Communicative Algebra
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Lecture – 26
Going- Down Theorem (Continued)

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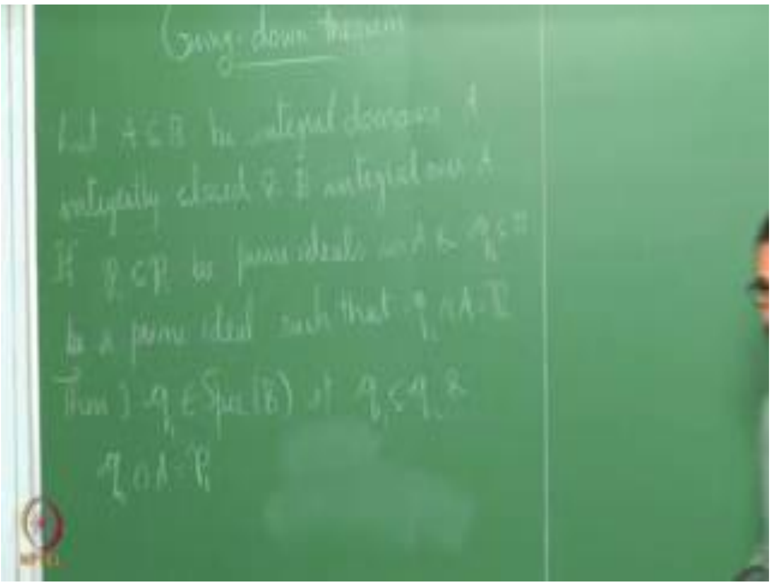

Proving the going down theorem. Let A to B be integral domains. A integrally closed and B integral over A. If p 1 contained in p 2 be prime ideals in prime ideals in A and q 1 be prime ideal q 2 prime ideal such that q 2 intersection A is p 2. Then there exists q 1 prime ideal of B such that q 1 is contained in q 2 and q 1 intersection A is p 2 p 1; that means, you start yeah.

Student: The statement is p is integrally closed.

Yeah.

Student: Integrally closed varies A.

Yeah.

Student: And you asked B integral over A.

So, when you say a is integrally closed it means it is integrally closed in B, in the fraction field of A. So, for example, if you take see Z this is integrally closed, but you have this is an integral extension right.

So, integrally close means Z to Q is now in Q there is no integral extension of Z, but it could be somewhere else. So, that is what we mean by this. B will not in fact be contained in fraction field of A. In fraction field of A there are no elements which are integral over A, but not contained in A. So, B is integral over A. So, I mean it is says you start with a prime ideal chain here, and you have a in above p 2 there is one, then you can go down and get a prime above p 1.

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So, we were halfway through the proof. So, enough to show that p 1 b q 2 intersection A is p 1, we showed that if x belongs to p 1 b q 2, then x is integral over p ,1 at this is something that we showed yesterday. Now suppose x by s belongs to p 1 b q 2.

Student: Sir we started with x by.

Sorry, this or this one right if x belongs to p 1 b then x is integral over p 1 that is what we showed yesterday.

Student: Yes, sir.

Right. Now suppose we have some x by s in the intersection.

Now, we wanted to say that this belongs to.

Student: p 1.

P 1 right p 1 is contained here. So, we need to show that this is in p 1. Now s, see here s belongs to B not in q 2 right. So, first of all this x belongs to p 1 B therefore, it is integral over p. So, let us write x the let x power n plus a 1 x power n minus 1 plus etcetera a n equal to 0. So, I will write this as you know minimal polynomial of or the minimal equation, minimal integral equation; minimal integral equation of x over A. See here one has to be slightly careful in the sense that, see there exists a minimal polynomial for x over A. Suppose it is reducible, so, this is something that if f is a monic. So, this is I will just write this an exercise, A contained in B integral extension. We do not need integral extension itself. Let f be a monic polynomial, sorry this is only a ring extension because does not monic polynomial in f x in A x. Suppose f x is equal to f x is equal to g x times h x, this is reducible in B x then the coefficients of g x and h x are integral over A.

