Commutative Algebra

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Lecture – 23
Integral Extensions

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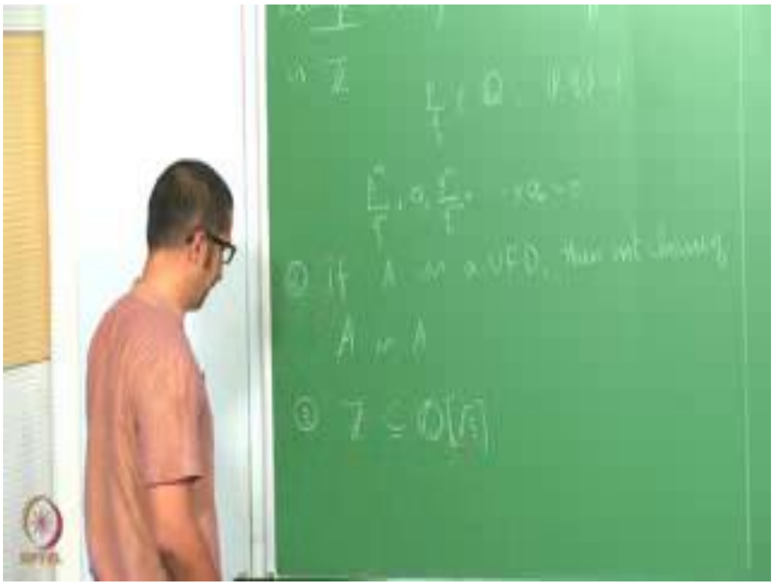

Let us see that first integral closure of Z in Q. So, when we are talking about domain, I do not really have to specify this and I say integral closure Z by default it is in Q is Z. This is easy to see if you take any p by q in Q, p, q relatively prime. And if you write down an integral equation p n by q n plus a 1 p n minus 1 by q n minus 1 a n. If this is 0 then that would say that p n is equal to some q times something; that means, p n is equal to Q times something that will say that p and q are not co prime.

And more generally if A is a UFD then integral closure of A is A itself. What we are using here is that you have a prime factorization for p and q; and you have this you know in fraction field every element can be written like this. Same thing here that will if I have something in the integral and fraction field of A, I have such an equation and ultimately we get there exist a prime which divides p as well as q, so unless q is 1.

So, therefore, we have integral closure of A or in other words a UFD is integrally closed. We say that, so integral closure of A in fraction field of A is A itself which means a is integrally closed. Let us look at some other examples. Let us take Z. So, we saw that integral closure of Z in Q is Z itself. What if I take root 3? What is the integral closure of Z in Q root 3, Z is certainly there. Can you tell me one more element?

Student: Root 3.

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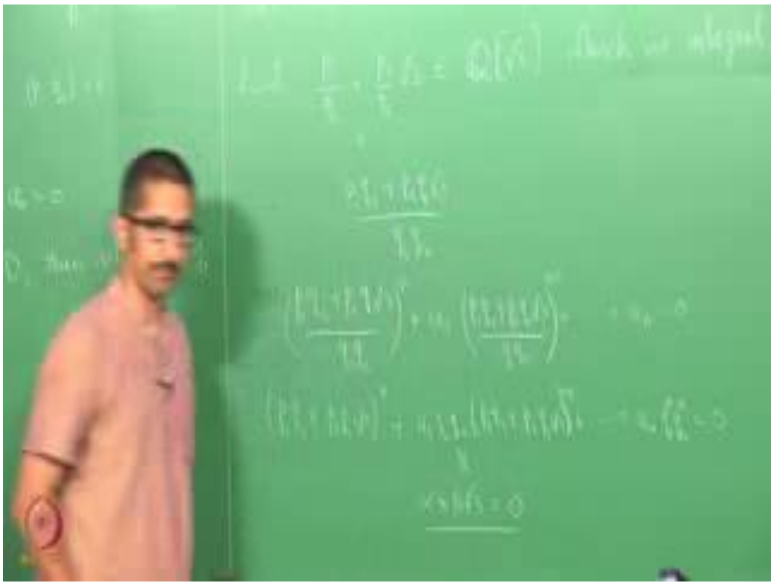

Root 3 is there; root 3 satisfies equation X square minus 3 right. So therefore, integral closure of a Z this contains Z root 3. Can we say that this is indeed Z root 3? How does any element in Q root 3 look like. Some a plus b root 3, where a and b are in Q. So, let us write let p 1 by q 1 plus p 2 by q 2 root 3 belong to Q root 3 which is integral over Z root 3.

