Commutative Algebra
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Lecture – 27
Chain Conditions

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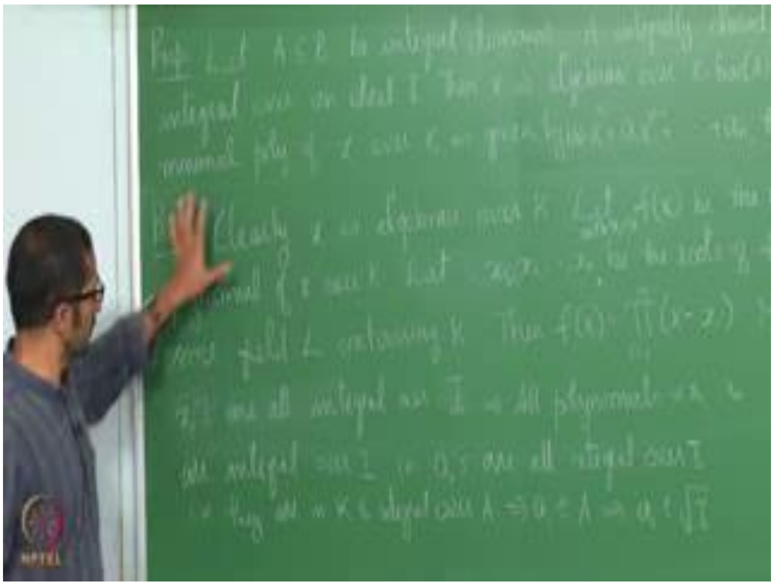

So we will quickly complete the proposition that was left over from the last class. So, this is the proposition that we looked at the end of the last class. So, if A contained in B are integral domains and A is integrally closed. Suppose we have an element x in B which is integral over an ideal I then x as an element in the fraction field of B is algebraic over K. What is meant by algebraic over K, it satisfies a polynomial when you are over fields. If it satisfies a polynomial it will satisfy a monic polynomial because you can divide by the leading coefficient and say make the polynomial monic, but that is not the case when over rings.

So, integral and algebraic have 2 different meaning in over rings, but in over fields they are all same anyway let us just assume that it satisfies a polynomial. So, it is algebraic over K this statement is fairly straightforward because it is integral over an ideal. So, it has a monic polynomial with coefficients coming from I which is already there in K. So, it is certainly algebraic over K, but the second statement is more important that if I take the minimal polynomial over K then all these coefficients are coming from the radical of I.

So the first statement is fairly straightforward because it is integral it is algebraic. Now I take the minimal polynomial. So, I have x in sum I mean extension in some field see, I mean as a fraction field of B fraction field of B will contain fraction field of A. So, as an element in fraction field of B I can think of element x in fraction field of B over K. This element is algebraic over K that is the first statement therefore, there exists a polynomial. So, I take the polynomial to be this f x minimal polynomial, what is minimal polynomial.

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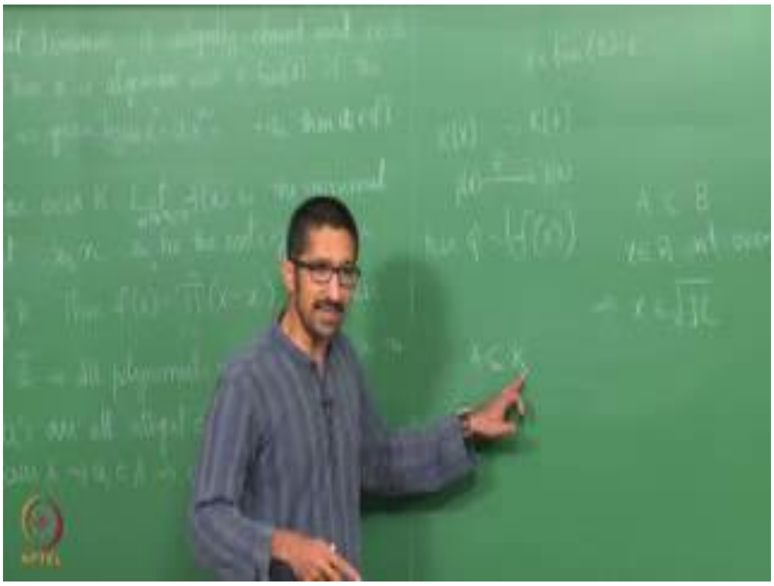

So, if I have I have an element x which is in frac B and this contains K. So, let me call this K prime now if I look at this K x and I have this surjective homomorphism the variable X going to the element x p of capital x going to p of small x this is a ring homomorphism. This is a field, there this is surjective therefore, the kernel is generated by and the reducible nonzero irreducible polynomial. There exists something in the kernel because x is algebraic right. X is algebraic therefore, there exist some polynomial which is which goes to 0. So, therefore, kernel is non 0 and it contains a polynomial, for this has to be generated by a non 0 irreducible polynomial because this is a field.

