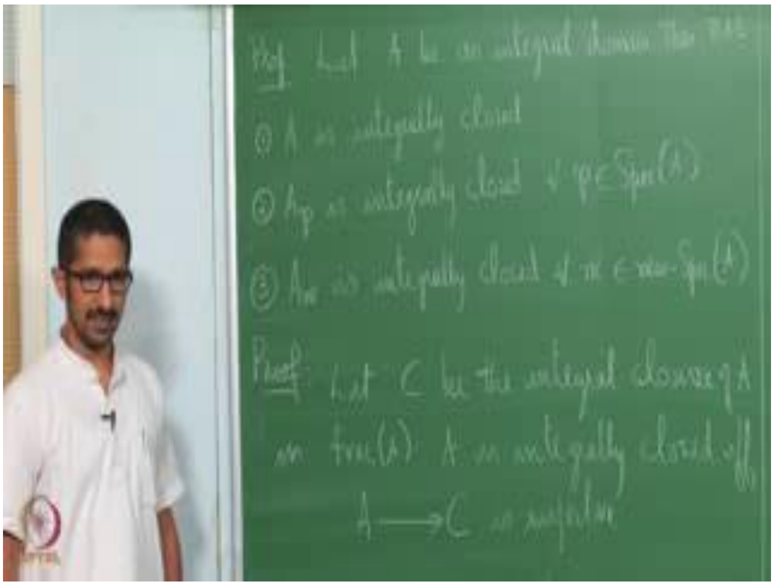
Commutative Algebra
Prof. A. V. Jayanthan
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 25
Going-Down Theorem

Let us prove that being integrally closed is also a local property.

(Refer Slide Time: 00:31)



That is let a ring A be an integral domain then the following are equivalent. A is integrally closed, A p is integrally closed for every prime ideal of A, and A m is integrally closed, m n; for every m in the max in maximal ideals of A. So, this max spec is usually written for the maximal ideals of A which are maximal elements of this set.

So this is follows, some what we saw some time back that. So, let C be the integral closure of A in the fraction field of A. Then what is meant by A being integrally closed? A is equal to C or in other words, one can say that A is integrally closed if and only if the natural inclusion map from A to C is surjective. But then we have already seen that such a map is surjective if and only if the corresponding localization with respect to p is surjective which is if and only if A m to C m a surjective.

(Refer Slide Time: 03:56)

So, this is equivalent to saying A p is surjective for every p in A. Now the last proposition that we proved says that this is indeed the integral closure of A p right, in k. This is again equivalent to saying that Am to Cm is surjective the max spec of A. So, this proposition says that being integrally closed is a local property.

Now, suppose I have a ring extension and I in A be an ideal. So, we have been talking about elements which are integral over A. Now if all the coefficients suppose I take an element x in B, and if it is integral over A and if all those elements all the coefficients of the integral equation belong to an ideal particular ideal i, then we say that the element is integral over the ideal i.

So, if x in B satisfies an integral equation, x power n plus a 1 x power n minus 1 plus etcetera a n equal to 0 with a i in i for all I then we say that x is integral over i. So, in particular, suppose I take an integral equation of x over B, and look at the ideal generated by all this a 1 up to a n, naturally it will be a i mean it will be integral over that ideal see it maybe that this ideal will turn out to be the whole of A. Sometimes it will be sometime it will not be.

(Refer Slide Time: 07:13)

So, let us look at one situation say for example, k x cube x square y y cube. Look at the ring k x cube x square y y cube. What would be the can you give me an element in let us say k x y which is integral over the maximal ideal for example. What is the maximal ideal here the ideal generated by x cube x square y y cube? Can you see that? See what is that I am asking about a maximal idea l what is a maximal ideal here? This will be if I take an a i mean if I take the quotient of this k modulo this what would the.

Student: Be k field.

It will be the field k right. So therefore, this is a maximal ideal now can you tell me an element here which is integral over this and which is not here of course, all the elements here they do.

Student: X.

This is indeed an integral extension right. So, x is integral over k x cube. Now what is the integral closure of this? What is the fraction field of this? So, I call this a; what is fraction field of a is this equal to k x y, is this equal to k x y does x belong to this fraction field.