Student: (Refer Time: 05:05).

Sorry, over Z. Can we say that this is I mean; I can write this element as p 1 q 2 plus p 2 q 1 root 3 divided by q 1 q 2. Now, the question is whether see we want to see whether integral closure is Z root 3 itself. So, the question is whether q 1, q 2 is equal to.

Student: Plus minus 1.

Plus minus 1, so how do we say that. So, let us write like this, you know let us. So, suppose this has an integral equation over Z, what is that mean p 1 q 2 plus p 2 q 1 root 3 divided by q 1 q 2 whole power n plus some a 1 p 1 q 2 plus p 2 q 1 root 3 over q 1 q 2 a n is 0. What is that mean? Here again you can do the same process q 1 q 2 power of n.

So, maybe slightly easier method would be if at all there is an equation. Now, I think let us do this. So, if I multiply by q 1 q 2 power n, what we get is p 1 q 2 plus p 2 q 1 root 3 whole power n plus, this is n minus 1 a 1 q 1 q 2 p 1 q 2 plus p 2 q 1 root 3 a n q 1 q 2, this is 0. So, see there are see in this one in each of them what would be a, what are the powers that will have only integers. See, this will again be equal to some alpha plus beta root 3.

Student: (Refer Time: 08:07).

I mean what are all those will be absorbed in alpha. This will be equal to I mean this will reduce to a form alpha plus beta root 3. Now, this says that alpha is 0, beta is 0. Now, you have to see what are alphas here alpha coming from each place and then try to reduce it. So, this is not that easy. Now, let us look at another method. So, this is you know one can do it, one can look at ultimately you will have you know some p 1 q 2, I mean you will have to get a contradiction with the fact that q 1 and p 1 are prime q 2 p 2 are prime to each other using this.

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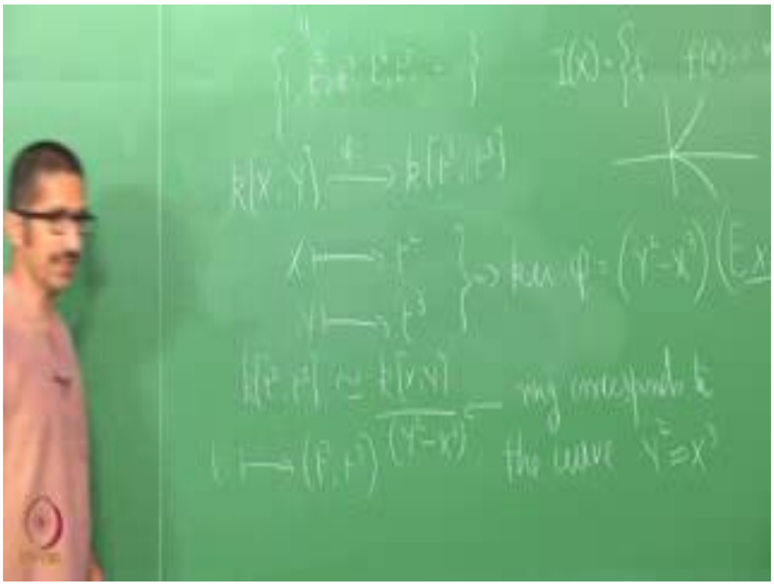

So, let us look at another example. This is look at A equal to k t square t cube this contained in the ring k t. What is mean by this ring, I am looking at all the polynomials with t square and t cube. So, the monomials here as a vector space basis, this is generated by 1, t square, t cube and then t 4, t 5 all of them are there it generated by this. Now, this ring is see if you look at the map k X, Y to k t square t cube X going to t square, and Y going to t cube. Then this k t square t cube is isomorphic to k X Y modulo Y square minus X cube.

The kernel of this map kernel phi is, so that is not very difficult to prove, you do not require too much assumption to prove that this k X, Y is. So, every this is certainly contained in kernel. Now, what we need to prove is that if a polynomial is map to 0, then you can indeed divide by Y square minus X cube. So, this ring corresponds to the curve Y square equal to X cube. See, if I have a I mean I said in the beginning, if I have an algebraic set which is satisfied by certain polynomials, then there exists some ring corresponding to that. That is if I have an algebraic set, I can look at I of X to be set of all f in the polynomial ring such that f of a is 0 for all a in X.