So therefore, this is the kernel is generated by kernel phi is generated by some polynomial f x which is because this is a pid kernel has to be a maximal ideal therefore, it is generated by a nonzero irreducible polynomial and hence this is irreducible. And that polynomial the monic polynomial which is irreducible which generates kernel phi is called the minimal polynomial. That is unique the monic polynomial which that is uniquely determined by this process. So, I take the minimal polynomial.

Now let us look at all the roots of this, in some it need not necessarily be in K itself not necessarily in a fraction field of B K prime itself it will be in some field somewhere. For example, if you take the polynomial x cube minus 2 and B to be q cube root of 2. Then all the roots of this is not here right it will be in somewhere. So, just look at those all the roots and I mean look at the field where all the roots or you just attach those elements to the base field K, look at that field. Then you can write this f x as product of x minus x i.

Now another thing that one can notice is that see f x will divide, see suppose x is integral over i right. So, that will satisfy an integral equation that may be different from this f x because this is f x is the minimal polynomial over K, but the other one is integral over i that may be a bigger. So, what I mean if I have an integral equation. See if the integral equation is given by x power m plus b 1 x power m minus 1 plus etcetera, b m equal to 0. Then this polynomial x power m plus b 1 x power m minus 1 plus etcetera b m, this polynomial will be there in this kernel. Therefore, f x will divide this polynomial which means all these roots x i's they are all integral over I because this is an integral equation with coefficients coming from i. We have this is b i in i we have taken the integral equation of x. So, this means all the x i's will b integral over I.

Now in the previous theorem, we showed that the set of all integral elements will form an ideal in the integral closure. So, what we are saying is that x i's are all integral over i, therefore, all the polynomials in x i's will also be integral over i. Now look at what are the coefficients of this. They are all symmetric polynomials on x 1 up to x n right. A 1 is x 1 plus x 2 up to x n a 2 is x 1 x 2 x 1 x 3 summation over all those. So, therefore, all the a i's are integral over i, all the a i's are integral over i.

All the x i's are integral over i. Now what did we prove in our earlier theorem. If I have an extension A contained in B, if I take x plus I mean x comma y in B which are integral over i, then x plus y x 1 I mean they all integral elements here over an ideal I form an ideal.

Student: Radical.

We precisely describe what the ideal is also.

Student: Radical of.

Yeah radical of.

Student: Extension.

Extension of this in the integral closure.

So these set of all elements share which are integral over an ideal forms an ideal in between somewhere. So, therefore, if 2 of them are integral over i their sum is integral over i their product is integral over i and so on. So, therefore, all the polynomials in x 1 up to x n form they are all integral over i. In particular, all these coefficients a 1 up to a n, they are all integral over i. Because a 1 is summation of x i's a 2 is summation over i j x i x j. What is see this is the minimal polynomial. Minimal polynomial is this. So, if you just equate the corresponding I mean degrees.

A i's are all polynomials in x i's. So, therefore, a i's are all integral over i. Now where are this a i's. This is the minimal polynomial of x over K; that means, all of them are in the fraction field of A.

Student: Integrally closed.

A is integrally closed. And we are saying a i's are all integral over i. In particular, they are integral over A, therefore, all of them have to be in A, but then see if I look at all the coefficients they are all integral over A, integral over i, then again applying the previous theorem that. So, I have a ring extension like this and x in B integral over A implies that x belongs to radical of I extension. What is I extension I, I c where c is the integral closure of A in B. Now apply that in the case of A, A to K. A i's here are all integral over I therefore, a i belong to radical of I c, where c is integral closure of a and k which is.

Student: A.

A itself. So, therefore, all the a i's are in radical of I. This requires little bit of field theory as well, but I these are something that you know that we can you know one can explain as one goes. So, that completes the proof of going down theorem. This was what was remaining. So, it would be nice exercise to go through the proof again try to write it down with all details.