Student: X y is equal to k x y.

So, the question is whether x is in of A. Is it can we write it as something you know some monomials divided by some monomials here. So, this is let me this is I leave this as an exercise. This is and it requires some amount of computation it is not difficult at all, but I i will I would really want you to do this find integral closure of A. So, well what I can tell you is that this is these 2 are not equal x.

See here we are talking about polynomial a variable x and variable y. For a variable x to be in the fraction field it has to be some monomial on the top divided by a monomial on the denominator; that means, there should the and like if I take a monomial here, it will be x power alpha 1 plus y power beta 1 divided by, it should be of the form x power alpha 1 minus 1 y power beta 1 it has to be of this form, but now what are the options how can you get this it has to be a polynomial combination of these 3 guys. So, how can you get this just try to make a analysis a alpha 1 has to be 3 a 1 plus 2 a 2 plus and that is all it has to be 3 a 1 plus 2 a 2.

Similarly, beta 1 can be come from 3 b 3 and 2 whatever see this x alpha 1 y beta 1 this will be some x cube power a 1 x square y power a 2 and y cube power a 3 right any monomial has to be some product of these guys. So, therefore, alpha 1 has to be.

Student: 3 a 1 plus 2 a 2.

3 a 1 plus 2 a 2 and.

Student: Beta 1.

Beta 1 has to be.

Student: A 2 plus 3 a.

A 2 plus 3 a 3 now this see below also it should satisfy the same equation; so you have to get 3 a 1 plus 3 a 2 as alpha 1 and a 2 plus 3 a 3 as a beta 1 and you have to get alpha 1 minus 1 and beta 1 minus 1 also in the same manner.

Student: Between.

See whether you can do that you should have alpha 1 to be this, beta 1 alpha 1 minus 1 should be again of this form, whether you can do it or not think about it and it is a easy computation, but. So, and then find the integral closure of a in it is fraction field.

(Refer Slide Time: 13:25)

So, let us come back to this. So, suppose I have a an extension A contained in B. Let A contained in B be a rings and C I be an ideal in in A and C be the integral closure of A in B. Now I am collecting I am trying to collect now all the elements of B which are integral over i. So, naturally it will be there in C they will all be there in C because they are C is the collection of all elements which are integral over a i is a subset of a therefore. So, now, see earlier when we looked at this, set of all integral elements the integral closure of A in B. We proved that this is a sub ring of B right. C is a sub ring of B. Now I am collecting all the elements which are integral over i is an ideal in C. If this is an ideal what does it can we have you know some way of expressing it.

So, let us look at let x be in C be integral over i. So, let me leave some space here I will complete this let c be x in c be integral over i. What does that mean; that means, there exists a 1 up to a n in I such that x power n plus a 1 x power n minus 1 plus etcetera a n minus 1 a x plus a n equal to 0. Where a i that we have already said. What does that mean this means x power n is equal to minus of this, but x is in C all these elements i 1 i 2 up to a 1 a 2 up to a n they are all in i. So, can you identify this as a part of an ideal?

Student: I will be in place.

In.

Student: C.

In C, or in other words i extended right. Therefore, this belongs to I extended in C right; that means, x belongs to the radical of I extended. So, what we are seeing here is that if x is integral over I then x belongs to the radical of I extended.

Student: That is what is I c.

I mean i.

Student: I c.

I mean this is the c is the interval closure. So, I am taking I mean by definition this is the extension of I in set of all linear combinations of the form a i x i where a i is in I and x i is in C. See I have an ideal I in A and if I have a ring C then I extended here is nothing, but this that is exactly what if denote. So, if x is integral over I then x is here. So, there is a natural question whether if I take any element here it is integral over I or not.

(Refer Slide Time: 18:35)

So, let x be in the radical of I extended. So, this is x integral over i. So, is meant by x belongs to I radical of I in extended this means x power there exists some n such that x power n belongs to I power e I mean sorry I extended. What is that mean; that means, there exists some b 1 up to b n let me denote by b m in c such that x power n is equal to summation b i x i and x 1 up to x n, x m in i, such that this is I from 1 to m. Consider the ring A b 1 up to b m right. This is b 1 up to b m all of them are in C therefore, this is called this b, this is a s generated a module. So, let me call this consider the a module this is a ring certainly m equal to this is finitely generated a module because all b is are part of a they are all in c and they are all integral over A, therefore this is finitely generated A module and x power in, what can you say about.

