let us look at let N be an sub module of M. If I say that N is finitely generated, I am through. Now how do I do this I know that M prime and M double prime are Noetherian. Now see we do not know anything about N. How do we bring in these 2 you know using N how can we get some sub modules of M prime and M double prime? Then N intersection M prime is a sub module of M prime. But M prime is Noetherian therefore, this is finitely generated.

Let N intersection M prime be generated by x 1 up to some x r. Now can you think of some other sub modules of M double prime? What is M double prime? M double prime you can think of it as isomorphic to M mod M prime. And consider the sub mod of m module M prime, N plus M prime mod M prime. This is a sub module of M mod M prime which is isomorphic to M double prime which is Noetherian.

Therefore, this is N plus M prime mod M prime is finitely generated. So let y 1 bar up to y s bar this a span of this be equal to N plus M prime mod M prime. This can also be seen as N mod N intersection M prime. So therefore, what should I claim? N is generated by x 1 up to x r y 1 up to y s. This is again like rank nullity theorem. The philosophy behind the proof is very similar to the proof of rank nullity theorem. So I start with an element y in,

So, let us start with some element n in N. Look at it is you know if this is if n belongs to N intersection m prime then n is n is summation a I x a i x i, i from 1 to r.

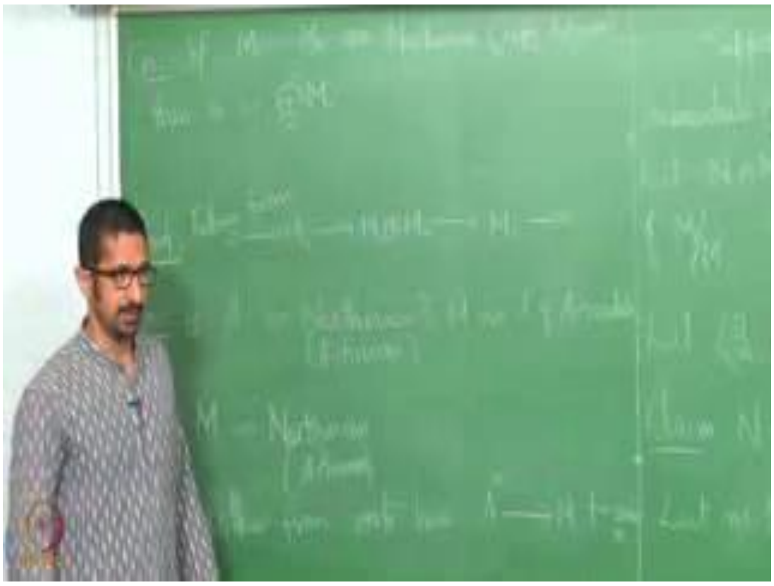
(Refer Slide Time: 34:28)



If x is not in sorry if n is not in N intersection m prime, then this is non 0 in N plus M prime mod M prime which is isomorphic to N mod N intersection M prime. Therefore, I can write n bar equal to summation a i y i, i from 1 to s, I will write this as b j y j j from 1 to s, and that implies that n minus summation b j y j belongs to N intersection M prime. Therefore, this implies, but n intersection M prime is finitely generated and it is generated by x 1 up to x r. Therefore, n minus summation b j y j this is summation a i x i, i from 1 to r. So what we have shown is that n is finite I mean n is generated by these x 1 up to x r and y 1 up to y s.

So, I will simply write like this, n is equal to for some b j in A for some a i b j in A so that implies n is finitely generated that implies M prime is Noetherian sorry M is Noetherian. So therefore, this is what we have shown is that then M is Noetherian, if and only if M prime and M double prime are Noetherian. And this is precisely the proof. I will leave it to you to verify that the same is true for Artinian. So this is Artinian, if and only if this is Artinian. If I have an exact sequence, then one is Noetherian if and only the other 2 are Noetherian. Middle module is Noetherian if and only if the other 2 modules are Noetherian. And similarly Artinian in fact, this is I will leave it as an exercise, that what we proved here is that suppose 0 to M prime to M to M double prime to 0 is an exact sequence of a modules. If M double prime and M prime are finitely generated a module then so is M, we have just simply created a spanning set for M from the spanning set for M prime and M double prime.