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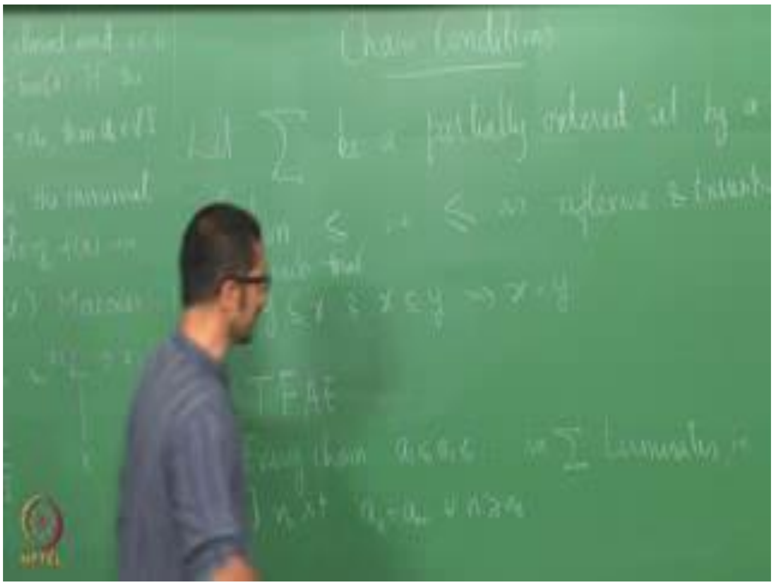

So now let us study what is called chain conditions. So, let sigma be a partially ordered set with, partially ordered set by a relation. I will denote this by less than or equal to, that is what is a partially ordered set. I have the relation that is reflexive and transitive. And why less than equal to x and x less than equal to y implies x equal to y and is such that then. So, first let us see, you know a nice characterization of what is meant by a chain condition. So, the following are equivalent.

Every chain a 1 less than equal to a 2 in sigma terminates. That is there exist n, n naught such that a i equal to a n equal to a n plus 1 for all n bigger than to n naught. That is a n naught equal to n naught plus 1 equal to n naught plus 2 and so on.

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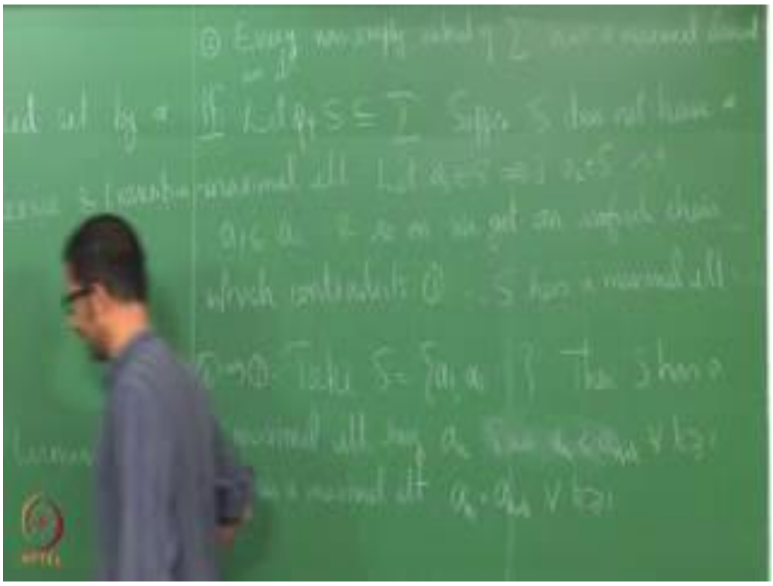

This is equivalent to saying that, every non-empty collection non-empty subset of sigma has a maximal element. So, this is fairly straightforward, but let us look at the proof. Let us assume that every chain in sigma terminates, and let us try to prove non-empty subset has a maximal element.

So let us take S to be a non-empty subset of sigma. I want to say that this has a maximal element. What is meant by it does not have a maximal element. I take every element there will exist something.

Student: Bigger than.

Bigger than that; See if S is finite then we are through. If S is not finite, then I start with an element a 1 and S. So, suppose S does not have a a maximal element then. What do we have? Let a 1 be in S this implies there exists a 2 in s a 1 in S cannot be a maximal element how can it not be a maximal element there exists something bigger than that were exists a to in S such that a 1 is less than equal to a 2 and then.

Student: a 2.

