Commutative Algebra

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Lecture - 18
Localization

So, we were talking about localization. This is a process where you know from a given ring. we obtain a new ring. So far we had seen many other ways of obtaining new rings that is by taking quotients adjoining variables and so on. This is another way of obtaining new rings by inverting certain number of elements.

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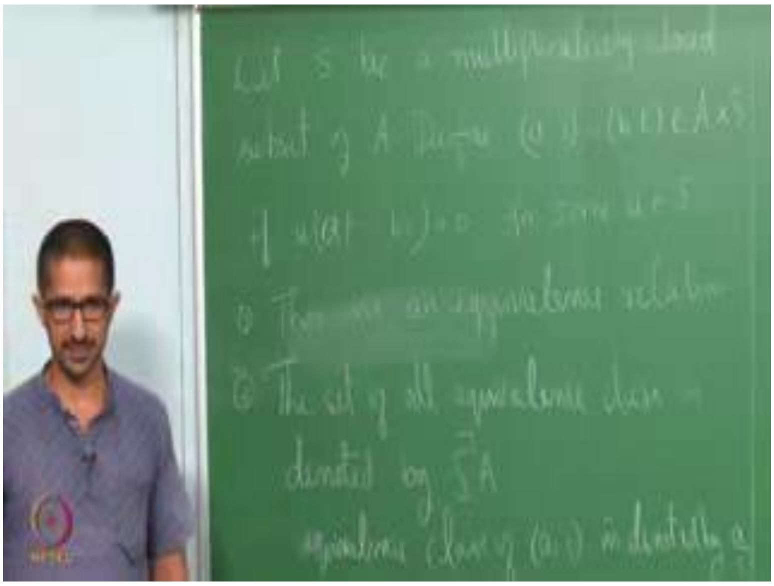

So, let me just quickly recall, what we were doing. Let S be a multiplicatively closed set closed subset of A. Then define a s equal to b t in A cross S; if a t minus b s, this is 0 for some u in S. We saw that you know this is kind of definition equivalent to the when we construct rational numbers from integers. But then we realized in rational numbers these a t minus b s is 0 is equivalent to saying a t minus b s u is 0, because it is a integral domain in integers this and both are equivalent. But in general, what we want is we want this is same as au su, we do not want to distinguish between them right that we want this to be same. So, therefore, we added this condition for the in general, this and this are not the same. So, we added this condition.

So, this is an equivalence relation. The set of all equivalence class is denoted by S inverse A, equivalence class of a comma s is denoted by as in the case of rational. So, I have this new object S inverse A. Now, from z we constructed q, and q turned out to be a ring field and so on. So, one can ask whether this is indeed a ring when we constructed A mod I, we defined a multiplication, addition and multiplication, and said well this is still ring with respect to these operations. So, in this case, if I take know collection of all equivalence classes, can we define an addition and multiplication and get this as a ring. So, there are you know obvious ways of defining these two.

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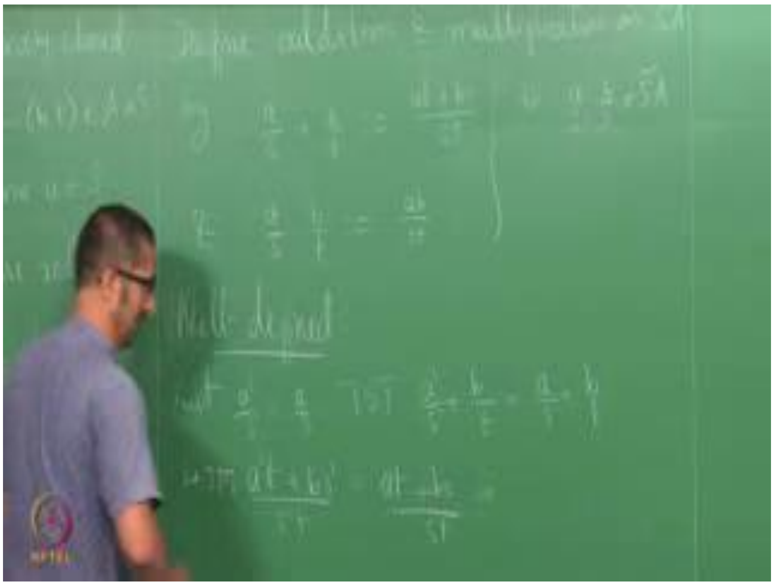

So, let us define addition and multiplication on S inverse A. How would you like to define, I have by so a by s plus b by t.

Student: (Refer Time: 05:13).