(Refer Slide Time: 39:26)



And if you look at the proof it is very similar to the proof of a, if A is Noetherian and M is a finitely generated A module. This is it could have stated immediately after first proposition. Noetherian Artinian, then M is Noetherian.

Student: How many finitely generated for Artinian also.

Yes, so we need one more property. I guess the before this, I need the one more property. Let me this is a corollary of the previous the exact sequence result. If M 1 up to M n are Noetherian respectively Artinian then, so if M 1 and M 2 are Noetherian then you have this exact sequence M 1 to M 1 direct some M 2 to, what is M 1 direct some M 2 modulo M 1?

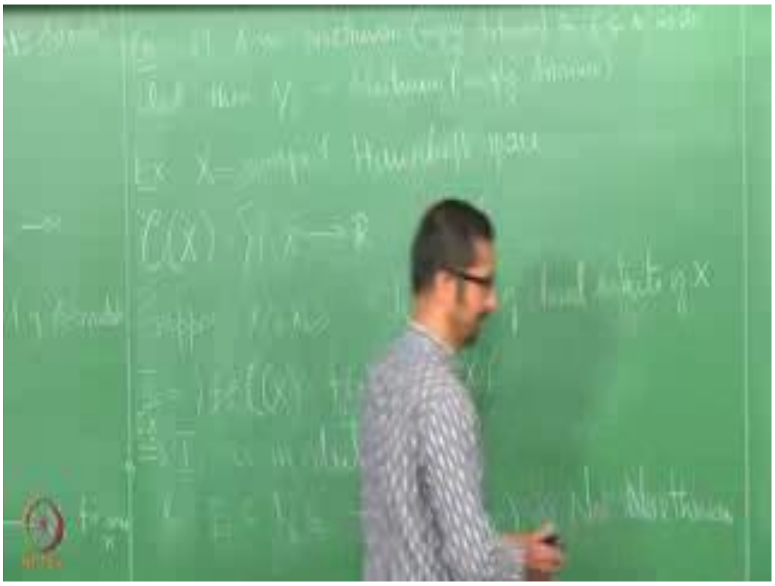
Student: Isomorphic.

Isomorphic to M 2 right. This is an exact sequence; this is if this is Noetherian. And this is Noetherian then this is Noetherian. Similarly, if this is Artinian, this is Artinian then this is Artinian. So another corollary is if A is Noetherian, and M is finitely generated, so here I will write Artinian, A module then M is Noetherian. Again it follows from how do you get this? How are finitely generated modules characterized?

Student: Quotient of some A power n.

Quotient of some A power n right. So from onto homomorphism right, so if A is exact sorry A is Noetherian implies, A power n is Noetherian, direct some of Noetherian modules is Noetherian. Therefore, A is Noetherian implies A power n is Noetherian. And you have a onto homomorphism. Therefore, M is Noetherian similarly if A is Artinian A power n is Artinian therefore, M is Artinian.

(Refer Slide Time: 44:02)



Another corollary is if A is Noetherian respectively Artinian, and I in A is an ideal, then A mod I is Noetherian. Again it follows from the exact sequence A to A mod I. So we have already seen some examples of rings which are neither Noetherian nor atinian and so on.

So, here another one more important example, that if I look at this X be a compact hausdorff space. And X 1 X 2 yeah, so look at this, C X to be set of all continuous real valued functions. This is a ring under point y sedation multiplication. We have looked at this for X equal to the closed interval a b, but one can do this for any con compact hausdorff space. And I look at this, suppose X 1 contained in X 2 be a chain of closed subsets of X. I write I i to be set of all f in C X such that f of a is 0 for all a in X i.

Vanishes on every points of X i. Then this is an artherian. One can check, exercise that I i is an ideal in C X and I 1 is contained in I 2 containing so on. So if this is a strictly decreasing chain, then this is a strictly increasing chain. And that would imply that C X is not Noetherian.

(Refer Slide Time: 47:40)

So is C X atrinian? Think about.