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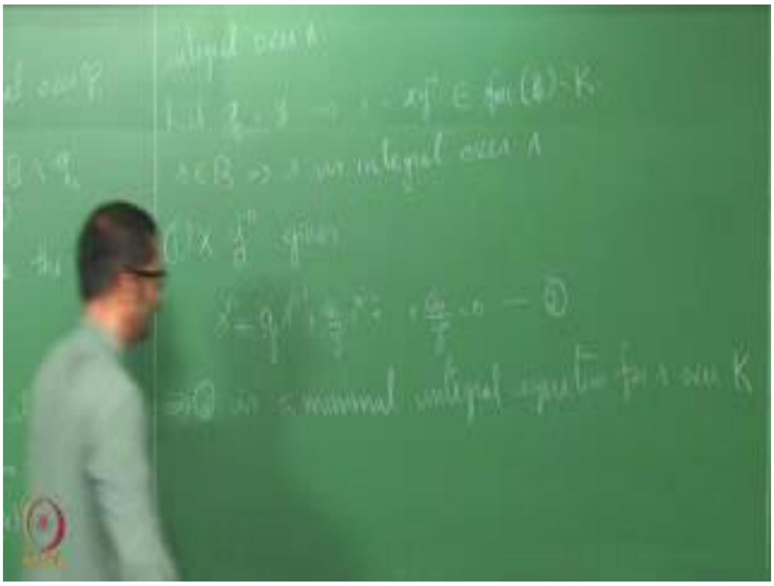

So, see in this case, I can talk about minimal integral equation. Because if I take polynomial, there exists see x, x is in p B 1 and we have shown that it is integral over p 1. So, it start with the equation.

Now, if this is integral, if this is reducible, then I can write it as a product of 2 polynomials each one would be suppose, it is strictly small I mean you can factorize into strictly smaller degree polynomials. Then by this each element is in each element would be integral over A. These are all polynomial these are all elements in p. So, I can say that if it is integral over A, it will belong to A itself. So, therefore, you can assume that this indeed is. So, I mean I am basically at each place if you are reducing it to into a smaller a polynomial each one you can bring it back to A. And assume there exists a minimal degree polynomial. I mean another way to see is look at this A x. So, there exists you know A x to x. So, I have a kernel. Kernel has you know collection of polynomials that will certainly have a smallest degree polynomial. If A is f then it will be generated by one polynomial, but in this case we do not really have to worry about that.

So, let us come back to this. This be a minimal integral equation of x over a now that would imply. So, see x by s, let x by s is equal to y. Then I can write s as x y inverse as an element in fraction field of A. It will be in the fraction field of A itself, right because y is in a sorry no fraction field of B. Now what is y? y is yeah sorry y is in A. Because it is x by s is y i am taking x by s to be y, y is in A x is in. So, this is in fraction field of B, but at the same time see s is in B itself s is in B itself. So, s is integral over A, s is in B therefore, it is integral over A. Last time I will we stopped here. Now s is integral over A. So, s satisfies a minimal equation. See again if I have an equation for x multiplying by y power the degree of this minimal equation we will get a equation for s.

So, let me call this equation 1, 1 multiplied with minus power n gives s power n plus a 1 s power n a 1 by y s power n minus 1 a 2 by y s power n minus 2 a n by y power n this is 0. Suppose I have a integral equation for s, from here I can get an integral equation for x by multiplying by y power n. From here if I multiply by y power n I get an integral equation for x. Similarly given in integral equation for x I can get a integral equation for s. So, therefore, we can assume that this is a minimal integral equation if and only if this is a minimal equation. Because you can you know from one you can get the other. From any one of them you can get the other one by multiplying by the correct power of y. So, this is a minimal integral equation if and only this is a minimal integral equation

So, this is 2, 2 is a minimal integral equation for s over K. Now we use another property here again I will state it I will prove it. That is a not very difficult.

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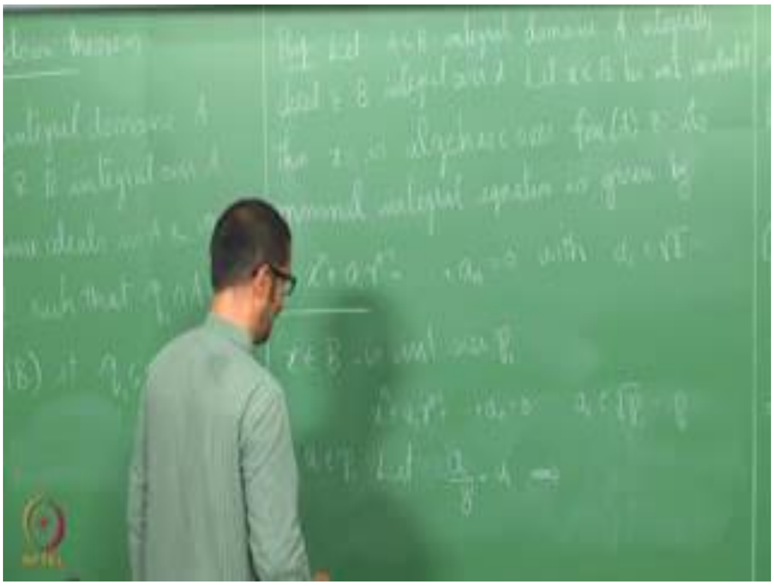

So, this is the hypothesis the first 2 lines same as going down theorem A B integral domains.

Student: (Refer Time: 15:31).