A t plus b s divided by s t. So, this is the definition. For any a by s b by t in S inverse A; and a by s times b by t a b divided by s t. Now, we have defined it as you know this is an equivalence class, it is a collection of elements not a single element right a by s is same as a u by s u for any u in S. So, the question is whether this definition depends on the representative that we choose.

Suppose instead of a s, I choose another representative which is say for example, in instead of 1 by 2, I choose 2 by 4 will this be same or different. One needs to attempt to this question that is these two definitions are well defined. You need to prove that the definitions are well defined. For this first of all, note that see I have to prove that this is well defined which means even if I choose different representatives, the sum remain the same. So, if I prove that I choose a different representative for a by s that is I choose a prime by s prime, which is equal to a by s, then for any b by t, a by s plus b by t is same as a prime by s prime plus b by t. If I prove that, we are through. We only need to prove for one because then you can next you can handle the other one and prove you know you choose any two representative different representatives for this as well as this you still get the same summation.

So, first let us prove that. I will prove this and leave this as an exercise. So, let a prime by s prime be equal to a by s. We want to show that a prime by s prime plus b by t is same as a by s plus b by t. Or in other words, what do we have? What does that we need to prove that is? We need to prove that a prime t minus b s prime is same as a t plus b s that is not enough. So, I sorry I need to prove this s prime t is s t or here plus. And this is how do you and how do you prove this that is.

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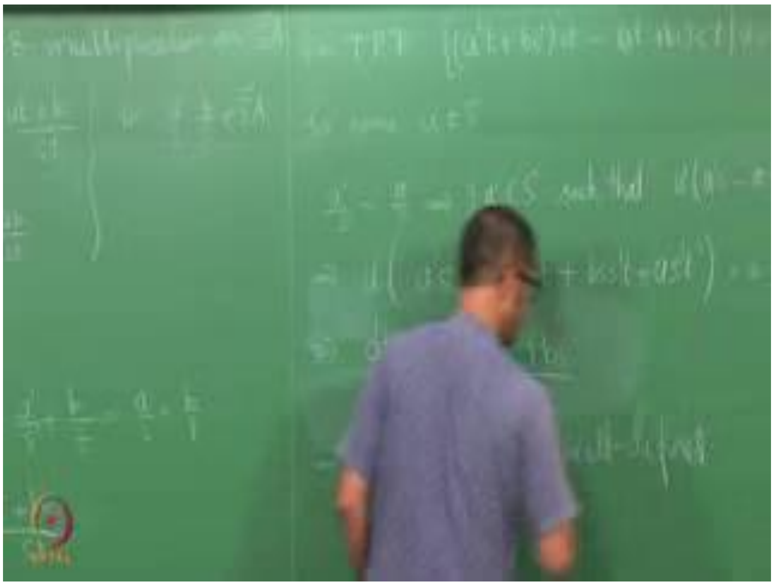

So, to prove that this is; when is this true?

Student: (Refer Time: 09:31).

This happens if a prime t plus b s prime times s t minus a t plus b s s prime t whole u is 0 for some u in S. Now, what we know is that a prime by s prime is same as a by s, a prime by s prime is same as a by s implies that there exists some let us call it u, let us see whether this u is same as this u. Let us take a u prime. Let us there exists u prime such that u prime a prime s minus a s prime is 0. So, this is zero implies that u prime times a prime s t square I forgot this minus b s s prime t plus b s s prime t plus sorry minus there should be minus right, a here a s prime t square, this is 0. I have just multiplied by t square, and added and subtracted this guy, but this is same as what we have there. Multiply this by t square, so u is same as u prime.

So, this implies a t a prime t plus b s prime by s prime t this is equal to a t plus b s divided by s t. So, this says that the addition is well defined, verifying that multiplication is well defined as even here.

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So, I will leave that to you to complete. So, verify that the multiplication is well defined. Therefore, S inverse A well there are more properties to be verified the addition is associative, the addition is associative, multiplication is distributive over the addition and the multiplication is also associative. So, I will just write here, and multiplication are associative, it is distributive of the addition and so on. So therefore, S inverse A is a ring. Well, there is what is the what is the identity, what is the additive identity, 0 by 1 that is I mean 0 by 1 is 1 representation we can have many representations. If you take any 0 by t that is going to be same as 0 by 1; so any 0 by t will be a representative for the additive identity, it has additive inverse, all these verifications, I leave to you to you know do it and convince yourself that this is indeed a commutative rings with respect to the addition and multiplication defined.

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So, we started by saying that you know construction of Q from Z and so on. So, let us look at that as the first example. So, how do you what is Q, I mean can you think of Q as S inverse A for some S in Z and A equal to z.