So, this is nothing but the ideal corresponding to the curve Y square equal to X cube or in other words, the curve this one. And one can see that this is singular at the origin, the curve is singular at the origin, this is we do not have a proper tangent at this point. So, it is singular at the origin. Now, let us that is the geometry behind it.

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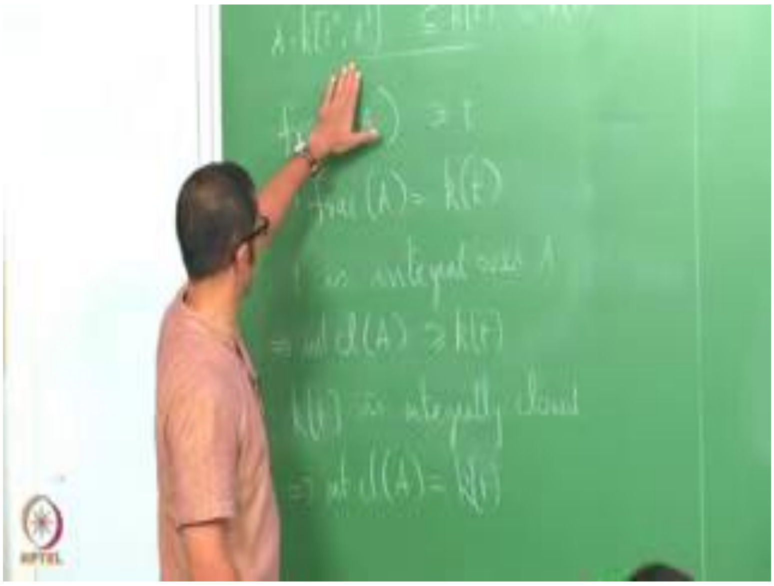

Let us come back to the algebra here. What is the integral closure of? So, what is the fraction field of this? See, this is contained k t, what is the fraction field of this. So, this is denoted by k round bracket t, you look at all you know polynomials f t by g t. Now, what is the fraction field of this? So, it will certainly be contained here. Now, you can see that if A is this, this has t in it, because t cube by t square is there in this one. So, the fraction field should contain k as well as t. So, it has to be this itself this is the smallest field containing k and t. So, therefore, fraction field of A is k t. So, now let us look at the integral closure of A in the in its fraction field which is k t.

Now, what about this extension what can you say about this extension, is that an integral extension is t integral over A t is integral over A right that means, integral closure of A contains k t. I want to say that ok. So, suppose I take a polynomial suppose I take A an element in. So, now, what can you say about k t, this is PID, and hence CFD, therefore, this is integrally closed by the exercise k t is integrally closed.

If I start with some element here, which is integral over A, it is naturally integral over k t. So, if I have something here which is integral over this, it is integral over this, but then this is integrally closed here. Therefore, that element has to be in k t itself that means, integral closure of this is k t itself. This implies that integral closure of A is equal to k t. So, the integral closure of k t square t cube is k t. So, here what we are seeing is that this ring is not integrally closed.

When the field k is algebraically closed, there is close relation between the integral closure and smoothness of the curve. See this ring corresponds to I mean another way to look at this is the curve is nothing but curve can be parameterized by t square t cube. The curve is nothing but t square t cube where t varies is over r. So, therefore, this is a parameterization for the curve. So, this is the ring corresponding to the curve or this is the ring corresponding to the curve.

Now, the normality or sorry integral closedness of this ring in fact, it is equivalent to saying that the ring the curve c is nonsingular, when the field is algebraically closed. I mean if you are taking over C or similarly algebraically close fields. So, this is again something that you will probably see if you take a advanced course in algebraic geometry. So, what we have used here is see if this is integral over this, this is integral over this, and an element here is integral over this implies it is integral over this and therefore, it belongs to here.

