Commutative Algebra
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Lecture - 20
Local Properties

Last time we saw 2 important properties of localization.

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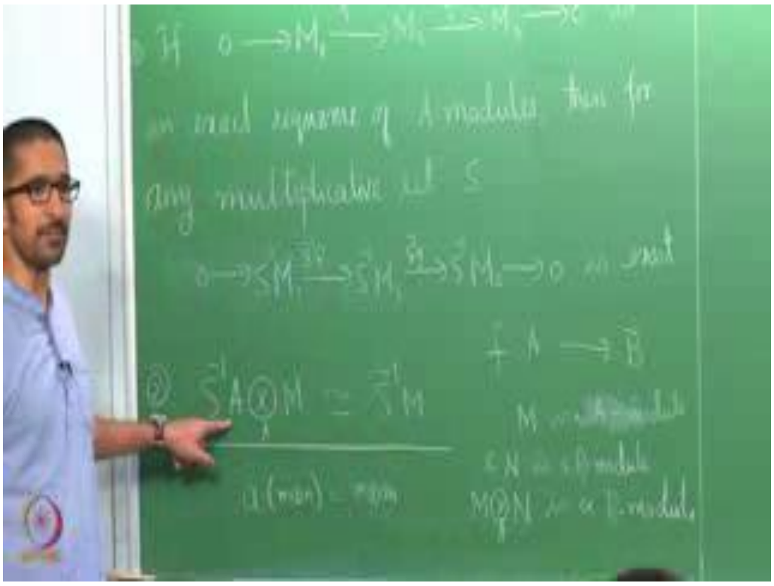

That if 0 to M 1 to M 2 to M 3 to 0 is exact there is an exact sequence of A modules, then for any multiplicative set S, S inverse M 1 to S inverse, M 2 to S inverse, M 3 to 0 is exact. So, I will call this f called this g and this is S inverse f S inverse g.

This is one important property that we saw; another property is S inverse A tensor M as A modules is isomorphic to S inverse M; this isomorphism is as S inverse A modules.

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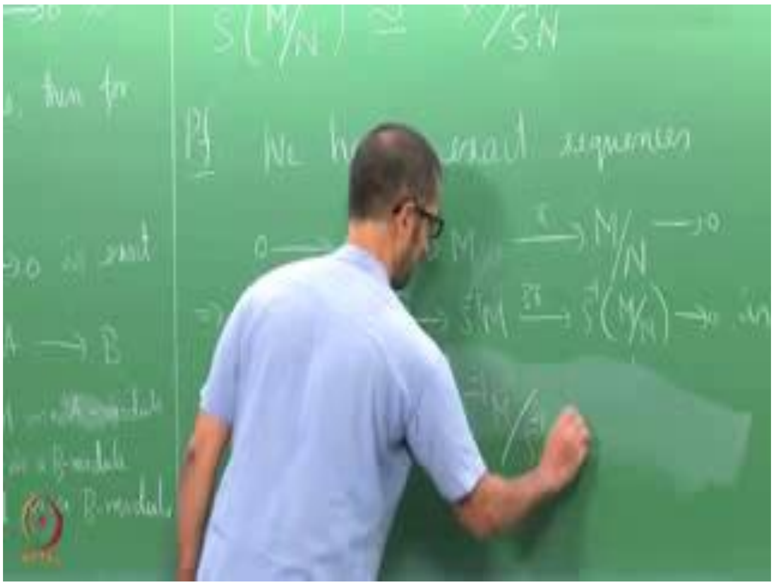

So, one immediate corollary is that if N is a sub module of M.

Student: (Refer Time: 02:32).

Sorry.

Student: How we can make M to be S inverse a module?

This is not S inverse a module this tensor product. See in this tensor product I can. So, this is in general true suppose I have a map f from A to B this is a ring homomorphism, and M is an A module and N is a B module then N is naturally an A module via f right. So, I can take; I can look at sorry this is a bi module this is this is something that we discuss right this is a bi module that is bi A B module.

If this M is a bi A B module N is a B module then M tensor N over A this is a B module; you can see you can define scalar multiplication like this a m tensor n this is m tensor a n right because this is a bi module so.