Yes.

Student: (Refer Time: 15:33).

At, this is integral domains A integrally closed and B integral over A. Let x in B be integral over an ideal I. Then sorry this is fraction field of A itself. See y what are we saying let come let me write down this, then x in B x is algebraic over fraction field of A and it is minimal integral equation is given by x power n plus a 1 x power n minus 1 a n equal to 0 with, with a i belong to. So, the see there is it is integral over an ideal implies that there exists a an integral equation with coefficients coming from the ideal, but the minimal relation we only can say that it belongs to the ring. If there are 2, I mean if it factorises into 2 polynomials, what we know is that the poly the coefficients will be integral over the integral domain A.

But here we are saying that the minimal relation will be with a i's coming from radical of the ideal not only from a, but it will b from the radical of i if it is integral over an ideal then the minimal relation will be with coefficients coming from the radical of i. I will prove this, but let us complete the proof of the going down theorem and then I will come back to prove this. So, this x is x in B, x in B is integral over p 1 this is what we proved in the beginning. So, the minimal relation, x power n plus a 1 x power n minus 1 plus etcetera, a n equal to 0 these are with a i because of this one we not only say that a i's are in A, but they are also in the minimal equation. So, we started with the minimal equation. Each a i coming from radical of p 2 which is p 2 of yeah p 2 which is equal to p 2 itself p 2 or p 1. This is integral over p 1; that means, each a i belongs to p 1. Now what we have is y power. So, this is a 1 I mean a i by y power. If I take this to be equal to u i. So, let a i by y i b equal to u i. Then what we have is y power I u i belongs to p 1. We have this integral equation.

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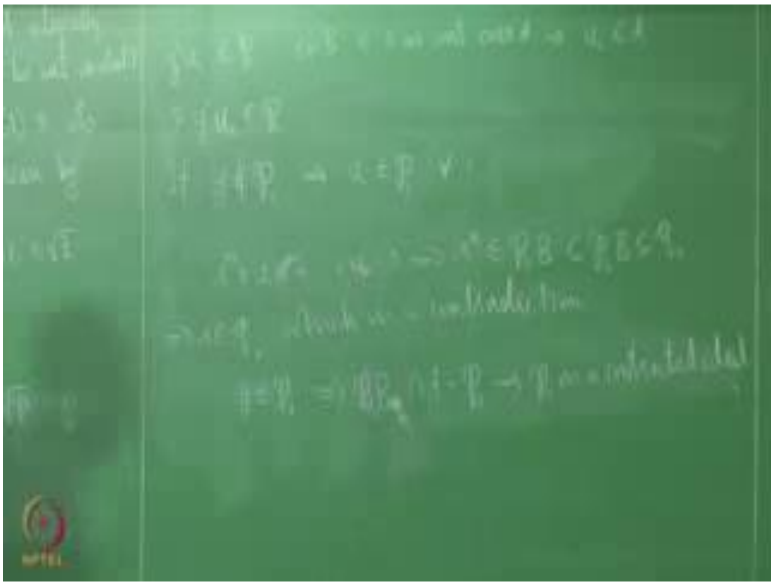

Now, s is so, this is again s belongs to B. We are applying this theorem again. S belongs to p and s is integral over A, therefore, the minimal equation. See this is the minimal relation right. So, therefore, each u i, u I is in A. Each u i is in A what does that mean? U i is in a and y power i, u i belongs to p 1. We want to say that y belongs to p 1. If y is not in p 1 that would imply that u i yeah u i belongs to p 1 for all i. Now we have this what is the relation what is the relation here.

Student: Why y power i?

Sorry.

Student: y power i does not belong to p 1.

Yes, y does not belong to p 1 is p 1 is a prime ideal. So, y is not in p 1 it is equivalent to saying that all none of the powers.

So, if all the u i's are in p 1, that would imply that this s power n. So, I have s power n belongs to p 1 B. Right this would imply that s power n belongs to p 1 B. Because you know s power n plus a 1 by y which is u 1 x power n minus 1 dot, dot, dot u n, this is 0 which means x power n is minus of this, but this portion belongs to p 1 B. All the a i's are in p 1, sorry u i's all the u i's are in p 1. So, therefore, s power n belongs to p 1 B. And that would imply that s power n belongs to p 2 B which is contained in q 2. And that would imply that s belongs to q 2, but how did we choose s, s is in b without q 2.

Student: s power 1 belongs to q 1.

So, we have the minimal equation right. What is the minimal equation?

Student: For s.

S x for s this s, s power n u 1 s power n minus 1 plus etcetera u n is 0. This is the minimal equation for s.

Student: Sir one more thing and you said x belongs to be integral over p 1 then why the coefficients are same a 1 a 2 a n its coefficient for s integral equation for s.

