Commutative Algebra
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Lecture - 21
Further Properties of Localization

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So, let us recall what we did last time that ideals of S inverse A, this is precisely S inverse I, I an ideal of A such that S intersection I is empty. Then we define we saw that for every prime ideal or so let me set this notation if you have already done some exercises from a (Refer Time: 01:06), you would have seen this. If A is a ring then I set this notation spec of A, this is set of all prime ideal, this is called a spectrum of A or in short spec A, so this is notation. Then what we showed yesterday was that spec of S inverse A, this is S inverse p, where p is in spec A with p intercession is the intersection s empty.

So, not only this, this is even more strong. Suppose I take, so I take these two sets, spec of S inverse A, if I send the map from p. So, I send this p. So, I collect this set, set of all p in spec A such that p intersection S is empty. From here, I send the map to spec of S inverse A that p is map to S inverse p.

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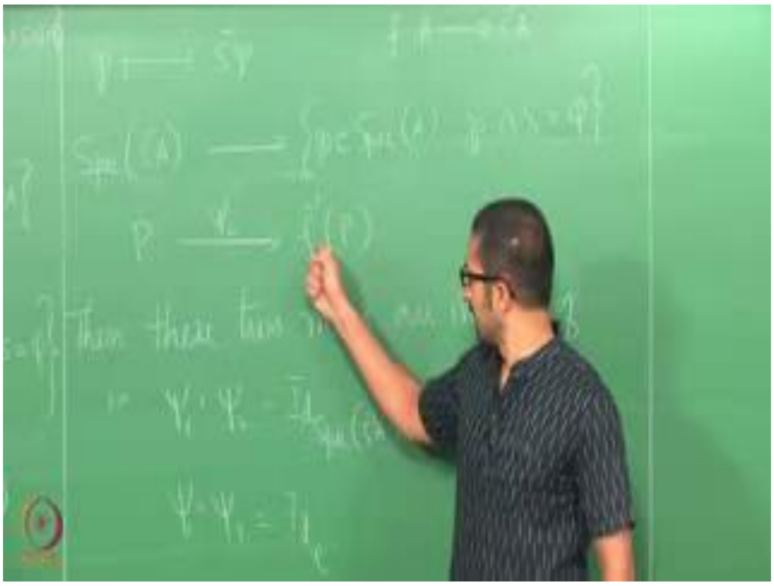

Then this map is and from spec of S inverse A to set off for p in spec A such that p intersection S is empty. I send p to. So, let us look at this natural map from A to S inverse A, I send p to f inverse of P. These two maps are inverses of each other. So, if I call this let us say psi 1 and psi 2, then let us psi 1 in composite psi 2, this identity on this set spec S inverse A; and psi 2 composites psi 1, this is identity on the other set let me call this some C.

Here, see yesterday we saw I start with an ideal p in S inverse prime ideal S inverse A, and then pull it back to A. What you get is a prime ideal. Moreover if you take S inverse of that ideal then you get back this P that is I did not prove it, but verify that its, so that is basically what I am stating here. So therefore, this is the map from here to here or this psi 1 or psi 2 both of them are inverses of each other that is there it is a bijective correspondence. So, here what we are saying is that this is the collection of ideals of S inverse A all are extended ideals with respect to this map.

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When you have a map f from A to B, and I is an ideal in A, then f of I need not be an ideal in B, but then you look at the ideal generated by this set, this is called extended ideal I e, ideal of extended ideal corresponding to the ideal I with respect to this map. What we are saying here is that every ideal of S inverse A is an extended ideal, you start with an ideal of A containing sorry which is contained in the complement of S, and then look at its extension S inverse A that is precisely these are precisely the ideals of S inverse of A.

Now, if so this is. And if you look at this set p in spec A, p intersection S is empty, what we are seeing here is that this p by the proof or you know this is the inverse of psi 1, you can see that this p is nothing but contraction of some prime ideal in S inverse A. So, in this case every ideal is every prime ideal that does not intersect S there is a contracted ideal. This is we have a more general statement.