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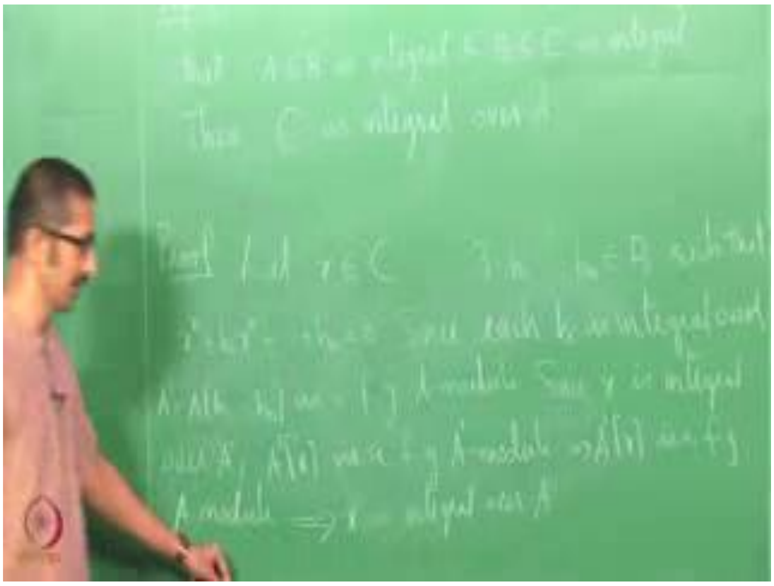

Now, one can ask the following question. Suppose A is I have A contained in B contained in C are rings, B integral over A, and C integral over B does that necessarily say that C is integral over A. So, this is called Tower law let A contained and B contained and C be rings such that A this is integral and B over C is integral. Can we say that A, I mean C is integral over A? Let us check. So, I will write rest proof and see where do we reach.

Let us start with an element C, when do you say something is an extension is integral every element is integral over the base ring. So, let us start with an element in C what we know is that X is integral over B. So, therefore, there exists b 1 etcetera up to b n in B such that X power n plus b 1 X power n minus 1 plus etcetera b n is 0. Now, what do we know about B, B is integral over A; that means, each b i is integral over A, this is equivalent to saying that. So, each b i is integral over A, this is equivalent to saying that this is a finitely generated A-module. Now, what we know is that X is integral over B, but you do not need whole of B, but we only need this ring, X is integral over this ring; or in other words if I attach X to this, it will be a finitely generated module over this ring, but this is a finitely generated.

Student: A-module.

A-module; therefore, you attach X to this that will remain to be a finitely generated A-module and that was one of the equivalent condition for X to be integral over A. There exists an A algebraic C containing A and containing the element X that was the third condition that we wrote down yesterday. So, what is the conclusion then C is integral over A. So, let us write this. Since x, so let me call this A prime. Since x is integral over a prime A prime x is finitely generated A-module, A prime is a finitely generated A-module and A prime x is a finitely generated A prime module.

Therefore, A prime x is a finitely generated A-module and that implies that x is integral over. So, what we have shown is that every element in C is integral over, so this is called Tower law.

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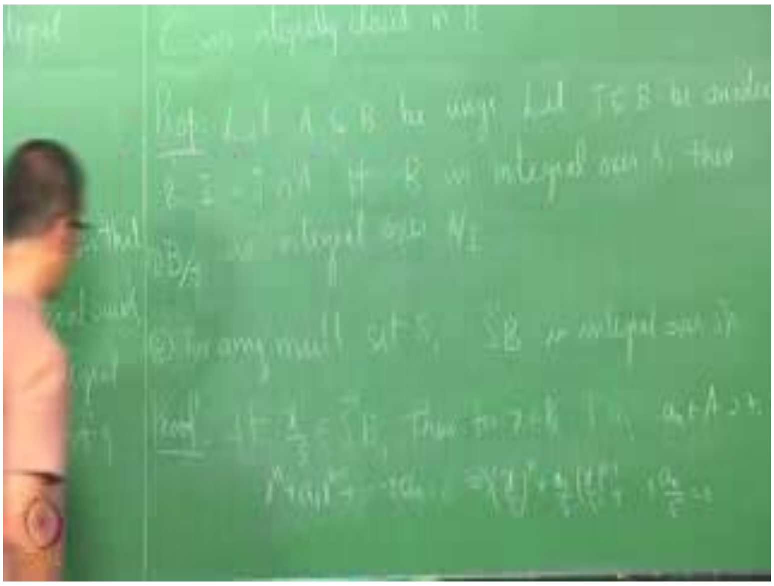

So, an immediate corollary is if C is the integral closure of A in B, then C is integrally closed in B, I mean the forward part is obviously true. So, if C is integral over A then obviously, every element of B is integral over A, and C is integral over B as well. For that you do not require any you know proof, I mean in the sense that that is trivially true. So, here what we are saying is that if C is integral closure of A and B, I collect all the elements in B which are integral over A, then you will take any element in C which is integral over any element in B which is integral over C, then by this that has to be integral over A. And hence it has to be in C itself. So, therefore, C is integrally closed in B.