Student: (Refer Time: 04:25).

I can just take this to be an A module. So, this M tensor M tensor N this is a bi module, then what we get is the M tensor N is a bi module is an A module via taking the scalar product on the first component and it is a B module via taking the scalar product on the second (Refer Time: 05:00).

Student: So, that will also B bi module S will also be.

Yes this will be a bi module. See any S inverse A module is also an a module via this homomorphism a to S inverse A. If N contained in M is a sub module is then S inverse M mod N this is I (Refer Time: 05:26) to S inverse M mod S inverse N. how do you prove this? Given a module M and the sub module N one can think of an exact sequence.

Student: (Refer Time: 05:58).

What is the exact sequence?

Student: (Refer Time: 06:04) 0.

0 to N to M to.

Student: M mod N.

M mod N right; so we have exact sequences 0 to N to M mod N to sorry N to M to M mod N to 0. This is the natural inclusion this is the natural surjection and here the kernel is N which is the image therefore, this is an exact sequence. Similarly, I have an exact sequence S inverse N going to S inverse M going to S inverse M modulo S inverse N to 0; this is exact implies that 0 to S inverse N to S inverse M to S inverse M mod N to 0 is exact. So, if this is I call this I and this pi. So, S inverse I this will be a S inverse pi. So, this is exact implies that S inverse of M mod N this is isomorphic S inverse M modulo S inverse N.

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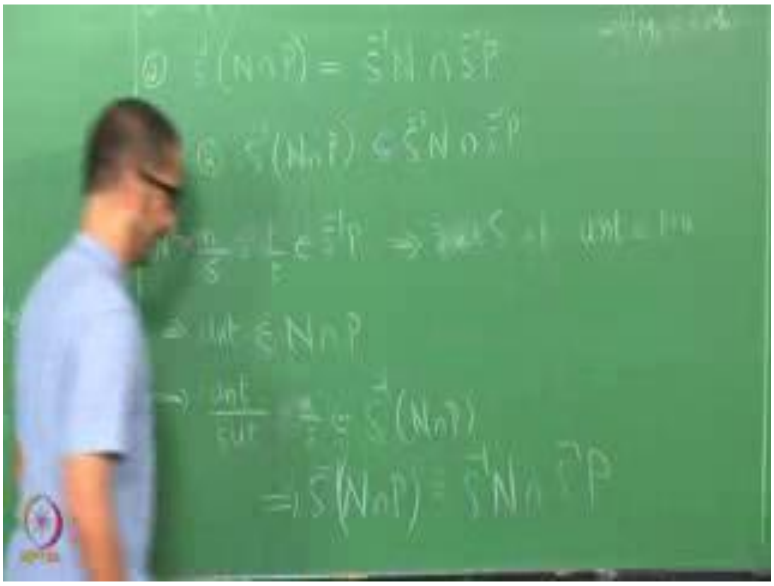

I will just put one or two exercises that if N and P are sub modules of M. Then S inverse N plus P is equal to S inverse N plus S inverse P, S inverse N intersection P is equal to S inverse N intersection S inverse P. The first one is easy verification; for the second one can you say that you know one set is contained in the other which is contained in the other. So, I have this if I have to.

Student: M 1.

M 1 is contained in M 2 then S inverse M 1 is contained in S inverse M 2 right. So, here N intersection P is contained in N therefore, S inverse intersection P is contained in this one similarly it is. So, S inverse of N intersection P is contained in S inverse of N intersection S inverse of P. Now, suppose I take an element here which is let me say n by s is in n by s. So, this is n by s this in S inverse N equal to P by t which is in S inverse P let us say; that what does that say that says that. So, there exists u such that you in S such that, u times n t is p S u. So, that says now look at n; n is in? This is in n and p is in p right.

So, u n t belongs to. So, this says that u n t belongs to; that is by since it is in n it is certainly in n; at the same time it is equal to p s u therefore, it is in p as well; which means this is in N intersection P; that means, u n t divided by s u t all of them are in S this is in, but what is this? This is n by s this is in S inverse of N intersection. So, therefore, what we have shown is that this n by s is in S inverse intersection S inverse of

N intersection. So, that proves S inverse N intersection P is equal to. Algebraically as well this localization is important in the sense that if you certain properties if they are locally true then they are globally true.