X power n m this is contained in I M right because x power n M x power n is some linear combination of elements from I and b 1 up to b m. Therefore, x power n is here therefore, all the corresponding coefficients are here. So, therefore, that means, if I consider the map consider the map mu x power n from M to M given by mu x power n of m is equal to x power n m then. Mu x power n M is contained in I M, M is a finitely generated a module I is an ideal in a.

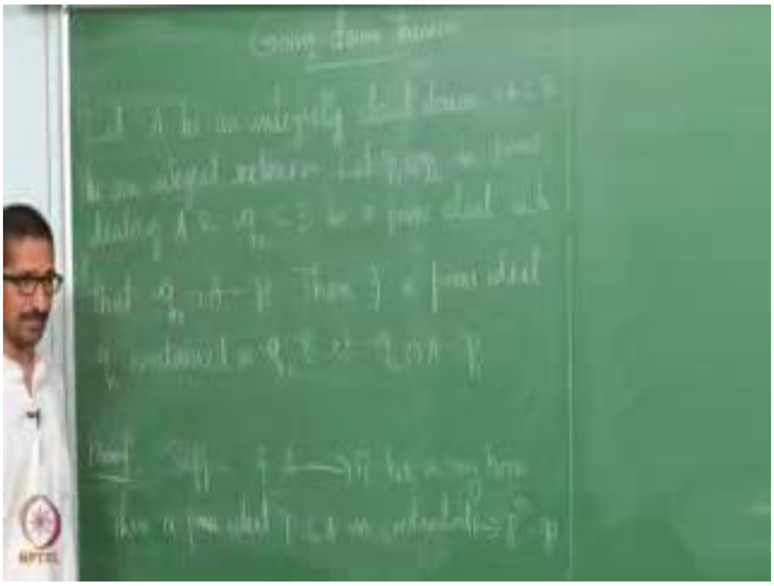
(Refer Slide Time: 22:29)

So, now you can apply the determinant trick to say that therefore, there exists a 1 up to a n in some. Let me write this as a 1 up to a r in I such that mu x n power r plus mu x n power r minus 1 a 1 a r is 0. Apply this on the element 1 in m what you get is, x power n r plus a 1 x power n n r minus; that means, x is integral over, is this clear, we are again and again using this determinant trick we are getting a map with the property that it is image is contained in I m therefore, the map satisfies an integral equation with coefficients coming from the ideal.

So, let me complete this proposition here. Then the set of all elements in B which are integral over A over I is the radical of I C or radical of I extended in C. And that is precisely what we proved now. So, if I have a ring extension and I is an ideal in a i have a ring extension a contained in b i have an ideal I in A if you take all the elements of B which are integral over i, then this is indeed an ideal in the integral closure. It need not necessarily be an ideal in B, but what we are saying that it is an ideal in in the integral closure and it is precisely given by the extension of I in the in a radical of the extension of I in the integral closure that is what the proposition say.

Now, let us prove another very important theorem in the theory of integral dependence which is called going down theorem.

(Refer Slide Time: 25:56)



That is see earlier we looked at this. Suppose I have a integral extension. If I start with an ideal here, then there exists an ideal here which contracts to p. Then we proved little more if I have an ideal p 1 contained in p 2 and if I start with an ideal q 1 here, then I can find an ideal q 2 which contains q 1 and contracts to.

Student: P 2.