Which one?

Student: This one x belongs to be integral over p 1. So, there exist an integral equation.

Yes.

Student: But why the coefficient a 1, a 2, a n are same for integral equation.

That is exactly what we are saying. So, I have what we showed is that, see we are not only talking about the integral equation. We are talking about the minimal equation satisfied by x. So, x is so, some we have p x is a minimal.

So, let me complete this. I will come back to this first this is this is over. So, this implies that s power n belongs to p 1 B which is contained in p to B which is contained in q 2 and that would imply that s belongs to q 2. That is a contradiction. Therefore, s is in sorry, the I mean the assumption is that y is not in p 1. That is wrong therefore, y belongs to p 1 and that is that says that x by s that is equal to y. So, that says that p 1 B q 2 intersection a is p 1. That implies p 1 is a contracted ideal and that is exactly what we wanted to do.

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Now, let me get back to your question. So, here I have x in p 1 B. So, I can even just take x to be in B what are is integral over A. In fact, integral over p 1. That is what we have proved. Now let us look at this this result. This result says that x is I mean this we can even talk it just take it to be B itself. Does not really matter x in B, is integral over p 1. Therefore, x is integral algebraic over the fraction field and moreover see what is mean by x is algebraic over fraction fit see I have A, I have B.

Now, look at the fraction field of A and look at the fraction field of B, x is integral over A and A is integrally closed means x is not here to start with x has to be here itself. So, therefore, I have a field extension, and x in here is algebra this, which means I have a monic polynomial, and for in the case of fields there is a unique monic polynomial which is satisfied by the given polynomial I mean given element x. So, what we are saying here is that if you take that unique minimal polynomial, it is coefficients come from the radical of this ideal. See here if I look at let me call this to be K.

Now, I look at this K contained in K x, K this x or what are you know both are equivalent in fact. I if x is algebraic over K then these 2 are same this is same as K x. This will be a filled in if I look at this ring homomorphism, p going to x the kernel will be generated by this will be a field this is a pid therefore, kernel is generated by a irreducible monic polynomial. And that is precisely the minimal polynomial of x by definition. Here what we are saying is that if you take that minimal polynomial all these a i’s are in radical of the ideal. So, in our case now, let us come back to our case. This is integral over p 1.

So, let me write this f x to be equal to x power n plus a 1 x power n minus 1 plus etcetera a n. If this is the minimal polynomial of x over A then each a i belongs to the radical of p 1, but that is same as p is a prime ideal therefore, that is same as p itself p 1 itself.

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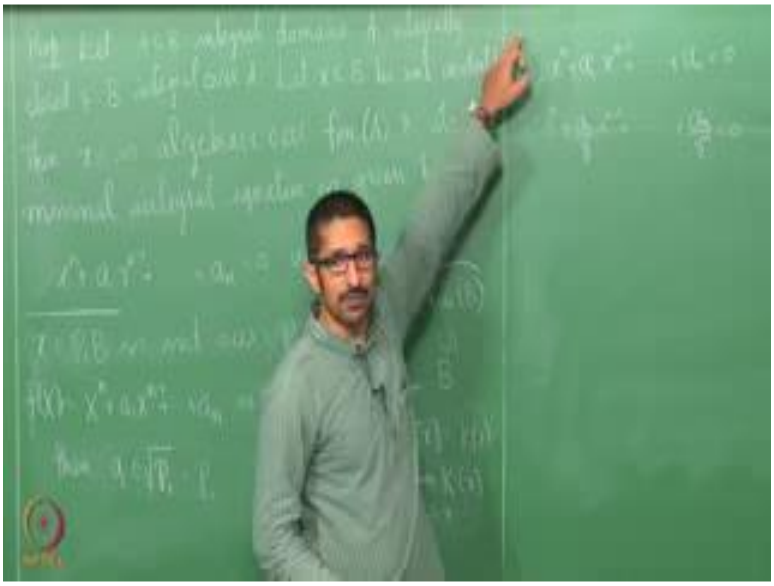

Now, as we saw x power n plus a 1 x power n minus 1 plus etcetera a n, this is 0 multiplying this by y power n what we get is s power n plus a 1 by y s power n minus 1 plus etcetera a n by y power n, this is 0. So, s is in B, s is in B therefore, it is integral over a now again the same thing you can apply it is integral over A, if you take any integral equation of s from that integral equation I can always obtain integral equation for x. And from any integral equation of x, I can obtain integral equation for s. So, therefore, degree of minimal polynomial of x is less than will degree of minimal polynomial of s. And similarly degree of minimal polynomial of s is less than will degree of minimal polynomial of s. Therefore, they are equal since they both are monic polynomial they have to be the same the polynomial this is the minimal polynomial of s over A. And that is exactly what we are getting. Is that clear?