Commutative Algebra
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Lecture - 19
Localization (Continued)

So, we were talking about tensor sorry localization and we proved localization, the uniqueness of localized localization of a ring at a certain multiplicative set.

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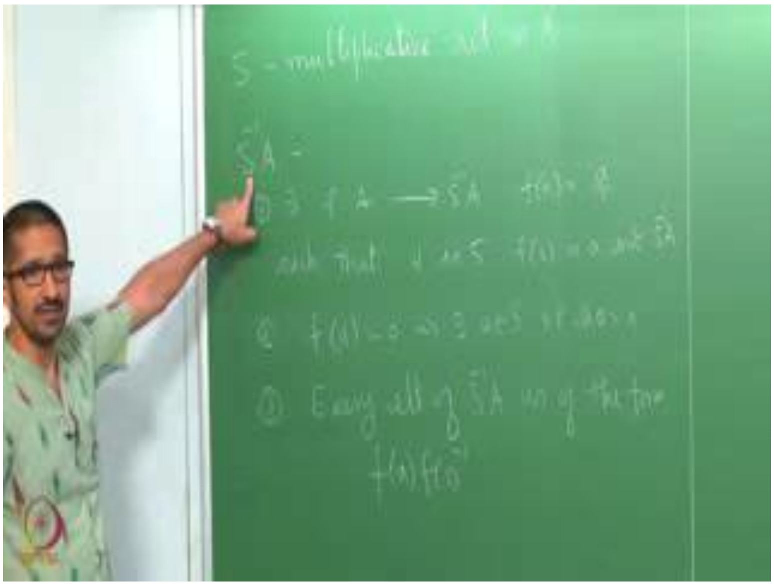

So, we said last time we showed that S is a multiplicative set in A, then the set S inverse A is uniquely determined by three properties, what are the three properties?

Student: (Refer Time: 01:08).

There exists a map f from A to S inverse A sending f of a to.

Student: a by 1.

a by 1 such that.

Student: for every (Refer Time: 01:24).

For every.

Student: S belongs to (Refer Time: :).

S belongs to s.

Student: f S is a unit

f S is a unit in S inverse a then second property.

Student: If f of a is 0.

f of a is 0 implies.

Student: There exists (Refer Time: 01:49).

There exists u in S such that.

Student: u (Refer Time: 01:54).

u a is 0 and the third property is.

Student: Every element (Refer Time: 01:58).

Every element of S inverse A is of the form.

Student: (Refer Time: 02:06).

f a times f S inverse right. So, the localization, this ring is called localization of A with respect to the multiplicative set S. Let us look at some interesting properties of more properties of S inverse A.

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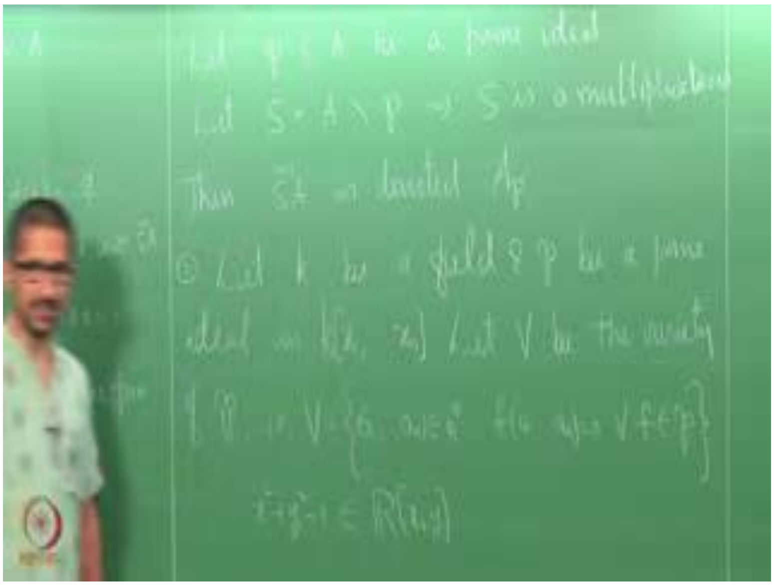

This is one of the most important applications of S inverse A that if I take a prime ideal be a prime ideal and if I take S to be A without p then 1 belongs to S right and if a and b are in S a b is also in S right, because if.

Student: a b (Refer Time: 03:22)

If a b is in S, if a b is not in S that means it will be in p, but p is a prime ideal therefore, either a is in p or b is in p or in other words either is not in S or b is not in S. So, therefore, this is a multiplicative set. It is a multiplicative set and one can talk about S inverse A. So, S inverse A this is also denoted by is denoted by A P, A localized P. So, this is a very important ring localization at prime ideals.