I keep saying this a 2 is not a maxim al element therefore, there exists something bigger than and keep going. So, we get an infinite chain which does not terminate which contradicts our hypothesis 1. So, therefore, S has a maximal element. Conversely to prove 2 implies one what do I do.

Student: Form the subset of a 1 a 2 collection of that.

Exactly. So, here I want to say that my assumption is that every non-empty subset of sigma has a maximal element. And I want to say that this chain terminates. So, take S to be the collection of all these elements then S has a, So, I should say every non-empty subset of sigma has a maximal element in it. Or every non-empty subset contains a maximal element. So, then S has a maximal element and that gives me. So, call it a n, but a n is less than equal to, So, a n says a n naught. Now by this what is given to us a is that a i is less than equal to a i plus 1. So, therefore, a n naught is less than into a n naught plus 1, but a n naught plus a n naught is a maximal element therefore, a n naught is equal to n naught plus 1 and. So, on since a n naught is less than equal to a n naught plus k for all k bigger than 1 and a n naught is a maximal element a n not is equal to a n naught plus k and that is exactly what the statement 2 statement 1 says.

So let us you know now let us look at some examples of such partially ordered set.

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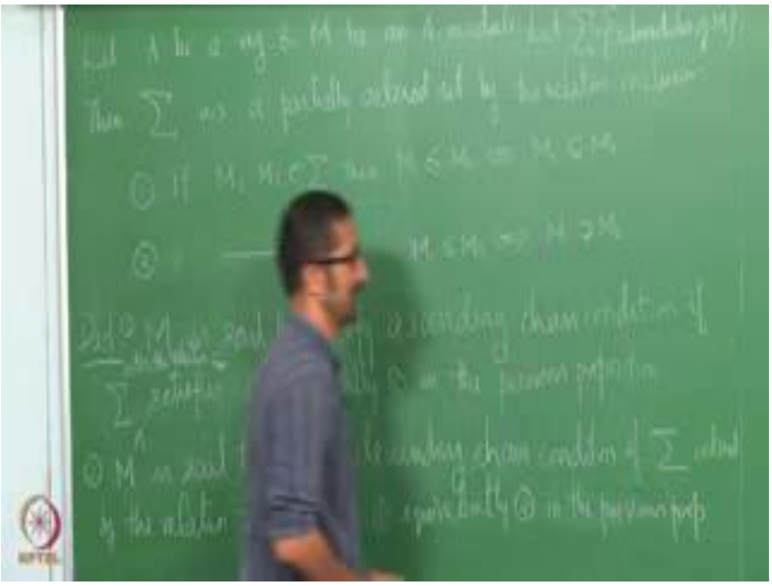

If I take a module, so let A be a ring and M be an a module. Take sigma to be a set of all sub modules of this is a partially ordered set right. It has partially ordered set with by what relation the inclusion. So, there are 2 ways you can say. One is then sigma is a partially ordered set by the relation inclusion. Now there are 2 ways you can say that, one is that if M 1 M 2 are in sigma then M 1 is less than equal to m 2 if and only if M 1 is contained in M 2. This is one way of saying another way is then M 1 is less than equal to M 2 if and only if M 1 contains M 2. M 1 is less than equal to M 2, if M 1 is a subset of M 2 that is one way another is M 1 is less than equal to M 2 if M 1 is a super set of M 2 or M 2 is a subset of M 1 both of them give a partial order structure to sigma.

Now we say that M is said to satisfy ascending chain condition if sigma satisfies 1 equivalently 2 in the previous proposition. That is every increasing chain in sigma terminates or every non-empty subset of sigma has a maximal element. So, there is one more sigma with the relation subset. This is 1 2 is M is said to satisfy descending chain condition, if sigma ordered by the relation by this I mean M 1 is less than equal to M 2 if M 1 contains M 2 the second one, which we described here satisfies one equivalently 2 in the previous proposition.

Now let us look at some examples of this cossets partially ordered sets which satisfy ascending chain condition descending chain condition.