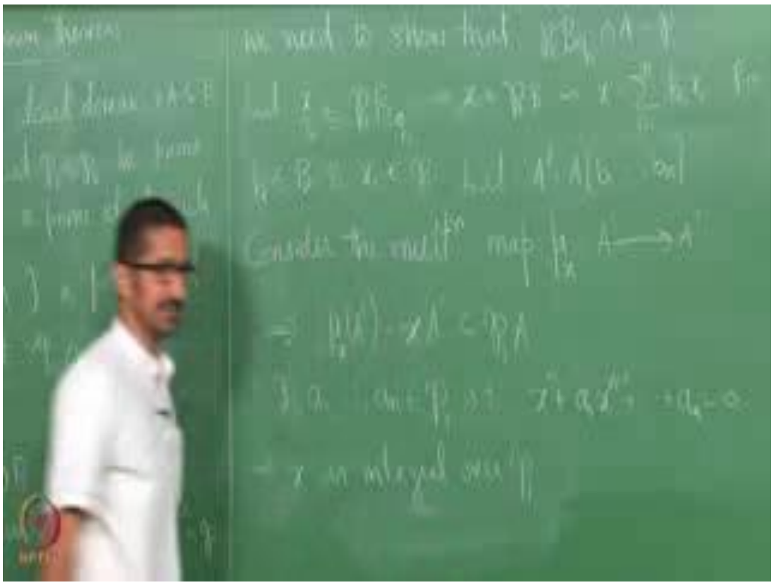
P 2; so if I have a chain here and if I have a starting point here I can go up. That is what we proved last time.

Now suppose I start with an ideal suppose I start with this ideal can we always find something here which contracts to p 1 let us prove that. That is called going down theorem. This requires little more stronger hypothesis. So, let a be an integrally closed domain and A contained in B be an integral extension. Let p 1 contained in p 2 be prime ideals of a and q 2 be a prime ideal such that q 2 intersection A is p 2 then there exists a prime ideal q 1 contained in q 2 and such that q 1 intersection a is p1.

So, let us try to prove this to start with let us recall one result that we proved sometime back. This is regarding prime ideals see a prime. Suppose I have a A to B be a ring extension. In fact, we do not really need to be a ring extension suppose f from A to b be a ring homomorphism. We had discuss the extension and contraction with respect to any ring homomorphism right. So, if I have a ring homomorphism and then a prime ideal p in A is contracted if and only if, p extended contracted this p itself right this is something that we proved.

If this is a contracted ideal; that means, if this is an ideal such that it is I have a prime ideal here which intersects with A to give the this prime ideal this is equivalent to saying that you extend and contract p you get p itself, this is something that we proved sometime back.

(Refer Slide Time: 30:45)



I have this ring homomorphism A p 1 to sorry I am looking for, So, A to B q 1. See I am looking for an ideal in contained I wanted to B q 2. I guess contained in q 2 right I am looking for an ideal contained in I should write p 2. I am looking for an ideal contained in q 2 what are the prime ideals here.

Student: S inverse b.

S inverse.

Student: P.

P where?

Student: A intersection s is empty.

So, but S here is.

Student: B minus q.

B without q 2 which means which should not intersect here means.

Student: That contains.

The prime ideal should be contained in q 2 right. So, the prime ideals of this ring are of the form p localized at q 2 where p is a prime ideal contained in q 2. So, here I am looking for an ideal here which contained in q 2. So, I am naturally looking for a prime ideal here which intersect to give me p1, so p 1 or in other words p1. So, if I if p 1 localized at p 2 is indeed a contracted ideal is amount to saying that there exists some q 1 here, q 1 which q 1 localized at q 2 intersected with a localized at p 2 gives me p 1 localized at p 2 which is equivalent to saying that if I take p 1 extended to this ring.

So, I will simply take A p 2 because it is p 1 is contracted is equivalent to saying that p 1 extended contracted is equal to p1, but then this p 1 extended contracted is nothing, but p 1 B q 2 intersected with A. So, to say that there exists a prime ideal here which contracts to p 1 are equivalent to saying that this is equal to p 1 right; p 1 is the smaller ideal. So, to prove this theorem we need to prove that p 1 B q 2 intersected with a is p 1 itself. So, let us try to prove this. We need to show that p 1 B q 2 intersected with a is p 1. We only need to prove this, if I prove that then because of this p 1 will be a contracted ideal in.

Student: A.

A, which means that there will exist something in b q 2 which contracts this and that will precisely be the prime ideal q 1. So, let us. So, let us first start with an element in p 1 b q 2 intersected with a this means what is this. So, let us first start with an element here what how does this look like. This means x is in p 1 B and s is in the complement of q 2.