One instance that I can immediately talk about is suppose I take you know a prime ideal in. So, let k be a field and p be a prime ideal prime ideal in k x 1 up to x n. Take V to be the variety of variety of p what does that mean? V is all n tuples in k n such that f of a 1 up to a n is 0 for all f in p. Look at all the elements of p and look at all the points in k n which vanish on this one.

So, let us look at a simple example I take the prime ideal, I take the ideal x square plus y square minus 1 in R x comma y. Can you tell me what are the elements in R 2? Such that you know variety correspond and I look at this ideal, this ideal; what are the elements in

R 2. So, this is a prime ideal, this is an irreducible element there are some you know one needs to prove this there is an irreducible element and it this is a, is it a p i d.

Student: u f d.

It is a u f d ok. So, therefore, this is prime and therefore, this is if this is prime.

Student: (Refer Time: 07:00)

It is a primate ideal the ideal generated by this is a prime. Now what does V corresponding to this can you can you imagine that.

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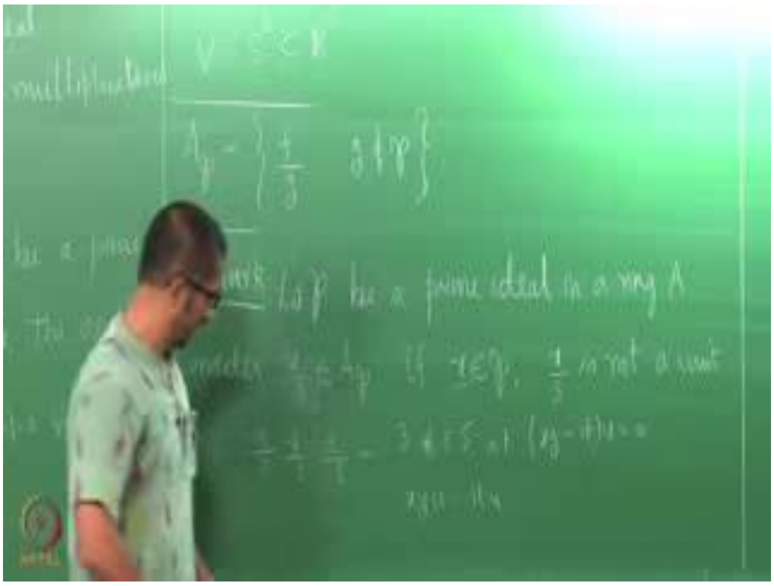

Student: Yes, sir.

I am looking at all points a comma b in R 2 such that a square plus b square minus 1 is 0 or in other words all a b such that a square plus b square is 1 which is.

Student: (Refer Time: 07:25)

The unit circle we call it S 1. So, these are all you know algebraic sets or you know these are all called the varieties. In algebraic geometry what one does is given algebraic sets one has to study lot of properties of this, algebraic geometric properties. So, the idea is create corresponding algebraic objects, study algebraically and take the properties back into the geometry. So, one of such objects is you know the correspondence between V and p ok.

Now, if I look at. So, here V is in this case V is S 1 in R 2. Now coming back to the a general case the ring, so I called this ring to be A, the ring A p here is set of all f by g such that g is not in p right that is by definition I am looking at S inverse A, A localized at p is S inverse A and S is?

Student: A minus p.

A minus p. So, this is by definition set of all f by g where g is not in p; g is not in p is equivalent to saying that g does not vanish on all the points of V. So, or in other words as a ring of rational functions from k into k this is irregular this is a well defined continuous function because this does not have any 0s this is no where vanishing right g is no where vanishing on V. So, therefore, this is A p is set of all it is called ring of regular functions on V and this ring plays an important role in algebraic geometry in the study of varieties. So, this is one such occasion that one you know encounters in as applications of this localization. So, this is one of the most important objects. There is more you know this is, there is more closer study that if you take local ring corresponding to a point you can further you know take a what is called a ring of regular functions which are at a point local ring corresponding to points in a variety. We will not get into that you will certainly see that if you opt to take algebraic geometry course in their next semester.

So, let us get back to this. So, this is one of the important applications of localization in. So, now let us look at this ring in general. So, we this is A is k x 1 up to x n and p is prime ideal and so on. Let us get back to another remark in general suppose I take A p be let p be a prime ideal in a ring A. So, I am not saying that a is k x 1 up to x n this is in general a be a commutative ring with identity and p be a prime ideal and consider A p take an element in a localized at the, so let x by S is in a localized at p. See if x is not in p, if x let us take; if x is in p can this be a unit? If x is in p x by S cannot be unit right x by S is not a unit, why is this not a unit? For this to be a unit I need an element let us say y by S such that y by t such that x by S by y by t this is 1 by 1 or in other words.

Student: (Refer Time: 12:56).

What does this mean?

Student: (Refer Time: 13:01)

why x a unit?

Student: (Refer Time: 13:09).

This will say that there exists t in, so u in S such that.

Student: x y minus (Refer Time: 13:22).

X y.

Student: minus (Refer Time: 13:26).