So, let me just state it let f from A to B be a ring homomorphism. An ideal prime ideal and p be a prime ideal of a spec of A. Then p is a contracted ideal if and only if p extended contracted is p itself. I take p extend it to B and then contract to A, it is p itself that is exactly what we are having here in this case. In the special case, A going to S inverse A, I have natural map. Every prime ideal, in fact, for us this is true for every prime ideal every prime ideal that does not intersect S in the case of S inverse. So, this is

I mean straightforward. If this is true then by definition see this is p is the contraction of p extended. So therefore, if p I mean if this is true then p is contracted.

Now, if p is contracted ideal that means, p is equal to some q contracted for some q in B. Now, f inverse of q is where need to do see that see p is this, this would imply that p extended contracted is q contracted extended contracted, but this is same as q c e c any ideal contracted extended and then contracted is contracted. So, this is q contracted this is. So, this is kind of straightforward. So, here when is in the natural case of A to S inverse A, when is an ideal contracted it is when it the prime ideal does not intersect the multiplicative set S. And when it does not intersect the multiplicative set, we have this is naturally true, more properties of S inverse A.

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So, if I have if I 1 up to I n is a finite set of ideal A, then S inverse of summation I j this is same as summation S inverse I j; j from 1 to n; S inverse of intersection I j is same as intersection S inverse I j. This we proved for two, but it is one can extended to the case of many. Another important observation is S inverse of radical of I is same as radical of S inverse of I. So, let us take an element here x by s in S inverse of radical of I, x in radical of I, this implies that some x power n with x in radical of I, and that implies that x power n, x power n is in I; that means, x power n by 1 is in S inverse I. And that implies x by 1 is in radical of S inverse I.

And that clearly implies that x by s is in S inverse of I, I mean radical of S inverse I. X power n is an I, this is an I this I can write it as x by 1 whole power n. Conversely, if x by s is here that means, x power n by s power n is in S inverse I that means, I want to say that x power n belongs to; so x by s is in radical of S inverse of I. That means, I can write x power n by s power n is in S inverse of I every element of S inverse of I looks like some element in I divided by this.

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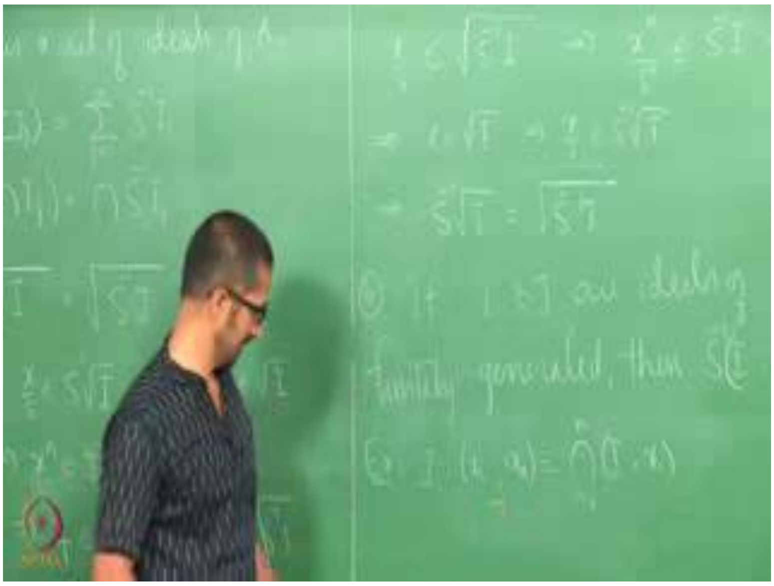

So, this I can write that x power n is in I this implies that x is in radical of I that implies that x by s is in S inverse of I. So, this implies S inverse of radical of I is equal to radical of S inverse of I. Another nice property of localization; I mean this localization is such a process that we are seeing now, where it commutes with I mean the commutes distributes with all operations, whatever we can think of you know summation intersection taking radical. And if I and J are ideals of A with J finitely generated then S inverse of I colon J is same as S inverse I colon S inverse J.