So, this implies x belongs to p 1 B; that means, x i will write it as some a i x i I from 1 to n for some a i let me use b i x i with b i coming from B and x i in p 1.

So, let me look at the ring A prime equal to a B 1 up to B n. I do this determinant trick again consider the multiplication map, mu x from A prime to A prime. This is any element mu x acting on any element A is x times A or a times x does not matter. Then what can you say about mu x of a prime? This is.

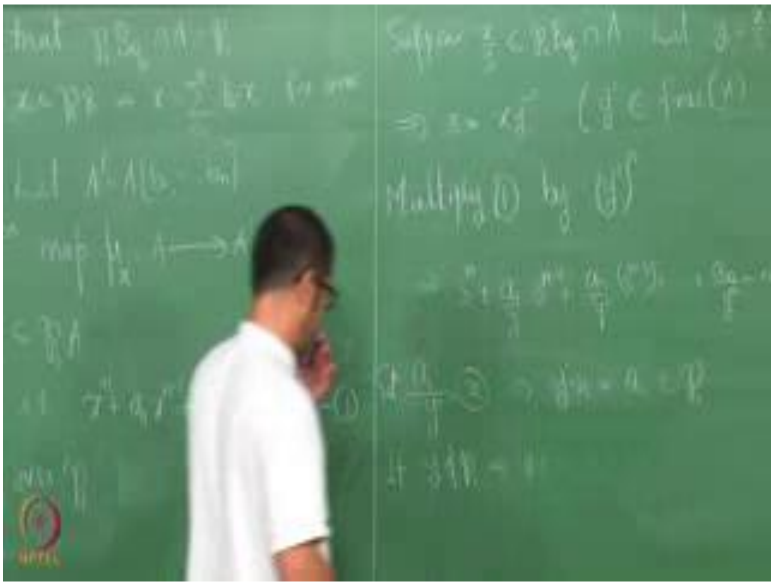
Student: Ideal generated by x b prime.

So, this will be x a prime, but that is contained in.

Student: P1.

This is equal to x a prime, but x is summation b i x i, where x i is coming from p and all these b is are in a prime. So, I can write this as this is contained in p; p is p 1 p1 a prime. Therefore, by the determinant trick mu x or you know I will just simply write x. Therefore, there exists a 1 up to a n in p 1 such that, by the way this is a finitely generated a module we are using that here, a 1 up to a n in p 1 such that x power n plus 1 x power n minus 1 plus etcetera a n is 0 which says that x is integral over p 1. If I take an element in the in p 1 extended then it is indeed integral over p1.

(Refer Slide Time: 39:21)



Now, now let us look at suppose this x by s, x by s belong to p 1 b q 1 intersected with A. See p 1 extended to B q 1 and contracted to a will certainly contain p 1 right, p for any ideal i I extended contracted will contain I; similarly I contracted extended will.

Student: It will be i.

Will be contained in i; so, here this will contain p 1 we only need to prove that, this is contained in p 1. Or in other words if I start with an element in p 1 B q 1 intersected a i need to show that this is in p1. So, let y let us write y as x by s. So, why is this? So, I can write x sorry s as see the problem is with this s right, to say that x by s is in p if I can get to a form where x by s is y by 1 for some, then we are through that is what we have to show. So, the trouble is with this s.

So, let us just study what is s? S is equal to x y inverse I mean these are all domains. So, y inverse in the fraction field of A. Now if I multiply this equation by the corresponding power of y inverse, I will get an integral equation for s right if I multiply by y inverse power. So, let me call this equation 1 multiply 1 by y inverse power n, then what do I get? Then I have s power n plus a 1 by y s power n minus 1, a 2 by y square s power n minus 2 a n by y power n 0.

So, this becomes integral equation. So this is see s you can see it as an element in the fraction field of A, and this is an integral equation as you know contained in the fraction field all these coefficients are in the fraction field, so we are not. Let us look at what are these coefficients. See a i by y y power i a i is in. So, let us write this is let us set this to be v i. This implies v i belong to a y power i v i is a i, but a i is in.

Student: P1.

P 1 if y is not in p 1 what happens if y is not in p1. That would imply that v i no. Maybe I will continue this in the next class. I will conclude this, next time the complete proof.