Minus S t times u is 0 right. What does this say? x y u is S t u, now u is not in p see this. So, see this is an element x y u, x y u is not in p because it is in s; look at the right hand side it is S t u all of them are in S therefore, this is.

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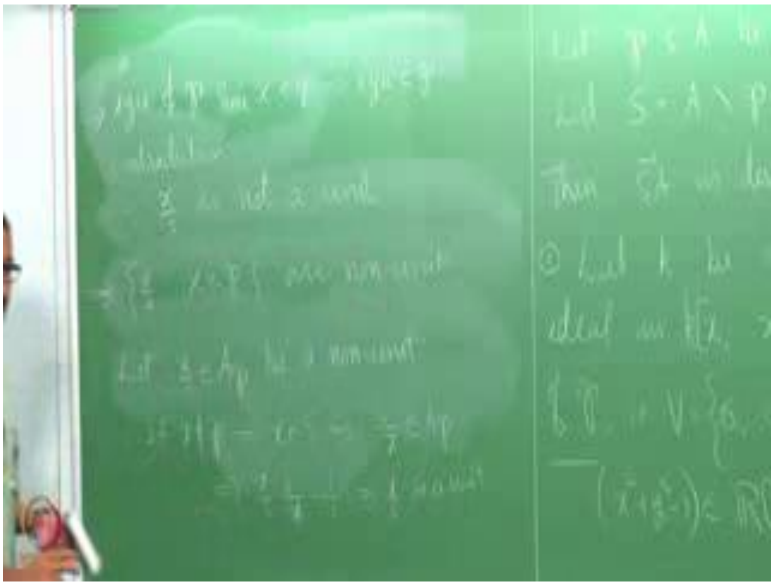

Student: (Refer Time: 14:18).

x belongs to p implies x y u has to be in p. So, these 2 this is one statement there is another statement contradiction. So, therefore, this implies x is also, since x is in p x y u is in p therefore, this is a contradiction therefore, what does the contradiction say? That x by S is not a unit that is the assumption that we have made right any element of the form x by S such that x is in p are non units.

Now suppose I take a non unit ok. Let x by S in a localized at p be a non unit. I claim that x is in p if x is not in p that implies x is in S and that would imply S by x is in A p and that would imply that x by s, S by x this is 1 by 1 that implies x by S is a unit. So, that is again a contradiction; that means, what we have proved now is that any unit in x by S any unit in A p is of the form x by S where x is an element of A p.

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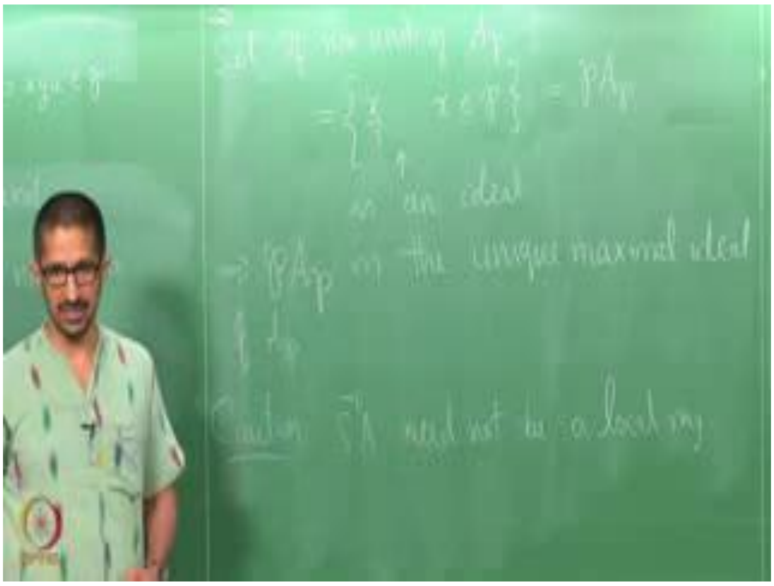

So, the set of all non units, we have proved that set of all non units of A p is set of all x by S such that x is in p right. We proved both the directions if x is in p then x by S is a non unit, similarly if x by S is not a unit then x has to be in p.

So, we proved that precisely these are the non units, this can be written as you know p a p I will write it as p A p right it is of the form p A p this is a this is an ideal x by S where x is in p is an ideal. So, this set is an ideal why is that an ideal? Immediately one can check 0 is there it is an abelian group if a by S is there then minus a by S is there ah then the multiplication with respect to r by S if x by S and r by t take r by t in a p then product is there in inside because you know that p is an ideal. So, therefore, this is an ideal.

Student: (Refer Time: 18:22).

Sorry. So, what we have shown is that set of all non units form an ideal therefore, it has to be.

Student: The unique maximal.