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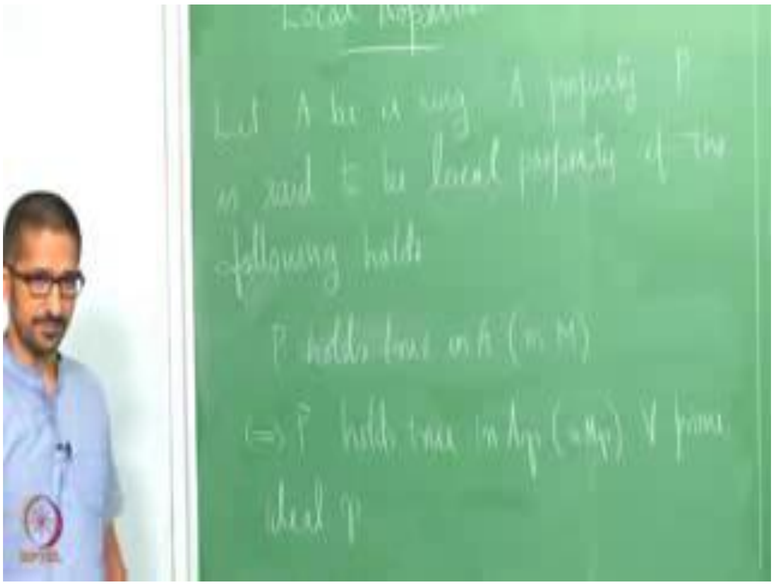

In the sense that. So, let me just write down this definition let A be a ring, a property P is said to be a local property, if the following holds, P is P holds true in A or in M, if and only if P holds true in Ap or in M p for every prime ideal p.

So, there are many properties which are local, you many times you can localize a ring and assume that the ring is local. So, let us look at an I will give you one example; being 0 a modulus being 0 is a local property what do I mean by that?

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Let me write it let A be a ring and M be an a module, then the following are equivalent. M is 0, M localized at p is 0 for every prime ideal p, and third is even more stronger this is localized at every maximal ideal. What we are saying is that in a ring if M you look at S inverse M, if S inverse M is a 0 for very maximal ideal a compliment S being complement of maximal ideals then the module has to be 0.

Or in other words if this is non-zero I can at least find one maximal ideal for which S inverse is non-zero localization gives you a non-zero module. So, this is 1 implies to trivial, 2 implies 3 trivial because every maximal ideal is prime. So, we need to prove 3 implies 1. So, assume that M localized it M is 0 for every maximal ideal; now suppose M is non-zero; that means, there exists a non-zero element x in M. So, since x is non zero annihilator of x cannot be equal to A right, and annihilator equal to A if and only if x is 0 otherwise annihilator of x is equal to A means one times x is 0, but one times x is always x. So, if x is non-zero this cannot be equal to A which means annihilator is a proper ideal of A.

So therefore, I can find the maximal ideal containing annihilator of x. Now look at the localization of M with respect to this maximal ideal; in that module x the image of x is 0 because that module itself is 0; we are assuming that M localized it the maximal ideal is 0 for every maximal ideal.

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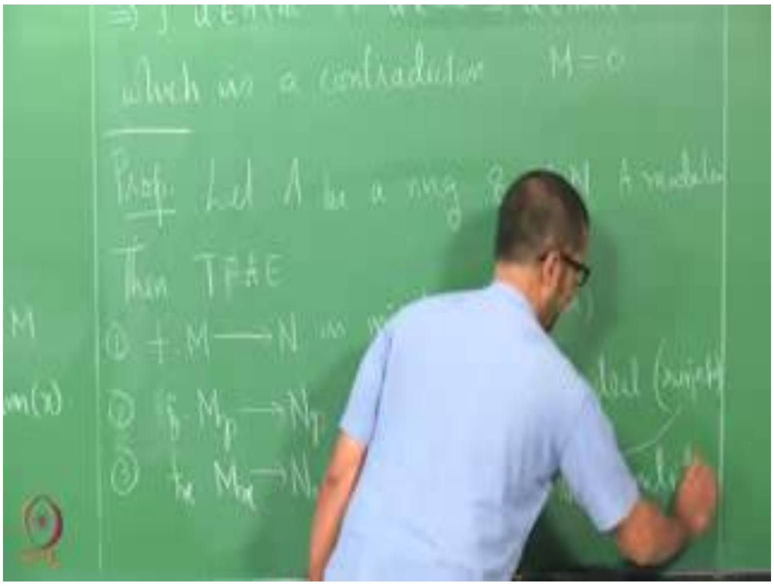