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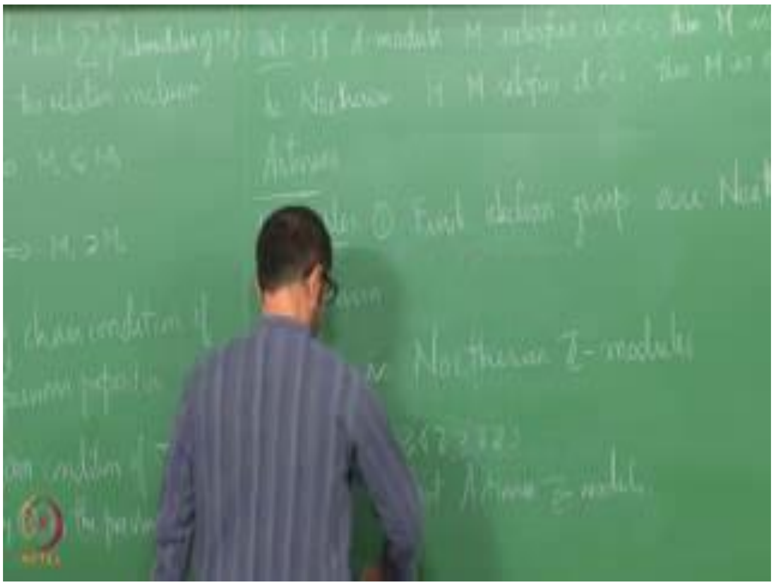

So, let me just make this definition once again if an a module m satisfies ascending chain condition then M is said to be Noetherian. And if M satisfies descending chain condition then M is said to be Artinian.

Student: Sir, this is Noetherian module.

Noetherian module, a ring is Noetherian if it is a module over Noetherian module over itself. We will see equivalent conditions and so on. So, let us look at some examples of

Noetherian Artinian and. So, on what can you say about finite abelian groups? They are all Z modules right. As a Z module can you say whether they satisfy ascending chain condition descending chain condition and so on?

Student: (Refer Time: 28:52).

There are only finitely many elements there. So, if you take any ascending chain it has to terminate there is no way. And similarly if you take any descending chain it has to terminate because there are only finitely many elements in that. So, therefore, are Noetherian and Artinian. Now what about Z? Does it satisfy ascending chain condition?

So let us start with let us say 36 Z. Can you tell me ideals that are you know that contain see we are looking at an ascending chain? So, what would be can you think of some?

Student: 18 Z.

18 Z.

Student: 6 Z.

6 Z, 3 Z; however, we take the factorization, if you take a chain like this it will terminate right. So, more general justification is if I have p 1 power alpha 1 up to p n power alpha and if your ideal any integer I mean this is a pid, therefore, every ideal is generated by some K Z every ideal is K Z. Now if I take k to be the prime factorization to be this maximum number of elements in the ascending chain can be alpha 1 plus etcetera alpha n right. If you take any ascending chain, it will have you know the see the there are finitely many ideals containing this one.

Student: What about the sigma n?

Sigma is set of all sub modules of you know what are the sub modules of Z.

Student: Ideals.

Their ideals n Z right sub groups of Z or ideals of Z they are all of the form K Z. So, if I take an ideal I there are only finitely many ideals containing this ideal. Therefore, if you take any ascending chain you start from here you go ahead there are only finitely that contain this one. So, you cannot have an infinite chain. So, therefore, this satisfies. So, this is Noetherian.

Student: Sir you say sir a Abelian group form a sigma.

Everything is ok, for I mean finite Abelian group is any Z module. My sigma whenever I consider module my sigma is set of all sub modules of the module under consideration I fix a module and then I look at sigma that sigma is collection of all sub modules of M. So, Z is Noetherian I mean does Z satisfied descending chain condition.

Student: 2, 2 power 4, 2 power 8.

You can write you can have 2 Z, 4 Z, 8 Z. This is a strictly descending chain that never terminates. So, therefore, Z is not Artinian.

So we have now seen an idea module which is both Noetherian and Artinian this is Noetherian, but not Artinian.

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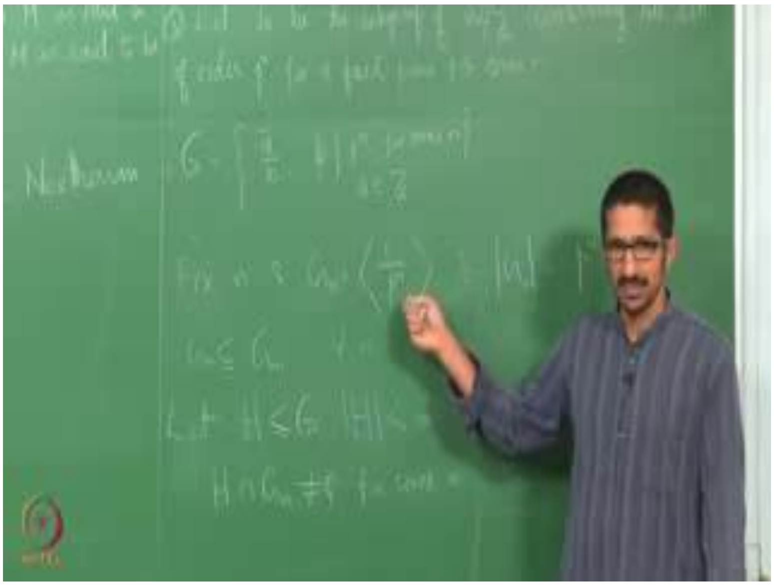