So, this is again I will this I will leave this exercise because you know if I take what is I intersection and ideal generated by x 1 up to x n, this is same as intersection of i from 1 to n. If J is finitely generated lets write this j like this then I colon j is same as intersection of intersection of this form, this is exercise this straightforward. See for J to be multiplied inside I every element each x i its equal to saying that some a multiplies J inside I is equivalent to saying that a multiplies all of them x 1 up to x n inside I that is exactly on the right hand side. These two are equivalent statements, therefore these two are equal.

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Now, you only have to proof using the intersection property I only have to prove S inverse of this to say that S inverse of J I only have to prove it goes inside that is know. How did we prove that nil radical of A is intersection of all prime ideals? How did we prove that? We prove this by saying that if I take any nilpotent element it is there in every prime ideal; conversely if I take an element which is not nilpotent.

Student: There exist.

There exists a prime ideal that avoids this. So, this second part can be viewed using the localization technique. See, if I take another correspondence between prime ideals of A and S inverse, I mean it is not really with the correspondence I take S to be equal to f power n. So, let me write x, x be an non nilpotent element, then 0 is not here. And that implies that S inverse A is nonzero, it is a nonzero ring. Therefore, there exists a maximal ideal m in S inverse A. If I take p to be f inverse of m, f is you know natural map from A to S inverse A, then this has to be a prime ideal. This is precisely the one-one correspondence that we talked about in the beginning of the class today.

So, this is a prime ideal can it contain f, this see f m is a maximal ideal proper ideal. So, inverse image because of the one-one correspondence the inverse image is again a proper ideal, therefore this cannot be entire ring which means f cannot be here. See, f is not here it cannot be here, I mean m what are the prime ideals of I mean maximal ideal is a prime ideal. What are the maximal or prime ideals of S inverse A, all S inverse p, where p does not intersect S.

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So, here x is this. So, x cannot be I x cannot be in p, this implies x cannot be in p. So, this is one direct application of that one-one correspondence, but nice way to view this property that if x is not a nilpotent element, then there exist a prime ideal that avoids this element. So, now if let us again property of nil radical of S inverse of A is equal to S inverse of nil radical of A follows directly from the definition of the follows directly from the property of bijection between prime ideals of A that not intersecting S and prime ideals of S inverse A. So, this is property five, I will call nil radical of S inverse A is S inverse of nil radical of A.

Now, let us move on to the module set up prove some of them over the modules. So, let M be a finitely generated A-module, then S inverse of annihilator of M is equal to annihilator of S inverse of M. So, localization commutes with you know when annihilator.

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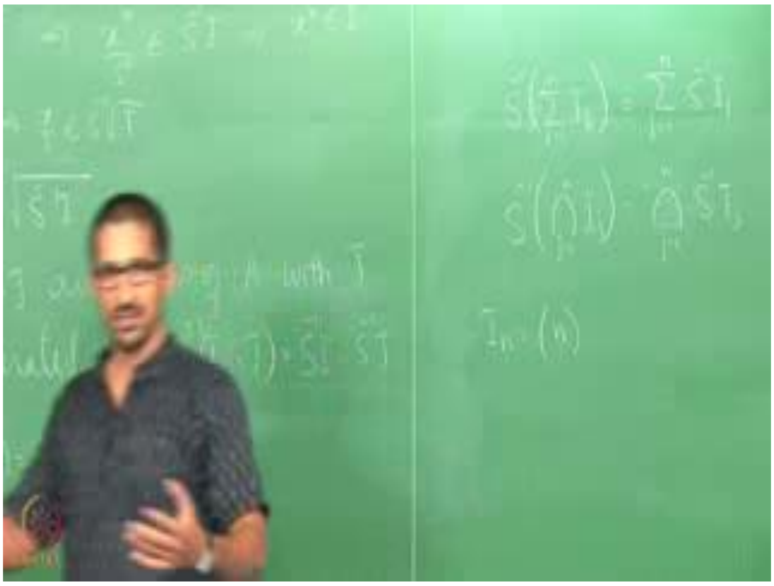