Now, suppose I have be rings. Let J be an ideal in B, and I be equal to J intersection A the contraction of J into A. Then we have this natural inclusion then we have this natural inclusion a mod I going to B mod j, x plus I map to x plus J, x in A. So, if I take two representatives x and y, x minus y is in I, if and only if it is in J, because X and Y are in A. So, therefore, this is well defined as well as injective map. Now, if B is integral over A, do we have some relation like this if B is integral over A, then B mod j is integral over A mod I. I mean straightforward right, do I need to prove that if I have an integral if I take an element X plus j in B mod j then X is.

Student: Integral over.

Integral over A; so I have a an equation with coefficients coming from A, now we just have to take that equation model of I. So, let us complete this. If x by s belong to S inverse B then for x and s x an B there exists a 1 up to a n and a such that x power n plus a 1 x power n minus 1 plus etcetera a n is 0. And that implies that x by s whole power n plus a 1 by s x by s whole power n minus 1 plus etcetera a n by s power n is 0.

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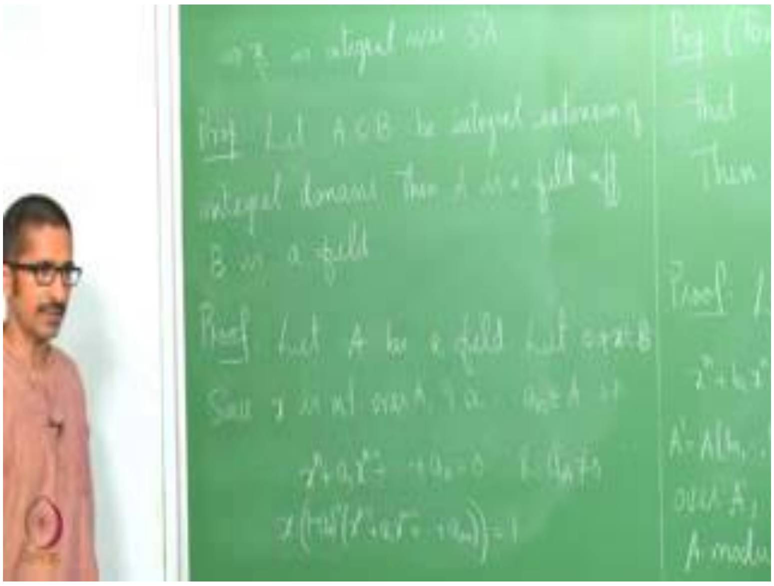

And that implies x by s is integral S inverse A, because each x I by s or A I by S power J A I by S power I belongs to S inversely. So, there are some nice you know applications of this. This is one of the important applications. Let A contained in B be integral extension of integral domains then A is a field if and only if B is the field. Well this is not a direct application of the earlier proposition, but it is an important result in this direction. So, what this says is that if I have an integral extension of integral domains either both can be fields or neither of the mesa field. It is easy to prove. So, in each case what we need to prove is that there exists multiplicative inverse. So, let us start with A be a field. So, I start with a nonzero element in A, I want to say that it has inverse now.

Student: (Refer Time: 31:10).

Sorry nonzero element in B, to say that it has multiplicative inverse. Now, x is integral over A, there exists a 1 up to a n in a such that x power n plus a 1 x power n minus 1 plus etcetera a n equal to 0. Now, we can assume that this you know this is an equation of minimal degree, and hence we can assume that a n is nonzero. See what happens if a n is 0; if a n is 0, I will have x times x n minus 1 plus dot dot up to a n minus 1 is 0, but it is a we are in an integral domain. Therefore, either x is 0 or the other one is 0. So, we can always reduce this equation degree of the equation ultimately get to a minimal equation. So, without loss of generality, we can assume that this is an equation of minimal degree and hence a n is nonzero.