Now, let us look at an example where this is Artinian, but not Noetherian. So, let us take consider the subgroup G with a subgroup of Q mod Z. I am looking at the subgroup of Q mod Z which contain containing the all elements of order p power n for some for a fixed prime p and some n; that means, when is it and what are the elements of G? I should have some say a by b bar such that p power n times a by b is an integer. Or in other words p power n times when is see what is the order here. Here this is additive subgroup right we are looking at the additive subgroup of Q mod Z. So, what is mean by an element having finite order a p power n order you sum this up p power n times it should be 0, but what is 0 here integer. Or in other words it should be p power n I mean b should divide p power n right. So, therefore, what we are looking at is and a is in integer.

Now we want to check whether you know this is this satisfied ascending descending chain conditions. So, if I look at the element 1 by p power n. This is what can you say about the I mean the bar of the equivalence class, I will remove the bar when I write one by p power n I mean the corresponding equivalence class here. What is the sub module generated by this? I fix this n fix n, and G n be the sub module or subgroup generated by 1 by p power n. And what would this be 1 by p power n 2 by p power n 3 by p power n and so, on right. And p power n minus 1 divided by p power n right. What is what would be minus 1 by p power n will this be an element here what would this be?

Student: p n power over minus n.

Ah?

Student: p power n upon 1 over n.

This is what?

Student: p power n minus 1.

By p power n. So, similarly minus 2 minus 3, so they are all here. So, therefore, this is a group of order?

Student: p power n.

P power n, right. Now g n is what can you say about g n and g n plus 1 can you say something about these 2 subgroups.

Student: Order of g n plus 1.

Of course order of G n plus 1 can you have some comparison. Order of G see this is for any n order of G n is G n is p power n.

Student: G n is contained in g n plus 1.

G n is contained in g n plus 1 right. And if you take any element in g it will be in one of those G i’s right. Moreover, suppose I take any subgroup H of G. I take a subgroup H of G, H be a subgroup of G. So, let us look at finite subgroup finite. If it is a finite subgroup I would say I am saying that H has to be. So, this means sees every element of H every element of G will be in one of those G n’s so; that means, H intersected with G n this is non-empty right for some n. This is non-empty for some n. So, let a by p power n be in H, the element of largest order, be an element here there could be many. See for example, in this one this is not the only generator of G n right. If I take any element which is co-prime with p right a if a is co-prime with p a by p power n will generate G n.

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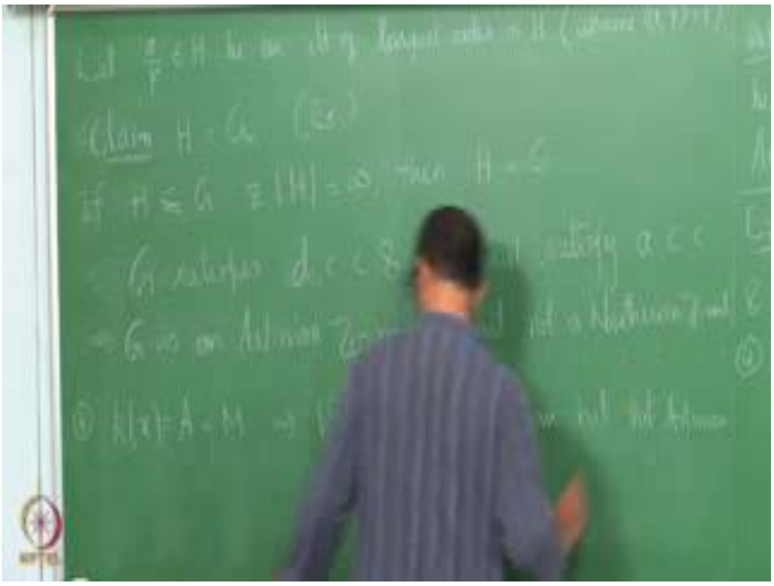

So, we do not really have to look at elements which are only one. So, there would be many generators similarly there could be many elements of largest order.