We will see. So, the first something that one should think about, we talked about these S inverse of summation I j, j from 1 to n this is same as j from 1 to n of S inverse I j. And two properties that S inverse of intersection I j, j from 1 to n this is same as j from 1 to n S inverse of I j.

Student: (Refer Time: 23:20).

Sorry, I am sorry this is intersection, where does it fail, if you take it to infinite sets. Can you think of some example where this is not true, if I take you know infinite sets? So, you can think of a ring a collection, where this is very, very small, but each of them is big. If you are still it is not sparking then you can think of where the intersection is 0, while each localization is whole ring, whole of S inverse A, think of this. And again similarly, this one there you have to think the converse like, whether this can we construct some of ideals where this is you know think about this, but in this one it is I thought it was fairly straightforward to think of.

Student: Z inverse.

In Z, you will take A equal to z. So, what is I j ideal generated by j maybe you know I n is ideal generated by n. What would this be I mean j from an n from 1 to infinity that will be.

Student: 0.

0, there cannot be any integer in that, so this will be 0. And what is S?

Student: (Refer Time: 25:50).

All you know Z star are n natural numbers. Then this will be 0. But each S inverse j will be whole of q, S inverse j will be q for each j, therefore, that will be this will be 0, while this is. So, similarly try to construct for this think of that. So, what I am trying to say is that the properties that you know S inverse commutes with they are you know we are assuming there finite intersection finite summation in some; without which it need not necessarily be.

Let us look at you know the proof will now reveal what happens. So, first of all, if I have suppose M is generated by a single element, suppose M is generated by a single element, then this annihilator of M is same as annihilator of this module element M. Anything that completely annihilates capital M should annihilate this one. Similarly, if an element annihilates m it annihilates the whole of M because every element is a scalar multiple of this. Now, therefore, I can have a map from A to M, a going to a m this is a surjective module homomorphism and what would be the kernel, kernel phi is annihilator of m small m or capital M which were one you want to write one can.

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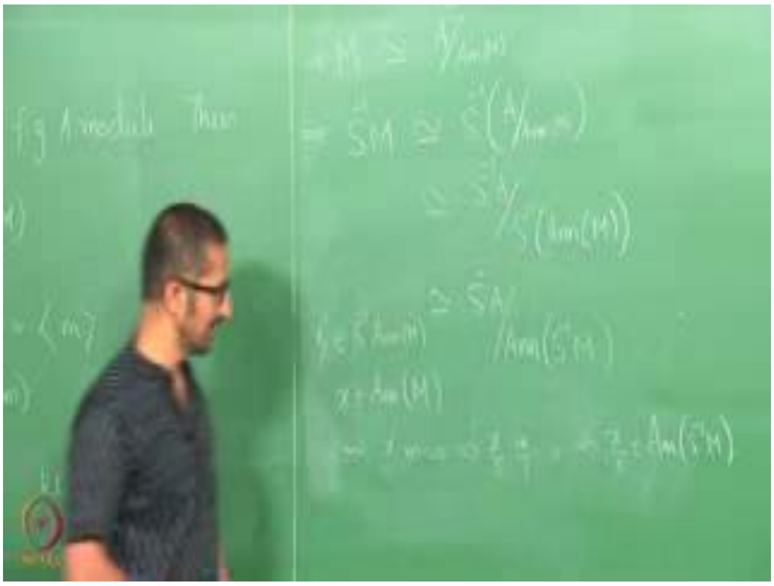

So, M is isomorphic to A mod annihilator of M in this is case right that would imply that S inverse M is isomorphic to S inverse A mod annihilator of M, but this is isomorphic to

S inverse A mod S inverse annihilator of M. Now, S inverse M is generated by again this m, and the map from S inverse A to S inverse M will be a m by, a going a by 1 going to a m by 1 that will that will give me a homomorphism with kernel as this one. Therefore, S inverse of and this is same as S inverse A S inverse M. So, this directly says that S inverse of annihilator of m.