The unique maximal ideal. So, this implies that p A p is the unique maximal ideal of A p; that means, A p is a local ring that is how the name localization came, we are from A we have come down to a ring with unique maximal ideal or you have obtained a local ring that is why the process is called localization. But you have to be slightly careful with the terminology S inverse A need not be a local ring for all. So, S inverse A need not be a local ring ok.

I will leave it to you to think about this for the time being, I can easily give an example where this is not local, but we will study little more properties of a localization that will immediately say about ideals here.

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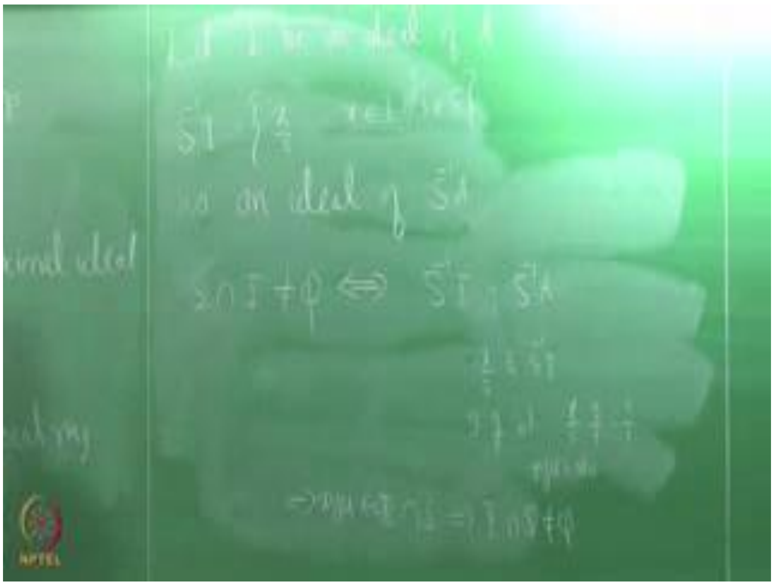

To start with, suppose let I be an ideal of A and look at S inverse I, I define this set like this x by S where x is in I and you know S in S, this is an ideal.

Student: Sir when you present the elements (Refer Time: 20:44) like this as well as.

Yeah.

Student: Now the summation.

So, what if I take x 1 by S 1 plus x 2 by S 2, what does this by definition?

Student: S 2 (Refer Time: 20:57).

s 2 x 1 plus S 1 x 2 by S 1 S 2.

Student: (Refer Time: 20:04).

Yeah, this is in S and when we are talking about ideals you know this is also there. So, it makes sense. So, this is for the same reason you know you can prove that this is an ideal, ideal of S inverse A.

But now the question is when will this be a, if I is a proper ideal does that mean that A inverse is a proper ideal in this case can you tell me you know if I is a proper ideal when do you when can we say that S inverse I will be a proper ideal. When will it be not a proper ideal? When will it be equal to S inverse A?

Student: When a unit belongs (Refer Time: 22:06)

S inverse I has to be S inverse A only if.

Student: (Refer Time: 22:13).

A unit belongs to S inverse I, a unit belongs to S inverse I means.

Student: Some element (Refer Time: 22:19).

Some element of S is in.

Student: (Refer Time: 22:25).

So, see any element of S inverse I looks like x by s. When can we say that and here x belongs to.

Student: I.

I and we want to say that this will be unit for that what I need is.

Student: (Refer Time: 22:48).

X should be in S as well as in.

Student: I.

I or in other words S intersection I is non empty then S inverse I is equal to S inverse A. So, S intersection I is non empty implies S inverse I is S inverse A, what about converse? Suppose S inverse I is equal to S inverse A; that means, I have an element x by S which is a unit in S inverse I or in other words, this says that there exists y by t such that x by S times y by t is 1 by 1 or in other words x y u is equal to S t u what is that we want to check? We want to see whether.

Student: (Refer Time: 24:12) x belongs to (Refer Time: 24:13).

X belongs to.

Student: s.

S, can we say that x belongs to s, x is in I, y is in I x y belongs to y from here can we say that S inverse I is sorry S intersection I is non empty.

Student: Yeah, since S t u belongs to s.

Yes.

Student: (Refer Time: 24:49) x y u belongs to (Refer Time: 24:51).

x belongs to I.

Student: x y u belongs to I.

X y u we do not want to say that x is in I we are not really bothered to say right we only want to say that I intersection S is non empty. So, here x y u belongs to.

Student: I.

I because x belongs to I and it is S t u therefore, it belongs to it is in S. So, therefore, I intersection S is non empty. So, the converse is also true. So, S inverse is non empty if and only if S inverse I is S inverse A. We will go into the properties little more properties. So, think about now I will give you one exercise to think about what are the prime ideals of S inverse A, think about it. We will come back to that.

Now we talk about localization of modules, we can do the same process.