Now, can you see an inverse here? I take minus a into the other side and multiply by the a n is in A, and A is a field. So, therefore, I can write x times minus a n inverse x n minus 1 plus a 1 x n minus 2 a n minus 1, this is equal to 1. I take minus a n and a into the other side and multiply by its inverse. So, therefore, this is an element in B therefore, I have an inverse in.

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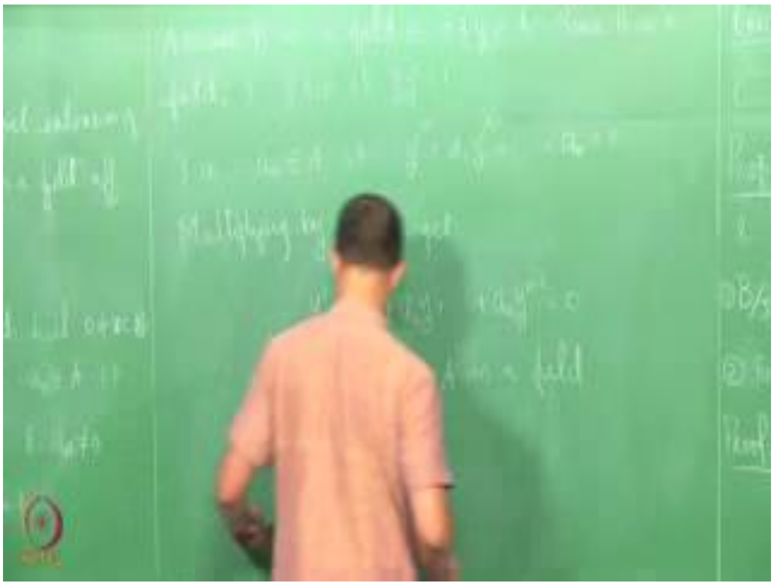

Conversely assume that B is a field and start with a nonzero element in A. Now, A is contained in B, B is a field therefore, there exist. Since B is a field, there exists y prime in B such that y, y prime is 1. Now, y prime is integral over A, therefore, there exist a 1 up to a n in A such that y prime power n minus n plus a 1 y prime n minus 1 plus etcetera a n is 0 multiplied by multiplied by y power.

Student: N minus 1.

N minus 1 multiplying by y power n minus 1, we get y prime plus a 1 a 2 y a n y power n minus 1, this is 0 or in other words Y prime is a linear combination of.

Student: Elements in A.

Elements in A; therefore, y prime is in A, this implies A is a field.

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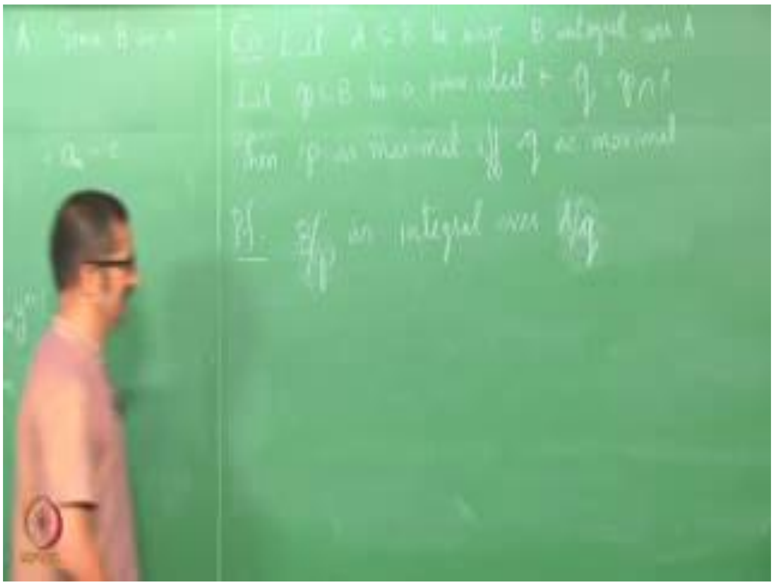

Now this has a very nice application. Let A contained in B be rings, with B integral over A. Let n and B be a prime ideal and, so p be a prime ideal and q be equal to p intersection A. Then p is maximal if an only if q is maximal, this is direct application of the cell that we proved earlier and this one, because A mod q is integral over B mod p that is what we proved just before, therefore B mod p is integral over A mod q.

So therefore, p is I mean B mod p is a field if and if only if A mod q is a field which is same as saying p is maximal if and only if q is maximal. We will continue in the next class.