See here there is one more catch that can you see that one of them is contained in the other S inverse of annihilator of m and annihilator of S inverse of M. If I take x in the annihilator of M that would imply x by 1, so x, x m is 0 that would imply x by 1 times m by 1 is 0 that would imply that x by 1 is in annihilator of S inverse of M. So, if I take x by s in S inverse of annihilator of M, this would imply this x by s is here. So, this is contained here, and we have this isomorphism says that these two have to be equal; otherwise this modulo, this will be nonzero.

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So, we prove this for we proved this for M equal to M to be a cyclic module. Now, we make an observation if N and P are sub modules of M then what can you say about annihilator of N plus P; this is annihilator of N intersection annihilator of P. If I take any element here, it annihilates elements of N as well as elements of P, therefore it has to annihilates elements of N, N plus P. Conversely, every element of N is contained in I mean N is contained in N plus P therefore, annihilator of N plus P is contained in annihilator of N, similarly this contained in annihilator of P, therefore, this is contained here. So, these this equality is a straightforward verification.

So, if I have S inverse of annihilator of N plus P, this is S inverse of annihilator of N intersection annihilator of P, but S inverse goes inside finite intersection. So, therefore, this is equal to S inverse of annihilator of N intersection S inverse of annihilator of P. Now if N and P where cyclic modules, so N this is equal to annihilator of S inverse N intersection annihilator of S inverse P, if N and P where cyclic modules. But annihilator of some module intersection annihilator of another module is same as annihilator of S inverse N plus S inverse P. The same equality I am using here that is what we proved only we proved for cyclic modules.

We have proved only for cyclic modules that S inverse of M annihilator of S inverse M is same as S inverse of annihilator. So, from here to here, we need N and P to be cyclic module. Now, take M to be a finitely generated module, take a finite generating set x 1, x 2 up to x n and write m 1, m i to be the sub module generated by x i. So, therefore, I can write M as m 1 plus m 2 plus etcetera m n apply this. So, now here we need S inverse distributes inside, for that we need finite intersection without which S inverse do not go inside, so that is where we are using that it is a finitely generated module.

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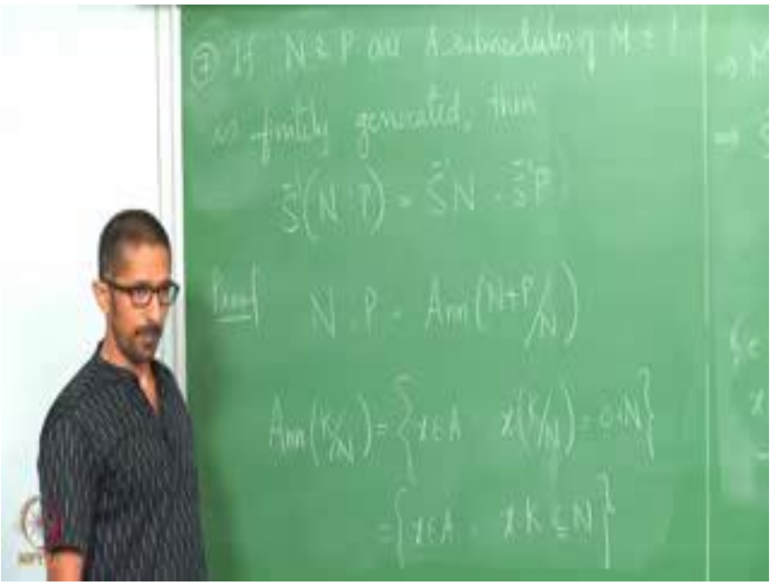