So I look at be an element of largest order in H. I will assume a p because now we are anyway talking about rational. If a by p power n if a has a factor of p I can always cancel this, then I claim that h is G n I mean that is kind of clear from what we have. So, I will I will leave this as an exercise. So, what we are shown here is that every finite sub group is G N. Suppose H is an infinite sub group what does that mean? It will have, it will contain a b a a by p power n for every n where a and p are co prime, but then that will imply that it will contain all the G n’s and that would imply that it is equal to G. So, therefore, if and this is infinite then H is equal to G. So, what we have shown here is that the only subgroups of G are of the form G n.

Now this is an ascending chain right. So, that will never terminate right. This is g 1 contained in g 2 contained in g 3 and so on. That is an ascending chain that never terminates. Therefore, our g does not satisfy the ascending chain condition. It is not a Noetherian module, but now what about the descending chain condition? I start with a sub module of G, how does any sub module of G look like?

Student: G n.

G n for some n, now you start with G n there are only finitely many G m contained in G n. So, therefore, it satisfies.

Student: Artinian.

Artinian property or descending chain condition. So, this implies therefore, G satisfies yeah descending chain condition and does not satisfy ascending chain condition therefore, G is an Artinian Z module, but not a Noetherian Z module.

Now if I take k x, what are the k x sub mod. So, this equal to A equal to M. What are A sub modules of k x?

Student: Ideals.

Ideals, right. So, if I take sigma is collection of all ideals. Now how does, can we say something about ascending chain condition, descending chain condition and so on?

Student: Exactly as Z chain condition.

Exactly like Z right. If I take any ideal here is a principle ideal and this is the unique factorization domain. So, therefore, I have a every ideal is some you know. So, there are given any element it has only finitely many factors. So, therefore, given any ideal there are only finitely many ideals containing the given ideal therefore, ascending chain condition is always true. But descending chain condition is not true because you can start with f x containing f x square containing f x cube and so, on. So, therefore, this is Noetherian, but not Artinian. So, we have seen examples where it is both Noetherian and Artinian they are either Noetherian, but not Artinian or it is Artinian, but not Noetherian.

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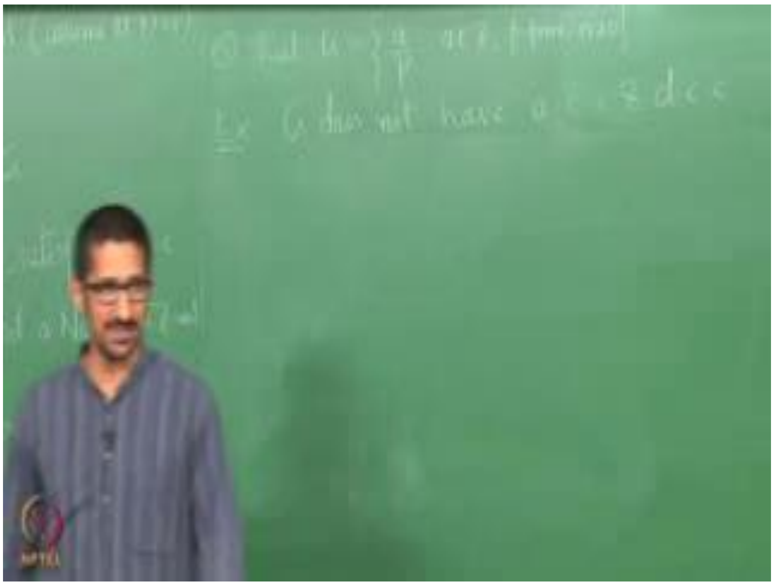

Now, if I have this example fix a prime p, sorry. G be equal to a by p power n a in Z p prime n bigger than into 0; that means, I am taking all the not fixed point p see I am writing p a prime here which means p is varying. I am looking at all the primes I am not looking at elements of the form say for example, 1 by 6, only some integer divided by power of prime. Now G cannot satisfy. So, this is let me write this as an exercise, G does not have and descending chain condition as a Z module G does not have acc as well as dcc. We will prove more about Noetherian and Artinian modules later.