So, this says x by 1 is 0 in M localized at this particular maximal ideal; that means, there exists u in A without M such that u times x is 0; that means, u is in the annihilator of x, but annihilator of x is contained in M; u is not in M and u is in M this is contained in M that is a contradiction. Therefore, M is 0. So, being 0 is a local property. So, if you want to prove that a module is 0 you can localize it you only need to prove that it localized at every maximal ideal is 0; or if you want to prove that something is non zero you only have to prove that there exists a maximal ideal for which this is non zero.

These are the you know tricks that comes up in often in even in research, that this is local global principle is something that is being used in the you know active research even nowadays. Another property another local property is of maths; being injective surjective etcetera that is a corollary of the exactness property of localization, but let us state that let A be a ring and M comma N A modules then the following r F from M to N is injective, in fact, you can just replace this by surjective, what should be the next statement?

Student: S inverse.

S inverse f from; so I will write this as M p. So, f p to N P this is injective for every prime ideal p prime ideal f M from M. So, this is you know again replace it by surjective here, this to N M plus injective or you know surjective for every M maximal ideal. Again one implies to that is property of localization right.

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One implies two property of localization f this injective, f is injective is equivalent to saying that 0 to M to N is exact where M to N the map is f. So, therefore, you localize you still get an exact sequence. So, the first 1 one implies to is a direct consequence of exactness of the localization fronting.

2 imply 3 trivially true because every maximal ideal is prime ideal 3 imply 1. So, I know that f from M localize at M to N localized at M is injective for every maximal ideal; I want to say that f from M to N is injective.

So, how do I do that again suppose there exists suppose f of. So, let us take M in the kernel of f; that means, f of M is 0 which means f of M by 1 is 0 by 1 N, N localized at m, but this is same as saying f of M by 1 is 0 f M of you know in N localized at m, but this is f M is injective this implies.

Student: M by.

M there exists some u in s. So, a without M such that M u is 0 we want to say that M is 0 I mean look at the proof that we did last time how did we arrive at that does that give you some indication see this is true for every maximal id ideal we have taken an arbitrary maximal ideal instead of that what we will assume what should be the maximal ideal that we will be taking see this says that u is contained in the annihilator of M right now if i. So, this is M is if M is non-zero I can always find a maximal ideal containing annihilator of M in that case I will have same as the last line there u is in a without M and u is in m. So, I added here if M is non-zero then there exists a maximal ideal M containing annihilator of M this is contained in m.

Therefore, which is a contradiction because u is in complement of M and this part says that u is in M therefore, M has to be 0 I will leave the second part that replace injective by surjective and prove that all are equivalent that as an exercise it is say again proving 1 implies to again applying the local localization property 2 implies 3 is always trivial we only have to prove 3 implies 1 there also you choose your maximal ideal appropriately you will get a similar contradiction.

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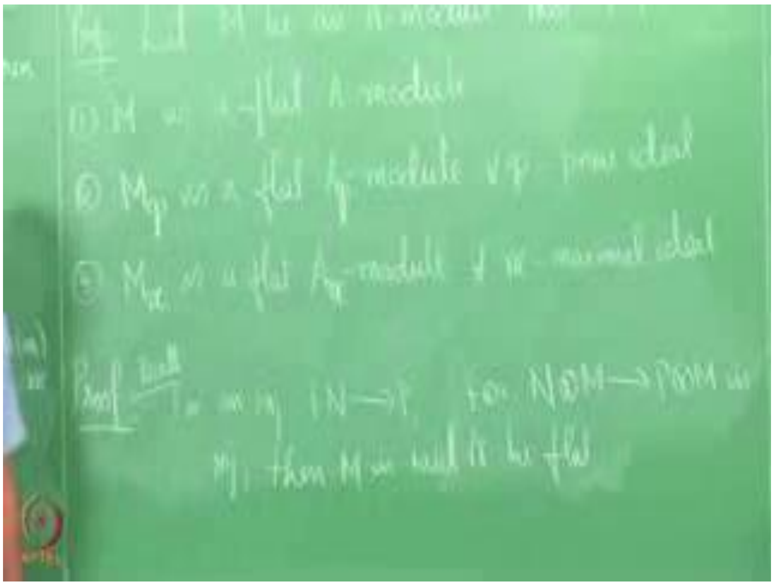