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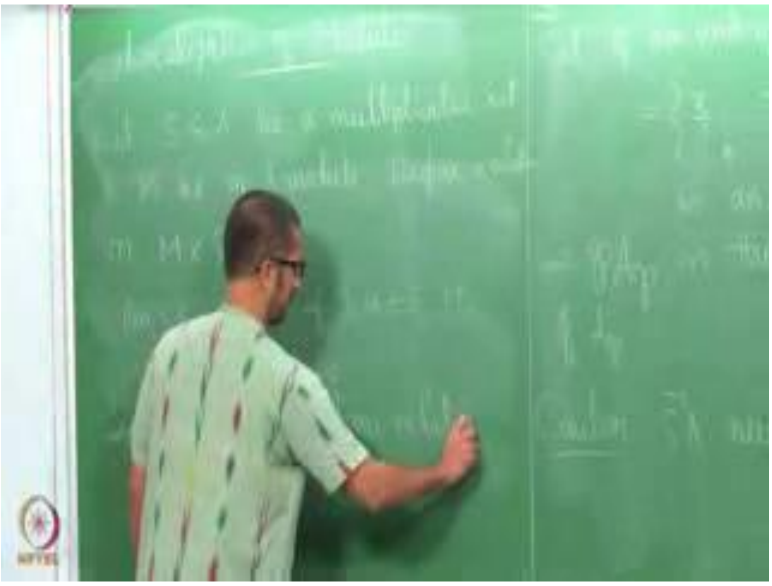

Now we talk about localization of modules, we can do the same process; S be A multiplicative set and M be as A module define a relation on M cross S by M s related to M prime s prime if there exists u in s such that u M s prime minus M prime s is 0. The same thing, we are instead of a here we are putting m, no difference at all. This is an equivalence relation; I will again this is an equivalence relation.

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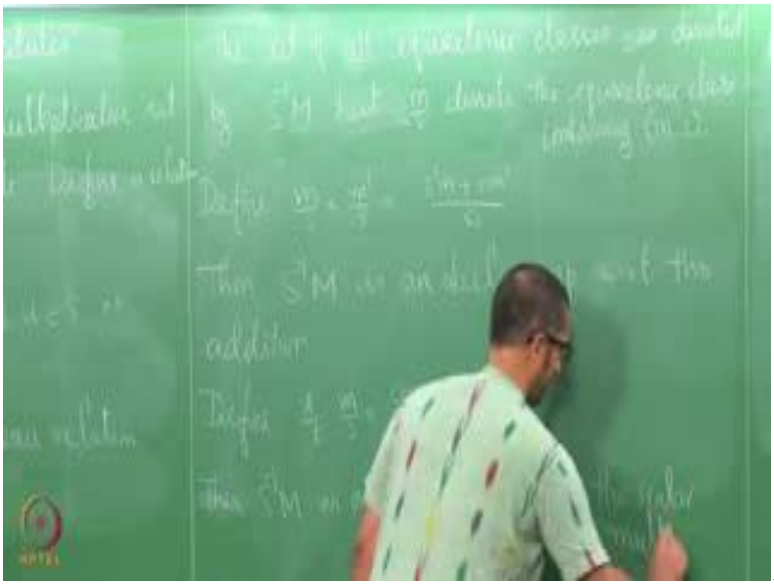

Therefore, equivalence relation partition the set and we can look at the set of all equivalence classes. The set of all equivalence classes is denoted by S inverse M naturally. And what do we want to say next.

Student: this is a (Refer Time: 28:15).

This is a?

Student: (Refer Time: 28:17).

We want to say this is an S inverse A module right. So, to start with; we need to define an addition on this. Naturally m by s; so let m by s denote the equivalence class containing. Now define m by s plus M prime by s prime by s prime M plus s M prime by s s prime. Then again first you have to show that this definitions well defined and prove that this is S inverse M is an abelian group with respect to this addition. And define scalar multiplication A by t times m by s equal to a m by s t.

So, this then S inverse M is an S inverse A module with respect to this scalar multiplication. We are just going through the same work that we did for S inverse A while proving S inverse A is a ring. So, I am not going to repeat all of them. These are the exactly the same. Here we have slightly less amount of work because it is you only need to prove that it is module. So, this gives the localization of M with respect to a multiplicative set.