Student: (Refer Time: 16:06).

So, Q is S inverse A, for A equal to Z and S equal to.

Student: (Refer Time: 16:23).

One can in fact, one can think of take n or take you know.

Student: (Refer Time: 16:34).

Need not necessarily S, I mean Z with.

Student: (Refer Time: 16:44).

Need not have if A has then S inverse A will have. Because well we are we are looking at we are always looking at I mean at least for this course we are looking at commutative rings with identity. And our multiplicatively closed sets will contain identity; one is part of S for any multiplicatively closed sets by you know our convention. So, therefore, in our situation, they are all there. So, this is one first example. So, let us look at one or two more examples. Suppose I take, so my A is Z itself, and S is 1, 2, 2 square, 2 cube all powers of 2. What S inverse A?

Student: (Refer Time: 18:23) dyadic.

Dyadic.

Student: Rational.

Rational; so this is one ring which is kind of important in the number theory this class of rings. So, this is set of our m by 2 power n, m in Z and n bigger than to 0, n in z plus Z is nonnegative.

Student: (Refer Time: 19:00).

Yes, so in fact, here we do not use any property of 2. So, therefore, one can instead of 2 I can even take any p, p square, p cube and so on periodic integers. So, this is p power n this has you know even better structure; I will come back to that little late. So, let us take A to be any ring, and S to be 1 comma 0, this is a multiplicative set by definition, one is there, if two elements are there their product is also there. So, this is there.

Now what is S inverse A. If I take let us take an element a by t in S inverse A, what do you expect this ring to be. We are trying to invert zero, we are trying to invert zero, I mean see what are the I mean what is I mean observe something that in S inverse A, all the elements of S are units, all the elements of S are units. In periodic integers, you take any p power n, so p power n by 1 times 1 by p power n is 1. So therefore, all the elements in S are units here. What we are getting is a new ring with all these elements of S being units. So, we here when we have this we are expecting a ring with all the elements here being units that means, 0 is a unit. When will 0 be a unit in a ring?

Student: (Refer Time: 21:30).

It is a zero there that is what we should be expecting. We should expect that this is 0. So, what does that we need to suppose I take a by t, I want to say that this is same as 0. Can we say can I say that this is equal to 0 by 1? How do you say that this is equal to 0 by 1, if I can find a u in S such that u times a is 0, can you find u in s?

Student: 0.

0 right; so therefore, this is trivially satisfied. So, every a by t is equal to 0 by 1 which means there is only one element S inverse A is 0. So, this let a by t then 0 times a is 0, which means a by t is same as 0 by 1 and that implies that S inverse A is there is only one. If 0 is there in the multiplicative set then S inverse A is 0.

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Conversely, suppose S inverse A is 0. So, let me if S inverse A is 0 then what kind of conclusion you want to make. So, let me write that 0 is in S, try to prove that. I will leave it to you to check.

Student: (Refer Time: 23:22).

Yes, correct, A not equal to 0. Now, let us look at one some of the properties of the localization. We will see few more examples maybe I will take one more example. Take A to be Z 6, and S to be 1, 3 is this is this multiplicative set, 3 times 3 in Z 6 is 3 again. So, 3 power n is 3 for any n bigger than to 1. So, therefore, this is a multiplicative set, what are the elements of S inverse A, 0 by 1, 0 by 1 and 0 by 3, they are the same. So, we do not have to write separate, 1 by 1 there are they I am not going to write bars here, they are all in z 6. 1 by 2, 1 by 3 there are many elements. So, let me not list them let us look at one or two you know elements here. I have let us look at 2 by 1 in this; this is a slightly different variation of this. See, here 0 times a is 0. Now, if I have if a is a 0 divisor, can you see here 2 by 1, this is same as 0 by 1. Why is it so?

Student: 3 times.

3 times 2 is 0, 3 is there in S. So, therefore, 2 by 1 is 0, 2 by 1 is 0 therefore, four by 1 is 0. So, there are more you know elements here which do not seem like 0, but they are actually 0, 2 by 1 to start with we do not expect we do not think that it is 0, but this is 0. So, this you know example says something more. First there exists a map in general there exists a map, let us call it f from A to S inverse A, f of a equal to a by 1 this is a map from A to S inverse A is this only a map is it something more is it a homomorphism both are rings. A plus b will be a plus b by 1 which is same as a by 1 plus b by 1; a b will be again a b divided by 1 which is a by 1 times b by 1. So, therefore, this is a ring homomorphism, f is a ring homomorphism. What more, is this an injective map? Here is an example right. This is not injective.