Another nice property that that is being is that its flatness which is local property flatness is a local property what do I mean by that a module is an a module is flat if and only if M localized at P is flat for every its a flat a P module for every prime and M m is flat for flat a M module for every maximum ideal let me write it up let M be an a module the N the following are equivalent M is a flat a module M localized at P is a flat a P module for every prime ideal P and M localized at maximal ideal M is a flat a M module for every maximal ideal.

Again 2 implies 3 is straightforward how do you prove the first 1 that is again you know property of the localization and 1 more property that we discussed. So, let us just recall what is meant by a flat a module a modulus flat a module if for an injective map f from

N to P f tensor 1 from N tensor M to P tensor M this is injective right this is the flatness then M S flat. So, I need to prove that M m is flat a module implies M localized at P is flat a P module this is this is the let f from a to B be an a be a ring homomorphism and M is a flat a module.

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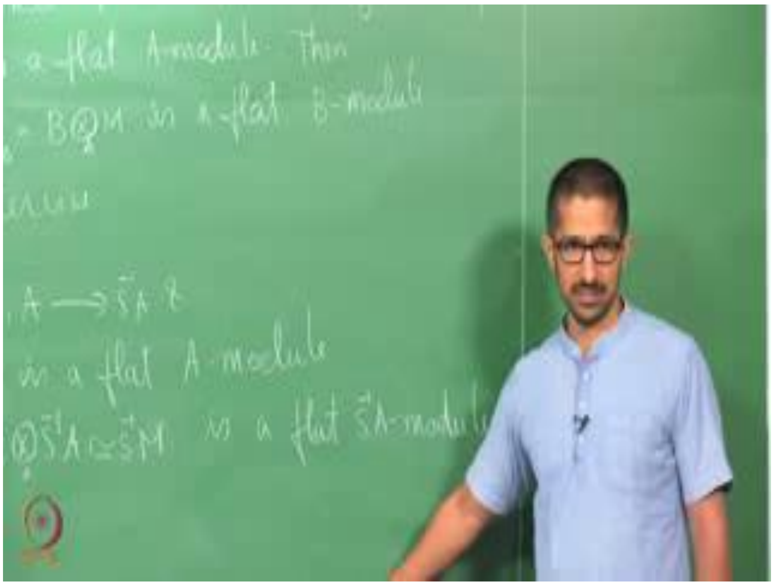

Then the module M B which is B tensor M this is a flat B module this is something that I i do not know whether I proved in the class, but I left it as an exercise in the did did we not you do not remember. So, let me leave this as an exercise now let us look at this situation I have f from a to S inverse a M is a flat. So, again 1 implies 2 M is a flat a module. So therefore, by this exercise M tensor S inverse a this is is a flat B module B here is S inverse, but what is M tensor S inverse a this is same as S inverse M. So, therefore, M tensor S inverse which is isomorphic to S inverse M is a flat S inverse a module and that is exactly what we wanted to prove. In fact, we are proving even more than 2 for any multiplicative set S inverse a is a flat now S inverse M is a flat S inverse a module.

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Two implies 3 as usual is trivial every prime I every maximal ideal is prime. So, let us prove 3 implies 1 how do you prove 3 implies 1 I i know that M m is a flat a module I want to prove that M is a fla flat a module. So, let us start with the sequence let 0 to 0 to N to P be exact use localization property to say that 0 to N P to sorry N M N M to M m this is exact sorry P M is exact and that would imply M is a flat a module that would imply that N M tensor M m to P M tensor M m this is exact for every maximal ideal, but this is see N M tensor M m this is same as S inverse a tensor with N tensor M this is another property that initially we stated S inverse a tensor N tensor M this is isomorphic to S inverse N tensor S inverse M using this property what we have is this is same as localizing. So, this is and which is same as saying S inverse of I think I wrote it in this form S inverse of N tensor M is same as S inverse N tensor S inverse m.