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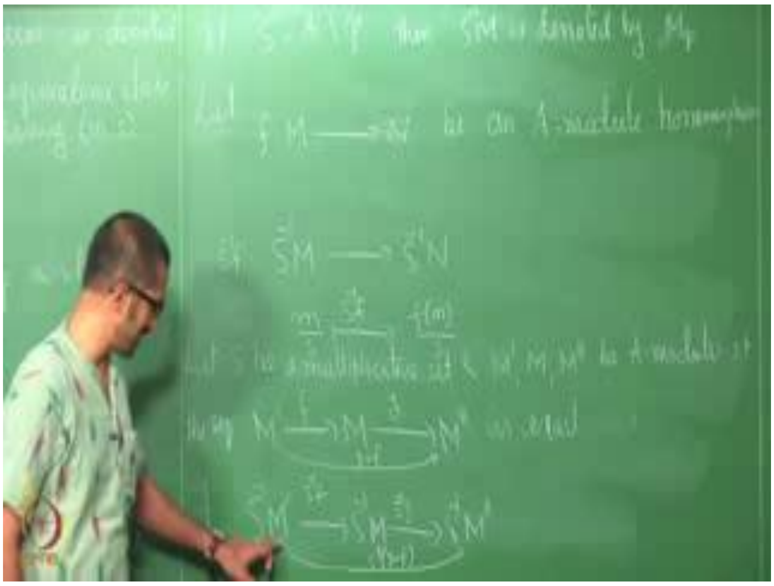

Just to set up a notation if S is A, I mean the complement of a prime ideal then S inverse M is denoted by M localized it p. Suppose I have a map from M to N. Let f from M to N be an A module homomorphism; this be an A module homomorphism. Can you think of this map inducing a map on the localization? Can we think of a map from S inverse M to S inverse N.

Student: (Refer Time: 22:07) m by s.

M by S being mapped to?

Student: F m by s.

F m by s; so this naturally induces a map between. So, I will denote this map by S inverse f, this naturally induces a map from S inverse M to S inverse N. Therefore, you can immediately ask questions: you have a map f and that induces a map S inverse f, so therefore it is natural to ask questions how the properties of f and properties of S inverse f are related. If this is injective can we say this is injective this is surjective, can we say this is surjective.m

More generally suppose I have a exact sequence M prime to M to M double prime we call this g call this f, then corresponding to this I have a sequence S S inverse M prime S inverse M to S inverse M double prime, this is S inverse f and this is S inverse g. Suppose this is exact, can we say this is exact? Natural question. Let us try to investigate this, so my our setup is. Let S be a multiplicative set and M prime M M double prime be A modules such that the sequence is exact. This sequence is exact, therefore there exists.

So, there exists this sequence; there exists an induced sequence we do not know whether it is exact or not, but let us look at some properties of this. So, I have a map from an M prime to M M to M double prime, so I have this composition right this is g composite f. Similarly, I have a map S inverse of g composite f this g composite f induces a map from S inverse M prime to S inverse M double prime its I denoted as S inverse of g composite f. And at the same time I have S inverse g composite S inverse f, coming from composition of these 2 units. How are this composition and this map related?

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Let us take 1 element M prime by s prime in M prime by S what does S inverse g composite S inverse f acting on this M prime by s prime. This is S inverse g acting on S inverse f of M prime by s prime which is S inverse g of this is what is the definition.

Student: (Refer Time: 36:22).

F M dash by s prime; and this by definition is?

Student: G of (Refer Time: 36:29).

G of?

Student: F M prime.

F M prime.

Student: By s

By S s prime, but again look at this one this is g composite f acting on M prime by s prime. But this is nothing but S inverse of g composite f acting on M prime by s prime. So, what have we proved? That S inverse of g composite f is S inverse g composite S inverse f. This sequence is exact; what does that mean? Kernel g is equal to.

Student: Image of f.

Image of f or in other words.

Student: G composite f is 0.

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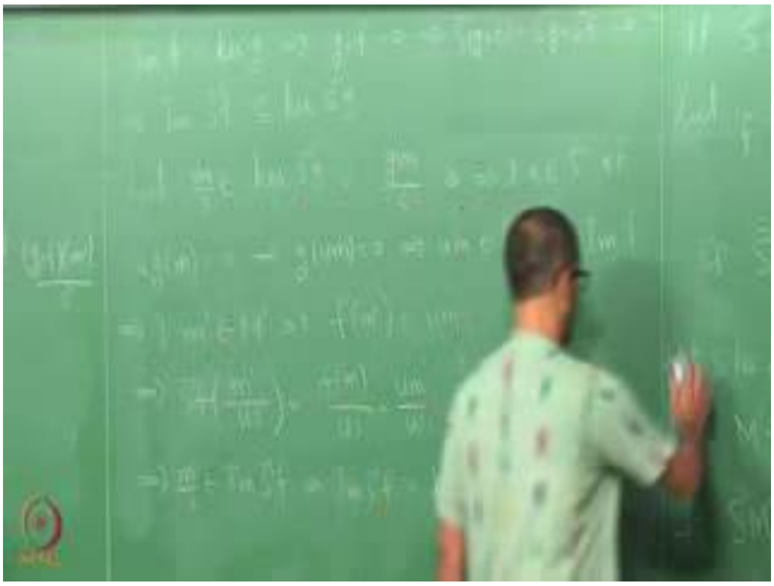

G composite f is 0; that means g composite f is 0 that means; S inverse g composite S inverse f is 0. Or in other words image of S inverse f is contained in the kernel of S inverse g. So, g come image of f is equal to kernel of g implies

Student: G (Refer Time: 38:04)

G composite f is 0 that will imply that S inverse of g composite f which is S inverse g composite S inverse f that is 0, and that would imply that image of S inverse f is contained in kernel of S inverse g. The question is whether

Student: Reverse (Refer Time: 38:29)

The reverse inclusion is true. Let us take an element here, let m by s be in kernel S inverse g, what does that mean? This implies S inverse g acting on this 0 which is g m by s is 0; g m by s is 0. And that implies that there exists u in S such that u is 0. But now u is a ring element, g is a an A module homomorphism, therefore u times g of M is same as.