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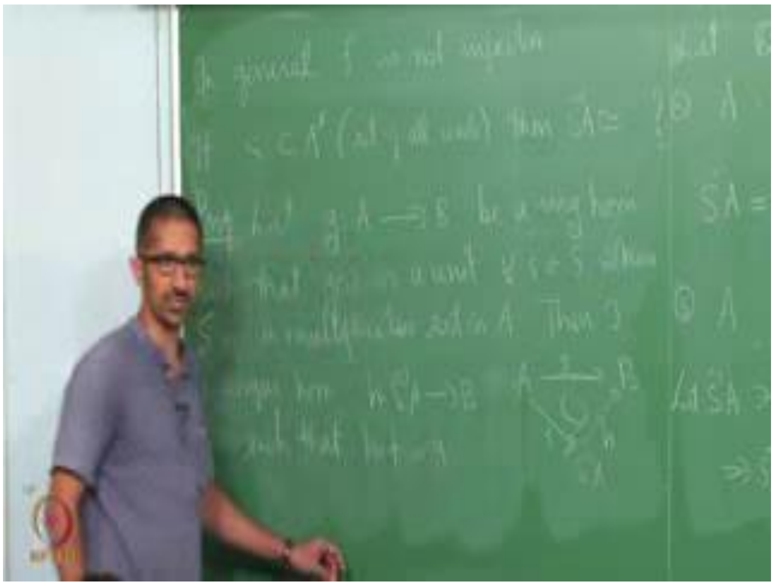

In general, f is f is not injective. Now, suppose I have a map, suppose I have a homomorphism. So, this is called universal mapping property of the localization. So, now S inverse A is called localization of A with respect to the multiplicative set S, one universal mapping property of the localization is the following.

Student: (Refer Time: 28:23).

Yes.

Student: (Refer Time: 28:26).

Yes.

Student: (Refer Time: 28:29).

Units in A; so let me write this as an exercise- if S is a subset of A cross set of all units then S inverse A, what would be S inverse A, try to see what would this be. We are trying to invert elements, which are already units there, and we are not trying to invert anything more. So, let us explore what this is I mean try to develop some idea and try to prove it. So, let f be a, so let g be a ring homomorphism such that g of s is a unit for all s in S. So, A be a s is a multiplicative set in A where s is a multiplicative set.

So, what we have is I have a map from A to B, this is g, I have a map from A to S inverse A, this is f, so this is g, and this is f. Then there exists a unique homomorphism h from S inverse A to B such that the diagram is commutative. Or in other words, h composite f is g. See, what it says is simple. If I have a ring homomorphism such that all the images of g, all the images of elements of s are units here, see they are all units here. So, you have a natural extension from here to here. If they are units here, then you have a natural extension that is all what it says is, so we have to just simply define a homomorphism and then prove the uniqueness. There is a very natural way of defining a homomorphism.

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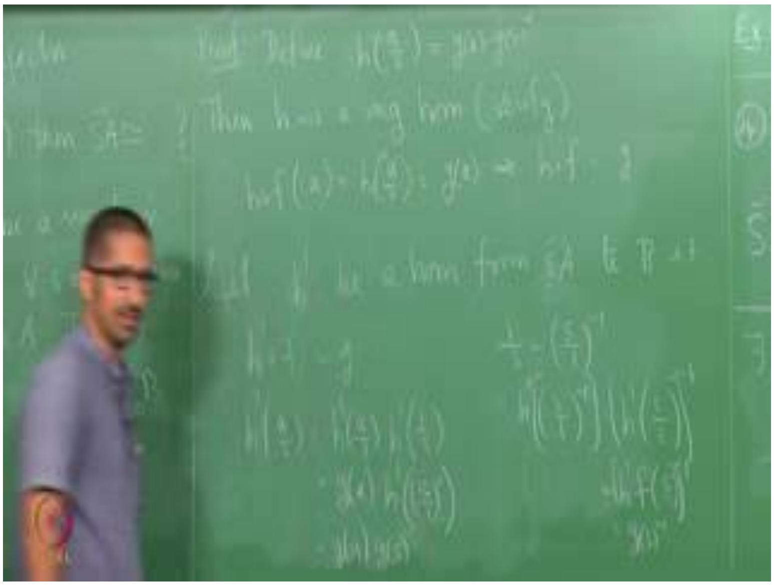

I want to say h of a by s g a times, will I want it to be g s. So, see what we want is basically a by s should go to I mean a by s should go to g a by g s that is what we the natural ring extension from here to here would be. But what is meant by g a by g s, it is basically g a times g s inverse. So, this is then h is a ring homomorphism, I will you know verify, there are easy ways to verify this. I will leave it to you to do that. h is a ring homomorphism. And what is h composite f of an element a, f of a is a by 1. So, h of f of a is h of a by 1, this is g a times, g of a is 1, where we are having maps between commutative rings with identity. So, therefore, this is one. So, this says that h composite f is g.