So, this is N tensor M localized at M, this to P tensor M localized at M, this is exact see this is exact for every maximal ideal M. This is exact for every maximal ideal M; this is injective for every maximal ideal.

Student: (Refer Time: 37:42).

So, that is the localization property of being a maps being injective or surjective is a local property that is what we proved earlier. So, this implies that 0 to N tensor M to P tensor M is exact I is injective that implies M is a flat a module from here to here. So, look at call this M 1 call this M 2 then what is what does this say 0 to M 1 localized at M 2 M 2 localized at M is injective for every maximal ideal, but the earlier proposition that we proved 3 implies 1 says that N 1 to M 2 is injective.

So, there are few these are some of the important properties that are you know local and this will be used during the course many times that localization a map is injective its localization is injective map is surjective its localization is subjective and so on.

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Let us look at more properties of S inverse a. So, we the other day we proved that you know ideals of S inverse a this set of all S inverse I such that I is an ideal of a and S intersection I is empty right these are the proper ideals of S inverse i.

This is this we proved right the other day we proved and I asked you to think about what are the prime ideals of S inverse a. So, to start with let suppose I take a prime ideal P if P is a prime ideal of a such that P intersection S t this condition has to be there because other without which S inverse P if I localize P that is going to be the whole ring. So, P intersection S is empty then can you say that S inverse P is a prime ideal how do you prove this 1 way is of course, you know you can take elements multiply see that 1 is there other is another more elegant way what is S inverse of S inverse mod S inverse P this will be isomorphic to S inverse of a mod P now a mod P is an integral domain need not necessarily be feel. So, this is this is isomorphic to if I this is isomorphic to S bar inverse of a mod P where S bar is the image of S in a mod p.

That will again be multiplicative set. So, I am see here what am I doing am inverting elements of s, but now we have already gone modulo a mod P this as ring itself these 2 will be isomorphic this will be a ring and as rings these 2 will be isomorphic that is what we are proving or by definition of elements in the quotients I mean multiplication of elements in the quotients in the mod going modulo this will be same as S inverse a mod P a mod P is a field and sorry a mod P is an integral domain therefore, S inverse a mod P will again be an integral domain. So, this is another way to look at it try to complete this argument that is see I am I am just trying to look at this as a ring this has a module structure I am see when I say S inverse a mod P S is not a subset of a mod p.

Student: (Refer Time: 43:53).

Exactly we are taking this is S mod P I mean in the sense that we are just taking the corresponding images in a mod P this is a module right I am taking this S inverse a mod P this is an a module we are thinking of this as an a module because we are doing the localization see S is a subset of a. So, this localization is as a module.

But here if I take S into a mod P that will again be a multiplicative set and this will become S inverse a mod P M S bar inverse a mod P a mod P is an integral domain therefore, this will again be an integral element that is something to be checked, but that is easy I will just talk about the converse. So, what we are saying is that if P is a prime ideal then S inverse P is a if a if P is a prime ideal then S inverse P is a prime ideal in S inverse a.

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Now, conversely if I start with let P be a prime ideal in S inverse a then I have this natural map a to S inverse a and I take P to be. So, I call this f f inverse of P then 1 can see that P is equal to S inverse of then P does not intersect S and P is S inverse.

So, this is. So, what we have shown here is a 1 to 1 correspondence between prime ideals of S inverse a and prime ideals of a not intersecting with S right for every prime ideal P which is not intersecting with S we showed that we have a prime ideal of S inverse a similarly for every prime ideal of S inverse a I have a. In fact, both of them are unique from here S inverse P is uniquely constructed and from here this P is uniquely constructed and hence this S inverse P is uniquely constructed.

Therefore, this gives a 1 to 1 correspondence between prime ideals of S inverse a and prime ideals of a not intersecting with S I will recall this in the next class.