Student: g of u M.

G of u m; that means u M is in?

Student: Kernel of g.

Kernel of g, which is same as?

Student: Image of f.

Image of f; what does that mean? That means there exist some M prime in M prime such that f of M prime is.

Student: U m.

U M; can you complete it now? The question is whether m by s is in the image of S inverse f.

Student: Since u belongs to the (Refer Time: 39:56) unit. So, if I can take it (Refer Time: 39:59).

So, what element, can you tell me the element which maps to m by s? F of that element should be m by s.

Student: (Refer Time: 40:13).

M prime by?

Student: (Refer Time: 40:16).

This implies that f of sorry, S inverse f of M prime by u S will be equal to f M by u is which is u m by u S; as an equivalence class this is same as m by s. So, this implies that m by s is in the image of S inverse of f, and that implies image S inverse f is equal to kernel S inverse g. So what we have proved is that, if M prime to M to M double prime is an exact sequence then the induced exile sequence S inverse M prime to S inverse M to S inverse M double prime is also exact.

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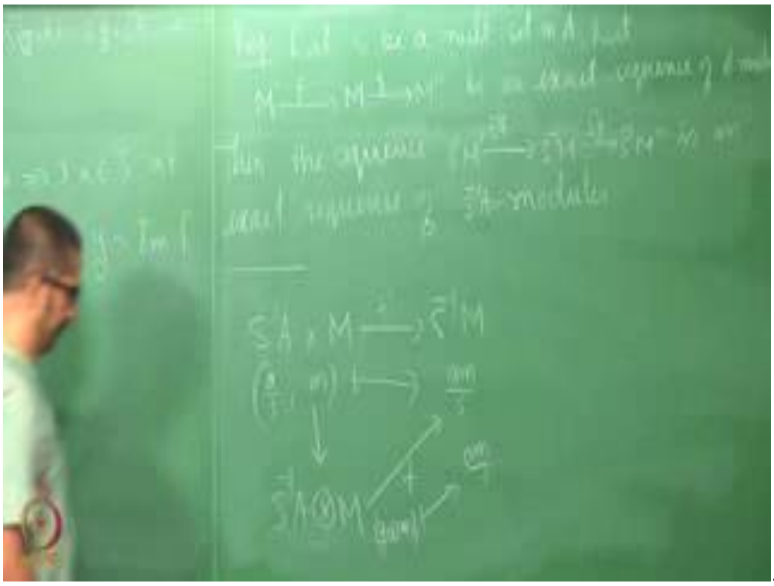

So, let S be a multiplicative set in A. Let M to M double prime be an exact sequence of; so let me call this f g, exact sequence of A modules. Then the sequence S inverse M to S inverse sorry, M prime to M to S inverse M double time this is an exact sequence of S inverse A modules; this is exactly what we proved now

Student: (Refer Time: 42:44) module (Refer Time: 42:47) equal g and how can we (Refer Time: 42:51).

Sorry.

Student: (Refer Time: 42:54).

U dot g M equal to 0; where is this equation happening, this equation is happening in.

Student: d (Refer Time: 43:02).

M double prime right; M double prime it is an A module. This is by definition when an element is 0 in m by s by definition it is this. And this equation is happening in the module M, and g is an A module; A module homomorphism that is why we. So, another important property I think I should be able to do that today.

So, I have this A module, I have S inverse; there is a natural you know map S inverse A cross M to S inverse M. A by S comma M is map to a m by s. And by the universal property of tensor products there is this extends to a map from S in S inverse a tensor M to S inverse M. So, A by S tensor M is map to a m by s. Can we expect that this is an isomorphism? Can we expect this is injective, surjective? Is it surjective? Any elements here is of the form m by s.

Student: So, 1 by S tensor (Refer Time: 45:02).

I take 1 by S tensor M is map to m by s. So therefore, this map let me call this f prime, I call this f; f is surjective. So the question is, is this injective? It is an onto module homomorphism, is this injective?