What we need to prove is this part now this is not only that this is unique once you define g this is unique. So, suppose there exists, so let g prime h prime be any be a homomorphism from S inverse A to B such that h prime composite f is g. I want to say that h prime is equal to h or in other words h prime of a by s is same as h of a by s for every s. Now, let us look at what would be h prime of a by s. I can write this as h prime of you know a by 1 times h prime of 1 by s. Now what is h prime of a by 1, this is a by 1 is f of.

Student: F of a.

F of a; so this is h prime of f of a which is g of k. Now, what is h prime of 1 by s same as h prime of s by 1 inverse. See 1 by s inverse of s by 1, 1 by s is same as s by 1 inverse because their product is one, but therefore now what is f of s by 1 inverse sorry. So, therefore, h prime of s by 1 inverse h prime is ring homomorphism. Therefore, h prime of s by 1 inverse is same as h prime of s by 1 inverse, but h prime of s by 1 is h prime of.

Student: F of s.

F of s h prime of f of S is g of s. This is h prime of f of s inverse which is g s inverse. So, what have you obtained here, this is g a times g s inverse. And that is precisely what is h of a by s. Therefore, this is unique. Is this clear? There exists a unique homomorphism which takes a by s to g a times g s inverse. So, now, this is you know this is a universal mapping property of the localization, and there is much more to it. This defines the localization.

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See I have this map f A to S inverse A. So, I have these properties there exists f such that if S is in S then f of s is a unit; obviously, that is how we have constructed. Now, if f of a is 0, what is meant by f of a is 0, a by 1 is 0 or in other words there exists u in S such that u times a is 0. And third one is every element of S inverse A is of the form f a times f s inverse; every element is of the form a by s, which I can write it as a by 1 times one by S that is of this one.

Now, the surprising thing is that this defines the localization uniquely. In the sense suppose I have a

Student: (Refer Time: 39:59).

No, this need not be surjective.

Student: (Refer Time: 40:04).

Is of the form f a times f s inverse.

Student: (Refer Time: 40:10).

Every element see what is this map this map is a going to a by 1 that can be far from being surjective. For example, Z to Q, but every element is of the form f a times f s inverse. See the thing is that S inverse need not be in A. When we talk about S inverse that need not exist in a see every element in Q, looks like m by n m times n inverse, but that n inverse is not here, this I cannot write it as A S inverse because this need not exist in a at all.

Now, suppose I have a ring homomorphism which satisfies these three properties that is let g from A to B be a ring homomorphism such that for S in S g of S is A unit in B. Second is if g of a is 0, then there exists u in S such that u times a is 0. Third one, every element of B is of the form g a g s inverse then what we are saying is that B is nothing but S inverse A.

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So, how do we write that in algebraic terms then? There exists a unique isomorphism h from S inverse A to B. So, I have such that, so I have A to B, A to S inverse A, this is h, this is f, this is g, such that h composite f is g. How do you prove this? Again we have to define an isomorphism from a map from S inverse A to B that is I need to define h of a by s as something.

Student: (Refer Time: 43:34).

G a times g s inverse then this is a ring homomorphism let us you know I will assume that you have to check that. Then h is a ring homomorphism, h is what does the third property said every element of B is of the form g a times g s inverse, therefore.

Student: (Refer Time: 44:08).

H is.

Student: (Refer Time: 44:11).

Third property why is that h is surjective. Now, what does the second property say, we want to check whether it is injective. So, suppose h of a by s 0 which means.

Student: G a.

G a g S inverse is 0, but what is g s inverse g s is a unit in B therefore, g a g s inverse is 0 will imply.

Student: G a is 0.

G a is 0, but the second property says that there exists u such that u times a is 0.

Student: (Refer Time: 44:57).

That means a by s is 0; that means, f is injective. So, if h of a by s is 0 that implies g a g s inverse is 0 that implies g a is 0. Since g s is unit in B that implies by second property there exists u in S, such that u times a is 0, and that precisely says that a by s is 0 in S inverse A that implies f is injective. So, therefore, f is an isomorphism.

Student: (Refer Time: 45:44).

Sorry not f, h is injected. Therefore, h is isomorphism uniqueness can be verified in the same manner.

Student: (Refer Time: 46:01).

Exactly from the previous proposition one can prove that it is unique. So, the localization is defined precisely using these three. We will see more properties of localization later.