Again if N and so this is 7. If N and P are A submodules of M, and P is finitely generated. Then S inverse N colon P is same as S inverse N colon, so here also the proof is well. So, in this case let me make another observation which I do not remember whether I stated this, N colon P is same as annihilator of N plus P mod N or you know P plus N mod N. See, here if I look at M mod N, I look at M mod N, and I am looking at the sub module generated by P that is precisely this N plus P mod N sub module generated by P. It is annihilator we are looking at the annihilator of P or in other words I am looking at all elements will which will you know kill P or in other words all elements x such that x p becomes 0. Here what is meant by 0 here, it is contained it goes inside N, or in other words all x such that x P is contained in N that is precisely the annihilator that is the meaning of this aspect.

See, annihilator of so you take any module K contain this, this as a sub module of M mod N. This is precisely set of all x in a such that x K by N is x dot K by N is 0 plus N. This is same as saying set of all x in a such that x K is contained in N and that is precisely N colon K. Now, in our case K is N plus P. This is so if I take an element that multiplies N plus P inside N, which is same as saying that multiplies P inside because the other one is anyway. So, therefore, this is equal to this.

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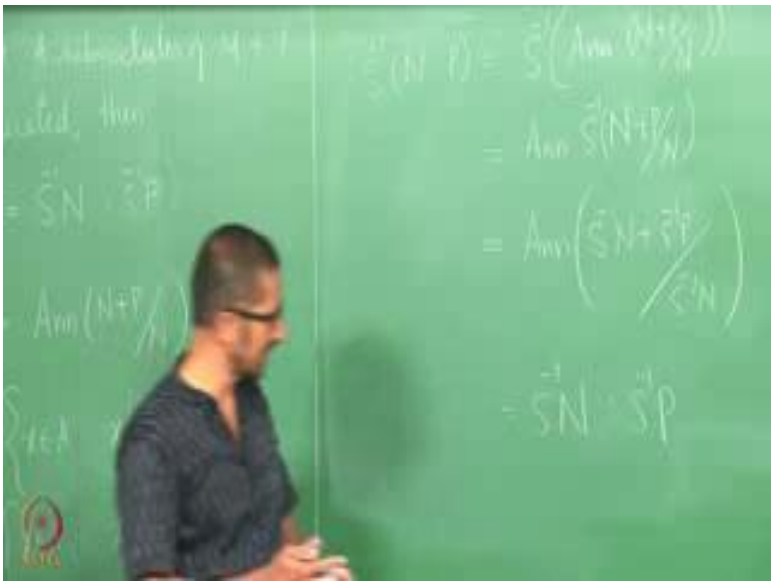

Now, we can use the previous property that we proved S inverse of N colon P is same as S inverse of annihilator of N plus P mod P, but this is annihilator of S inverse of N plus P mod N. So, here we are using the fact that P is finitely generated. P is finitely generated, therefore this is a finitely generated sub module of M mod N, and therefore this is equal to this by the previous property. But this is same as now I can you know take this further this is same as S inverse N plus S inverse P modulo S inverse N, but now I use this property once again to say that this is same as S inverse N colon S inverse. So, these are the some of the most important properties of localization that we will be using.

Now, we will further you know many times will be using what is called localization with respect to a prime ideal that is where as we saw in the last class that many properties if you want to prove A-module is 0, you only have to prove that localize that every maximal ideal is 0. Or some map is injective you can localize, and say that for every localization this is for every local localization at any maximal ideal, it is injective. This is of you know use when your ring is local, see if you are ring is local there is only one maximal ideal.

So, if you just localized at that, and say these are you know these are the properties we are through flatness, injectivity, surjectivity being 0 there are many properties that we have already discussed. So, most of the time, we will be using you know localization with respect to a prime ideal or maximal ideal. We will see that in the forth-coming classes.