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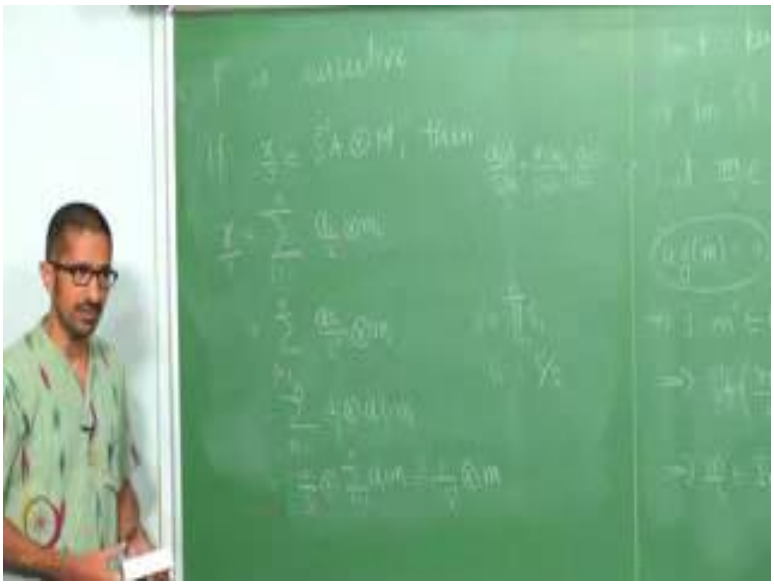

First let us look at an element of S inverse A. How does any element of S inverse A look sorry S inverse A tensor M look like, it is a linear combination of the basic tensors; the elementary tensors. So, if x by s is an S inverse a tensor M then x by s is of the form summation a i by s i tensor M I some finite combinations of the form like this.

Now if I take this, look at this summa that localization a i by s i is same as; I write a i t i by s prime tensor M i let me explain what are s prime. S prime is the product of s i i from 1 to n and t i is s prime by s i. See, i mean multiply and divide by the remaining s i’s; from S 1 to S n s i is present here, so I multiply and divide by I mean; I multiply both numerator and denominator by the remaining product. See suppose you 3 a 1 by S 1 plus a 2 by S 2 plus a 3 by S 3; what I do is I write this as S 2 S 3, S 2 S 3, here I put S 1 S three here I put S 1 S 3, here I put S 1 S 2, this is S 1 S 2. This is same as a 1 by S 1 by definition of localization. So, this becomes an element of this form sorry, a i. Now this I can take it to the other side; this is same as summation i from 1 to n 1 by s prime tensor a i t i m i, but here 1 by s prime is?

Student: Unit.

Is unit this is same for all summation. So, this I can write it as 1 by s prime tensor I from 1 to n a i t i m i.

So, this is some m. So, what did we show? Every element of S inverse a tensor M is of the form 1 by S tensor with some element of M. Every element of S inverse a tensor M can be represented ah in this manner. So, this is sorry, I mean some element x, I should write only x here. Some element x in this is of this form.

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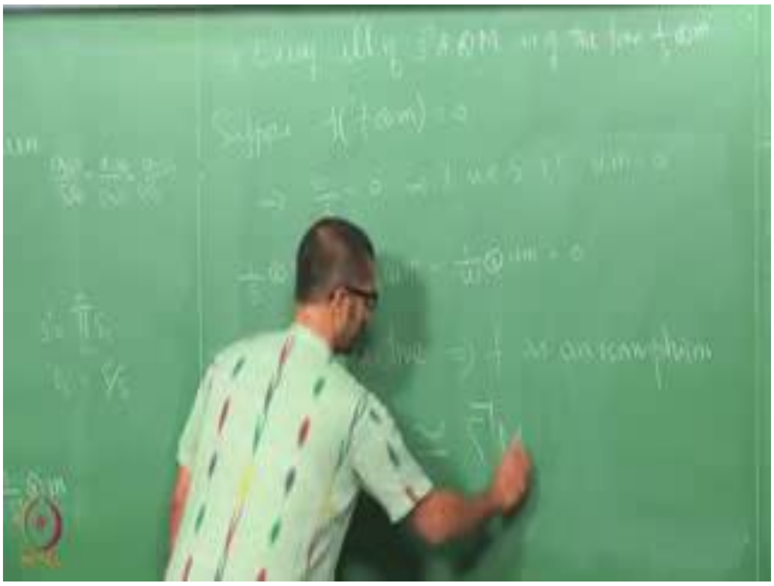

So, this implies every element of S inverse a tensor M is of the form 1 by s tensor m prime. Now it is much more easy to handle. Suppose some element is in the kernel and element is in the kernel I can take that element to be of this form. Suppose f of 1 by S tensor m is 0, that implies that m by s is 0 that implies there exists u in s such that u m is 0. Now can you see that this is 0? 1 by S tensor M, I can write it as u by u s tensor m u is in s. So therefore, this is same as this. This is equal to 1 by u S tensor u m, but u m is 0 therefore this is 0. So, this implies f is injective. So, that implies f is an isomorphism. And so what we have proved is S inverse A tensor m.

Student: Is isomorphic.

Is isomorphic to S inverse M.